QUIVERS IN REPRESENTATION THEORY (18.735, SPRING 2009) LECTURE 1

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0. Motivation

Quivers are directed graphs. The term is the term used in representation theory, which goes along with the following notion: a representation of a quiver is an assignment of vector spaces to vertices and linear maps between the vector spaces to the arrows.

Quivers appear in many areas of mathematics:

(1) Algebraic geometry (Hilbert schemes, moduli spaces (represent these as varieties of quiver representations); derived categories $D^b(\text{Coh } X)$, where X is a projective variety or a scheme of finite type (give an equivalence with the derived category of dg representations of certain quivers with relations))
(2) Representation theory (quiver algebras, groups, local systems/vector bundles with flat connection on curves)
(3) Lie theory (Kac-Moody Lie algebras are related to the combinatorics of representations of the associated quiver; quantum enveloping algebras of Kac-Moody algebras can be constructed from the category of quiver representations)
(4) Noncommutative geometry (computable examples, Calabi-Yau 3-algebras in terms of quivers with potentials)
(5) Physics (Calabi-Yau 3-folds X and their branes: one can represent $D^b(X)$ as a derived category of certain quiver representations; or in the affine case $X = \text{Spec}\, A$, one can sometimes present A using a quiver and a potential)

1. Introduction

Wikipedia (`http://en.wikipedia.org/wiki/Quiver_(mathematics)`) has the following to say about quivers:

A *quiver* is a directed graph, in which multiple edges and loops are allowed. The edges are called *arrows*, hence the name "quiver." For example, see Figure 1.

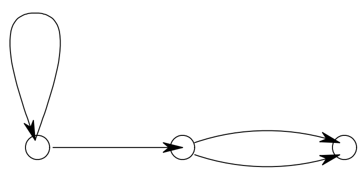


Figure 1. A quiver

Definition 1.1. For any quiver Q, let Q_0 denote its set of vertices, and Q_1 its set of arrows. Let $t, h : Q_1 \to Q_0$ be the maps assigning to each arrow, a, its head $h(a)$ (the vertex a points to), and tail $t(a)$ (the other endpoint of a). We will use the notation $a : t(a) \to h(a)$ to indicate the head and tail of a.

The term "quivers" is frequently used in representation theory, with the following definition in mind. Fix a field $\mathbf{k}$; all vector spaces will be assumed to be over $\mathbf{k}$.

Definition 1.2. A *representation* $(V_i, \rho_a)_{i\in Q_0, a\in Q_1}$ of a quiver Q is an assignment of a vector space V_i to each vertex i and a linear transformation $\rho_a : V_{t(a)} \to V_{h(a)}$ of vector spaces to each arrow.

Representations of a quiver Q form an abelian category $\mathcal{R}_Q$ in the following obvious way: objects are representations, and morphisms $\Phi : (V_i, \rho_a) \to (W_i, \tau_a)$ are collections of linear maps $\Phi = (\Phi_i : V_i \to W_i)_{i\in Q_0}$, such that $\Phi_{h(a)}\rho_a = \tau_a \Phi_{t(a)}$ for all arrows $a \in Q_1$. There are some standard examples of representations of Q:

Example 1.3. The *trivial representation* is the zero representation, i.e., (V_i, ρ_a) such that $V_i = 0$ for all i.

Example 1.4. For any single vertex $i_0 \in Q_0$, the simple representation S_{i_0} is given by $S_{i_0} = (V_i, \rho_a)$, with $V_i = 0$ except when $i = i_0$, where $V_{i_0} = \mathbf{k}$, and $\rho_a = 0$ for all $a \in Q_1$.

Example 1.5. For any two representations, we may form their direct sum: $(V_i, \rho_a) \oplus (W_i, \tau_a) = (V_i \oplus W_i, \rho_a \oplus \tau_a)$.

The category $\mathcal{R}_Q$ is the same as $Mod(P_Q)$, the category of left modules over the following ring P_Q, called the *path algebra*: P_Q is the vector space generated by paths in the quiver Q, with reverse concatenation as the multiplication: here, if $p : j \to k$ and $q : i \to j$ are paths, then $pq : i \to k$ is the concatenation; see Figure 2.

FIGURE 2. Product of paths p and q

This gives another explanation why $\mathcal{R}_Q = Mod(P_Q)$ is an abelian category. It now makes sense to consider *indecomposable* representations of Q: these are indecomposable objects of $\mathcal{R}_Q$, i.e., indecomposable P_Q−modules. Equivalently, these are representations V not representable as a direct sum $V \cong (V_1 \oplus V_2)$, where V_1, V_2 are nontrivial representations.

Definition 1.6. The dimension vector of a representation $V = (V_i, \rho_a)$, dim $V \in \mathbb{Z}_{\geq 0}^{Q_0}$, is given by dim $(V) = (\text{dim } V_i)_{i\in Q_0}$.

A third definition of $\mathcal{R}_Q$ is the category of contravariant functors $\mathcal{C}_Q \to \text{Vect}$, where $\mathcal{C}_Q$ is the category freely generated by the arrows Q_1.

A central problem of the representation theory of quivers is to classify all indecomposable representations for a given quiver. The following is the first main theorem in this direction (and the first goal of the course): Let $\text{Indec}(\mathcal{R}_Q) \subset \mathcal{R}_Q$ be the subclass of indecomposable representations.

Theorem 1.7 (Gabriel's Theorem)**.** *Let Q be a connected quiver.*

(i) *There are finitely many isomorphism classes of indecomposable representations if and only if Q is a Dynkin quiver (i.e., if we forget the orientation of arrows of Q, the resulting undirected graph is a Dynkin diagram of type A, D, or E).*

(ii) *In this case, the indecomposable representations are in bijection with the positive roots Δ_+ of the associated root system. Specifically, fixing a choice of simple roots $\{\alpha_i\}_{i\in Q_0} \subset \Delta_+$, the map* dim *yields a bijection*

$$\mathrm{Indec}(\mathcal{R}_Q)/\sim\ \xrightarrow{\sim} \Delta_+, \quad V \mapsto \sum_{i\in Q_0} \dim\,(V)_i \cdot \alpha_i. \tag{1.8}$$

(Recall that the root system of a Dynkin diagram is the set of weights of the associated Lie algebra $\mathfrak{g}_Q$ under the action of its maximal abelian subalgebra $\mathfrak{h}_Q$. We will also give an explicit combinatorial definition of Δ later.)

This gives the first sign that Dynkin diagrams are fundamental in the study of quivers. As a reminder, the Dynkin diagrams of type A, D, and E are depicted in Figure 3.

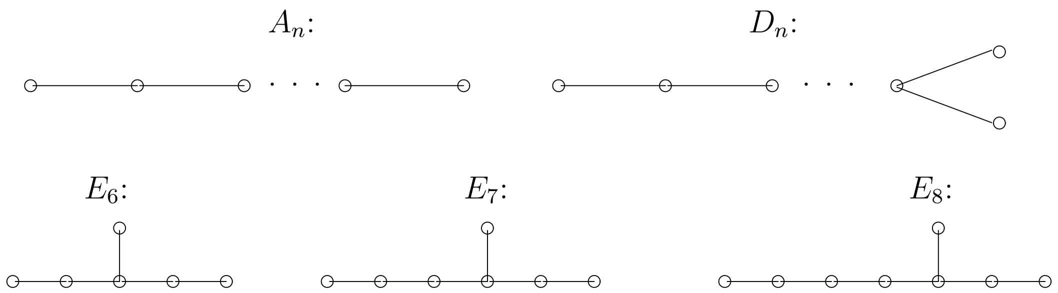


FIGURE 3. Dynkin diagrams: the subscript equals the number of vertices

In particular, for Q Dynkin, the set of isomorphism classes of indecomposables, $\mathrm{Indec}(\mathcal{R}_Q)/\sim$ does not depend, up to isomorphism, on the choice of orientations of the arrows of Q. It is a deep fact that the same holds for any graph. Indeed, there is a vast generalization of Gabriel's theorem to arbitrary quivers, called Kac's theorem, explaining that, for instance, the dimension vectors of indecomposables are positive roots of the associated Kac-Moody Lie algebra. We will get to this in a moment.

2. The McKay correspondence

In fact, Dynkin diagrams pop up all over the place in mathematics (quiver representations, Lie theory, platonic solids, etc.) The McKay correspondence is the name given to this broad-ranging dictionary and the study of direct connections between the different ways in which Dynkin diagrams appear. Here, a Dynkin diagram is considered as an *unoriented* graph.

Definition 2.1. Given a subgroup $G < \mathsf{SL}_2(\mathbb{C})$, the *McKay diagram* of G is given as follows: the vertices are labeled by irreducible representations of G, and given two irreducible representations ρ_i, ρ_j, the number of edges between ρ_i and ρ_j is $\dim(\mathrm{Hom}(\rho_i, \mathbb{C}^2 \otimes \rho_j))$. The *truncated McKay diagram* is given in the same way, but removing the vertex corresponding to the trivial representation.

Theorem 2.2 (The McKay correspondence)**.** *The following are all classified by Dynkin diagrams:*

(1) *Underlying unoriented graphs of quivers with finitely many indecomposable representations;*
(2) *Diagrams appearing as follows, for some finite subgroup $G < \mathsf{SL}_2(\mathbb{C})$:*
  (2a) *The truncated McKay diagram of G;*
  (2b) *The intersection matrix of the components of the exceptional fiber $\pi^{-1}(0)$ of the minimal resolution $\widetilde{X} \xrightarrow{\pi} \mathbb{C}^2/G$ of the associated simple rational surface singularity;*

(3) *Dynkin diagrams of simply-laced finite-dimensional Lie algebras, i.e., diagrams whose associated simply-laced root system is finite, i.e., whose associated Kac-Moody Lie algebra is finite-dimensional;*
(4) *Diagrams whose adjacency matrix A has the property that $2 \cdot \mathrm{Id} - A$ is positive-definite (i.e., $v^t A v > 0$ for all nonzero column vectors v, where v^t is the transpose of v).*

The matrix $2 \cdot \mathrm{Id} - A$ is called the *Cartan matrix*, and coincides with, in (2b), the intersection pairing on second homology classes, and in (3), with the Cartan form (inner product) on positive roots, in the basis of simple roots.

There are many proofs of the above theorem. One main goal is to not merely prove the theorem, but find as many explicit bijections between the different items above as possible. We explain now some of the known ones.

The equivalence (1) $\Leftrightarrow$ (3) is a consequence of Gabriel's theorem (Theorem 1.7).

For the remaining parts, it will turn out to be more convenient to prove an *affine* version of the above correspondence, which uses *nontruncated* McKay diagrams, *affine* Lie algebras, and positive-*semidefinite* Cartan matrices (A is positive-semidefinite if $v^t A v \geq 0$ for all column vectors v), which are not positive-definite. As we will see in the exercises, the advantage is that, not only are the graphs with semidefinite Cartan matrices easier to classify, but also one may show that any proper subdiagram has a positive-definite Cartan matrix and any strictly larger diagram has an indefinite Cartan matrix, and this classifies all graphs.

The map (2a) $\Rightarrow$ (4) follows by showing that the nontruncated McKay diagram must always correspond to a *positive-semidefinite* Cartan matrix. Namely, the matrix of the operator $\rho \mapsto (\mathbb{C}^2 \otimes \rho)$, acting on representations of G written in terms of the basis of irreducible representations, has eigenvalue 2 with eigenvector corresponding to the regular representation, $\mathbb{C}[G]$. (This can also be stated using the language of tensor categories). By the Perron-Frobenius theorem, all other eigenvalues have absolute value less than 2. One can then see explicitly (using the previous paragraph) that this map is bijective.

The equivalence (3) $\Leftrightarrow$ (4) is explained in any course on Lie algebras. For (3), *simply-laced* means that, for two distinct simple roots $\alpha \neq \beta$, we have $(\alpha, \beta) \in \{-1, 0, 1\}$. The implication (3) $\Rightarrow$ (4) follows because the finiteness condition implies that the inner product is positive-definite. For the opposite implication, we can use extended Dynkin diagrams, which are obtained by adding a new vertex labeled by the *negative root* $-\delta'$, where $\delta' \in \Delta_+$ is the maximal root; equivalently, we replace the truncated McKay diagram of (2a) with the nontruncated one. Namely, if Γ is a Dynkin diagram and $\widetilde{\Gamma}$ the corresponding extended Dynkin diagram, one may show that, for a special positive root $\delta \in (\Delta_{\widetilde{\Gamma}})_+$ (which corresponds to δ' plus the elementary vector of the extending vertex), every coefficient of α in the basis of simple roots is less than or equal to the corresponding coefficient of δ. See the exercises for details.

The arrow (3) $\Rightarrow$ (2b) may be given via Slodowy slices, which are transverse slices to so-called sub-principal orbits (=orbits of codimension 2) in the nilpotent cone of the given Lie algebra. This gives $\mathbb{C}^2/G$; its resolution may be obtained from the Springer resolution of the nilpotent cone. We may discuss this in more detail later in the course.

2.1. **Hilbert schemes, quiver varieties, and (2b) (Sketch, may be omitted).** The equivalences (2b) $\Leftrightarrow$ (2a), (4) may be realized using quiver varieties. This goes roughly as follows: $\mathbb{C}^2/\Gamma =$ the set of Γ-orbits in $\mathbb{C}^2$. The problem is that the zero orbit is degenerate. To fix this, consider

$$\mathbb{C}^2/\Gamma = \{\Gamma\text{-orbits in } \mathbb{C}^2\} \subset \text{unordered } |\Gamma|\text{-tuples in } \mathbb{C}^2\} = \mathrm{Sym}^{|\Gamma|}\mathbb{C}^2. \tag{2.3}$$

Now, $\mathrm{Sym}^{|\Gamma|}\mathbb{C}^2$ is itself singular. A resolution may be obtained using Hilbert schemes, as follows. Consider unordered $|\Gamma|$-tuples to be divisors in $\mathbb{C}^2$, i.e., formal sums $\sum_{z \in \mathbb{C}^2} \lambda_z z$, where $\lambda_z \in \mathbb{Z}_{\geq 0}$ and almost all λ_z are zero. Then, replace each weighted point $\lambda_z z$ by a subscheme of $\mathbb{C}^2$ concentrated at z, of length λ_z. That is, $\lambda_z z$ is replaced by an ideal $I \subset \mathbb{C}[x,y]$ such that $\mathbb{C}[x,y]/I$ is an algebra with unique maximal ideal $\mathfrak{m}_z$, and has dimension λ_z. The result is that $\mathrm{Sym}^{|\Gamma|}\mathbb{C}^2$ is replaced by $\mathrm{Hilb}_n(\mathbb{C}^2) = \{\text{ideals } I \subset \mathbb{C}[x,y] \text{ of codimension } n\}$, and the proper transform of $\mathbb{C}^2/\Gamma$ under the resolution $\mathrm{Hilb}_n(\mathbb{C}^2) \twoheadrightarrow \mathrm{Sym}^{|\Gamma|}\mathbb{C}^2$ yields the minimal resolution. This proper transform consists exactly of those I such that $\mathbb{C}[x,y]/I \cong \mathbb{C}\Gamma$, the regular representation, as representations of Γ.

Thus, the resolution of $\mathbb{C}^2/\Gamma$ is a moduli space of Γ-equivariant modules over $\mathbb{C}[x,y]$ which are isomorphic to the regular representation of Γ. To proceed, one restates this in terms of modules over the so-called *preprojective algebra* of the McKay diagram of Γ. This means that we consider the *McKay quiver*, obtained from the McKay graph by replacing each unoriented edge with two arrows going in each direction, and look at representations satisfying certain relations corresponding to the commutativity condition $xy = yx$.

Then, we see that both $\mathbb{C}^2/\Gamma$ and its resolution as above are obtained as varieties of modules over the McKay quiver satisfying these relations. Namely, $\mathbb{C}^2/\Gamma$ is the variety of modules, up to isomorphism, of dimension vector $\delta =$ the minimal positive vector in the kernel of the Cartan form (explained below as the symmetric Ringel form). The resolution is obtained by defining a *stability condition* on modules, and restricting to *stable* modules of this dimension vector before modding by isomorphism. This latter variety is called the *Nakajima quiver variety*, and it generalizes to give a description not only of the Hilbert schemes $\mathrm{Hilb}_n(\mathbb{C}^2)$ and its completion $\mathrm{Hilb}_n(\mathbb{CP}^2)$, but of the moduli spaces of torsion-free sheaves E on $\mathbb{CP}^2$ of rank r with $c_2(E) = n$.

Looking at the resulting quiver representations yields the desired description of $\pi^{-1}(0)$, and hence the equivalence (2a) $\Leftrightarrow$ (2b), and hence (2b) $\Leftrightarrow$ (4).

3. Kac's Theorem

Now, we give a precise statement of Kac's theorem, alluded to earlier. Let Q be a quiver *without loops*, i.e., without arrows a such that $h(a) = t(a)$. Let $\mathbf{k}$ be an algebraically closed field. As you may know, there is a Kac-Moody Lie algebra $\mathfrak{g}_Q$ associated to the underlying undirected graph of Q (we will recall the definition later in the course, and will not need it at the moment). There is an associated root system, $\Delta \subset \mathbb{Z}^{Q_0}$, consisting of those weights under the adjoint action of the Cartan subalgebra $\mathfrak{h}_Q \subset \mathfrak{g}_Q$ occuring in the decomposition of $\mathfrak{g}_Q$. The positive roots are $\Delta_+ := \Delta \cap \mathbb{Z}^{Q_0}_{\geq 0}$, and $\Delta = \Delta_+ \sqcup -\Delta_+$. The positive roots Δ_+ may be further divided into the *real* and *imaginary* roots, explicitly described as follows.

Definition 3.1. The *Ringel form* on $\mathbb{R}^{Q_0}$ is given by

$$\langle \beta, \gamma \rangle = \sum_{i \in Q_0} \beta_i \gamma_i - \sum_{a: i \to j} \beta_i \gamma_j. \tag{3.2}$$

The *symmetrized Ringel form* is $(\beta, \gamma) = \langle \beta, \gamma \rangle + \langle \gamma, \beta \rangle$.

Note that the symmetrized Ringel form is nothing but the Cartan form on root space, i.e., $(\beta, \gamma) = \beta^t C \gamma$, where C is the Cartan matrix of the underlying undirected graph of Q. We will mainly apply the Ringel form to the dimension vectors, $\mathbb{Z}^{Q_0}_{\geq 0}$, where the result is obviously an integer.

For $i \in Q_0$, let ε_i be the elementary vector corresponding to i.

Definition 3.3. The *fundamental region* $F \subset \mathbb{Z}^{Q_0}_{\geq 0}$ consists of those dimension vectors $\alpha \in \mathbb{Z}^{Q_0}_{\geq 0}$ such that

(1) $(\alpha, \alpha) \leq 0$,
(2) α has connected support, and
(3) $(\alpha, \varepsilon_i) \leq 0$ for all $i \in Q_0$.

We will need to consider the *simple reflections* of dimension vectors:

Definition 3.4. If $i \in Q_0$, we consider the simple reflection $s_i : \mathbb{R}^I \to \mathbb{R}^I$ given by

$$s_i(\beta) = \beta - (\beta, \varepsilon_i)\varepsilon_i. \tag{3.5}$$

(Note that the group generated by all simple reflections is called the Coxeter group of the graph, and it is important all over mathematics.)

Definition 3.6. A real root is an element of $\mathbb{Z}_{\geq 0}^{Q_0}$ which is obtainable from some vector ε_i by a sequence of simple reflections.

Definition 3.7. An imaginary root is an element of $\mathbb{Z}_{\geq 0}^{Q_0}$ obtainable from an element of the fundamental region by a sequence of simple reflections.

Theorem 3.8 (Kac's Theorem)**.** *Let Q be a quiver without loops. We work over an* algebraically closed *field* $\mathbf{k}$.

(i) *There is an indecomposable representation of dimension vector α if and only if α is a root.*
(ii) *If α is a real root, there is a unique indecomposable representation up to isomorphism; otherwise, there are infinitely many.*

Remark 3.9. The requirement that Q be loopless is actually not needed, if we change our definitions: we only use simple reflections at vertices which do not have loops, and the real roots are the images of elementary vectors at only these vertices under the action of these simple reflections, while the imaginary roots are the images of elements of the fundamental domain under only these simple reflections. In particular, the elementary vector of any vertex which has a loop becomes an imaginary root (which it should be, since there is $\mathbf{k}$-worth of indecomposable representations with that dimension vector).

The proof involves showing that $\beta \rightsquigarrow s_i\beta$ is accompanied by a bijection on the level of indecomposable representations. To show this, we define *reflection functors*, acting as follows:

$$\begin{array}{ccc} Q & \longmapsto & s_iQ \\ \\ V & \longmapsto & s_iV \\ \\ \dim V = \beta & \longmapsto & \dim s_iV = s_i\beta. \end{array} \tag{3.10}$$

QUIVERS IN REPRESENTATION THEORY (18.735, SPRING 2009) LECTURES 2,3 AND 4

TRAVIS SCHEDLER, TYPED BY IVÁN ANGIONO

1. Some examples

(i) A_2 : $\circ \longrightarrow \circ$. We have three indecomposables:

$$\begin{array}{cccc} & \circ^{\mathbf{k}} \longrightarrow \circ^{0} & \circ^{0} \longrightarrow \circ^{\mathbf{k}} & \circ^{\mathbf{k}} \xrightarrow{\sim} \circ^{\mathbf{k}} \\ \text{vector dim} & (1,0) & (0,1) & (1,1) \end{array}.$$

(ii) A_3 : $\circ \longrightarrow \circ \longrightarrow \circ$. We have 6 indecomposables: those with dimensions $(1,0,0), (0,1,0), (0,0,1)$ are the S_i's; the indecomposables with dimensions $(0,1,1)$ and $(1,1,0)$ are analogous yo the previous case, and the indecomposable with dimension $(1,1,1)$ is $\circ^{\mathbf{k}} \xrightarrow{\sim} \circ^{\mathbf{k}} \xrightarrow{\sim} \circ^{\mathbf{k}}$.

(iii) *Kronecker quiver*: $\circ \overset{a}{\underset{b}{\rightrightarrows}} \circ$. If $\mathbf{k}$ is infinite, there are infinitely many indecomposables $V = (V_1, V_2)$ of dimension (n, n), for each $n \in \mathbb{N}$.

To construct indecomposables, we fix ρ_a an isomorphism and look at $\rho_b \in \operatorname{Hom}(V_1, V_2)$. We also fix bases in V_i such that ρ_a is represented in such bases by the identity matrix. For each $\lambda \in \mathbf{k}$, we consider V_λ, where ρ_b is given by the matrix

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}. \tag{1.1}$$

V_λ is indecomposable because ρ_b is an indecomposable matrix, and for $\lambda \neq \lambda'$, $V_\lambda \not\cong V_{\lambda'}$.

(iv) $\widetilde{A}_0$: $\circlearrowright$. In this case, the n-dimensional representations up to isomorphism are in correspondence with the conjugate classes of $n \times n$ matrices. So the indecomposables up to isomorphism include (1.1), and they are infinitely many if $\mathbf{k}$ is infinite.

(v) $\widetilde{D}_4$:
$$\begin{tikzcd} & \circ \arrow[d, "a_3"] & \\ \circ \arrow[r, "a_1"] & \circ & \circ \arrow[l, "a_2"] \\ & \circ \arrow[u, "a_4"] & \end{tikzcd}$$
. For $\alpha = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}$ and each $V \in \operatorname{Indec}_\alpha Q$, ρ_{a_i} is injective, $i = 1, 2, 3, 4$, and $\operatorname{Indec}_\alpha Q$ is in correspondence with the 4-uples of 1-dimensional subspaces of $\mathbf{k}^2$

such that at least three of them are distinct. Geometrically,

$$
\begin{aligned}
&Conf(4,\mathbb{P}^1) \subseteq \text{indecomposables},\\
&(z_1,z_2,z_3,z_4) \sim (z_1',z_2',z_3',z_4') \Leftrightarrow \text{they are related by } P\in PGL_2,\\
&Conf(4,\mathbb{P}^1)/PGL_2 \cong \mathbb{P}^1 = \{[z,0,1,\infty]\}.
\end{aligned}
$$

2. Digression on representation varieties

Let A be an associative algebra over a field $\mathbf{k}$, and $n \geq 1$. Consider

$$
\begin{aligned}
\mathrm{Rep}_n(A) &= \{\text{affine variety of representations of } A \text{ of dimension } n\}\\
&= \mathrm{Hom}_{alg}\left(A, \mathrm{Mat}_n(\mathbf{k})\right),\\
B(\mathrm{Rep}_n(A)) &= \mathrm{Hom}_{alg}\left(A, \mathrm{Mat}_n(\mathbf{k})\right), \quad B\,\mathbf{k}-\text{algebra (commutative)}.
\end{aligned}
$$

This functor of points Aff Sch $\to$ Sets is representable by $\mathrm{Rep}_n(A)$. For example,

$\mathbf{A}$	$\mathrm{Rep}_{\mathbf{n}}\mathbf{A}$
$\mathbf{k}\langle x_1,...,x_m\rangle$	$\mathrm{Mat}_n(\mathbf{k})^{\otimes m}$
$\mathbf{k}[x,x^{-1}]$	$GL_n(\mathbf{k})$
$\mathbf{k}[x,y]$	$\{(x,y):[x,y]=0\}\subseteq \mathrm{Mat}_n(\mathbf{k})^{\otimes 2}$
$\mathbf{k}\langle x,y\rangle/(xy-yx-1)$	$\{\mathrm{ch}\ \mathbf{k}+n\mathrm{tr}(\rho(x)\rho(y)-\rho(y)\rho(x)-1)\neq 0\}$.

We want a description of $\mathrm{Rep}_n A$ for any finitely generated $\mathbf{k}$-algebra A; we write:

$$I \hookrightarrow F = \mathbf{k}\langle x_1,...,x_m\rangle \twoheadrightarrow A = F/I.$$

In this case, $\mathrm{Rep}_n(A) \subseteq \mathrm{Rep}_n(F)$ is the zero locus of $\mathrm{ev}_n^F(I)$,

$$
\begin{aligned}
ev_n^A : A &\to \mathbf{k}[\mathrm{Rep}_n A]\otimes \mathrm{Mat}_n(\mathbf{k}),\\
a &\mapsto (\rho\mapsto \rho(a)).
\end{aligned}
$$

One can check the following universal property:

$$
\begin{tikzcd}
A \arrow[r] \arrow[dr] & \mathbf{k}[\mathrm{Rep}_n A]\otimes \mathrm{Mat}_n(\mathbf{k}) \arrow[d, "\exists!\phi\otimes\mathrm{id}"] \\
& B\otimes \mathrm{Mat}_n(\mathbf{k}).
\end{tikzcd}
\tag{2.1}
$$

So the zero locus of $\mathrm{ev}_n^F(I)$ represents our functor of points; i.e. we can write $\mathbf{k}[\mathrm{Rep}_n A]$ as

$$\mathbf{k}\left[(x_1)_{ij},...,(x_m)_{ij}\right]_{i,j\in\{1,...,n\}}/\widetilde{I} = \mathrm{ev}_n^F(I).$$

Our goal now is to show that $\dim(\mathrm{Indec}/\sim) > 0$ for Q non Dynkin. Let us define more precisely what this means:

Proposition 2.2. *Let X be an affine variety of finite type, and G a reductive algebraic group acting on X. We have a bijection*

$$\textit{closed } G\textit{-orbits on } X \textit{ of closed points} \xrightarrow{\sim} \textit{closed points of } \mathbf{k}[X^G].$$

Proof. (follow Ginzburg) Note that $\mathcal{O}' \subseteq \overline{\mathcal{O}}$ implies $I_{\mathcal{O}'} \supseteq I_{\mathcal{O}}$.

Let $X = \operatorname{Spec} B$. If $\mathcal{O} = G \cdot x$, x closed in X, then $I_x \subseteq B$ is maximal. We have

$$I_{\mathcal{O}} \cap B^G = I_x \cap B^G = I_x^G \quad \implies \quad B^G/I_x^G = B^G/I_x \cap B^G \hookrightarrow (B/I_x)^G .$$

B/I_x is a finite extension of $\mathbf{k}$, so $B^G/I_{\mathcal{O}}^G$ is a finite extension of $\mathbf{k}$; i.e. a field. Therefore $I_{\mathcal{O}}^G$ is a maximal ideal of B^G. That is, we have a well defined map

$$\text{closed } G\text{-orbits on } X = \operatorname{Spec}(B) \xrightarrow{\sim} \operatorname{Max}\operatorname{Spec}(B^G).$$

Now we prove that it is a bijection. To prove it is surjective, consider $\mathfrak{M}$ a maximal ideal of B^G, and $B \cdot \mathfrak{M} \subseteq B$. We have $(B \cdot \mathfrak{M}) = \mathfrak{M} \subsetneq B$, so $B \cdot \mathfrak{M} \subsetneq B$. Therefore $\exists x \in \operatorname{Spec} B$ closed such that $I_x \supseteq B \cdot \mathfrak{M}$, so $I_x^G \supseteq \mathfrak{M}$. As $\mathfrak{M}$ is maximal, $\mathfrak{M} = I_x^G$.

To prove the injectivity, we need to show that if $\mathcal{O} \neq \mathcal{O}'$ are closed, then $I_{\mathcal{O}}^G \neq I_{\mathcal{O}'}^G$. Equivalently, we have to show that $I_{\mathcal{O}}^G + I_{\mathcal{O}'}^G = (1) = B^G$, since $I_{\mathcal{O}}^G, I_{\mathcal{O}'}^G$ are maximal. Since G is reductive (this is the only place we use this), we get that $(I_{\mathcal{O}} + I_{\mathcal{O}'})^G = ((1))^G = (1)$, as desired (we used also that $\mathcal{O}$ and $\mathcal{O}'$ are distinct, and hence disjoint, and then used Nullstellensatz, to obtain that $I_{\mathcal{O}} + I_{\mathcal{O}'} = (1)$). $\square$

Proposition 2.3. $\dim(\operatorname{Spec} B//G) \geq \dim(\operatorname{Spec} B) - \dim G$.

Proof. More generally, for any $Y \subseteq \operatorname{Spec} B$, $G \cdot Y$= image of $G \times X \xrightarrow{act} B$, so

$$\dim(G \cdot Y) \leq \dim G + \dim Y.$$

$\square$

Quiver situation: Given $Q = (Q_0, Q_1)$ and $\alpha \in \mathbb{N}^{Q_0}$ dimension vector, consider

$$\operatorname{Rep}_\alpha Q/GL_\alpha, \qquad GL_\alpha = \prod_{i \in Q_0} GL_{\alpha_i}.$$

Note that $\mathbf{k}^\times$ acts trivially, so we have $\operatorname{Rep}_\alpha Q/GL_\alpha = \operatorname{Rep}_\alpha Q/PGL_\alpha$.

Proposition 2.4. $\dim(\operatorname{Rep}_\alpha Q) = \sum_{a:i\to j \in Q_1} \alpha_i \alpha_j$.

Proof. Each $\rho_a \in \operatorname{Hom}(V_i, V_j)$ and we have to pick all the ρ_a's, so $\operatorname{Rep}_\alpha Q = \oplus_{a:i \to j \in Q_1} \operatorname{Hom}(V_i, V_j)$. $\square$

Proposition 2.5. $\dim GL_\alpha = \sum_{i \in Q_0} \alpha_i^2, \qquad \dim PGL_\alpha = \sum_{i \in Q_0} \alpha_i^2 - 1.$

Note 2.6. Any $V \in \operatorname{Rep}_\alpha Q$ can be decomposed $V \cong V_1 \oplus ... \oplus V_n$, with V_i indecomposable, unique up to reordering the isoclasses of the V_i's.

If $\dim(\operatorname{Rep}_\alpha Q/PGL_\alpha) > 0$, then $\dim(\operatorname{Indec}_{\alpha'} Q/\sim) > 0$ for some $\alpha' \leq \alpha$, because there are finitely many $\alpha' \leq \alpha$.

Theorem 2.7. *If Q has finitely many indecomposables up to isomorphism ($\mathbf{k}$ infinite), then Q is Dynkin.*

Proof. By the previous considerations, it is enough to show that $\dim(\operatorname{Rep}_\alpha Q/PGL_\alpha) > 0$ for some α when Q is Dynkin. Using Propositions 2.3, 2.4 and 2.5,

$$\dim(\operatorname{Rep}_\alpha Q)/PGL_\alpha) \geq \dim(\operatorname{Rep}_\alpha Q) - \dim PGL_\alpha) = \sum_{a:i \to j \in Q_1} \alpha_i \alpha_j - \sum_{i \in Q_0} \alpha_i^2 + 1 = 1 - \frac{1}{2}(\alpha, \alpha).$$

Therefore if $(\cdot, \cdot)$ is not positive definite, there exists α such that $(\alpha, \alpha) \leq 0$ and then $\dim(\operatorname{Rep}_\alpha Q)/PGL_\alpha) > 0$.

We have two ways to proceed at this point:

(1) If Q is an extended Dynkin diagram, there exists $\delta \in \mathbb{N}^{Q_0}$ such that $(\delta, \alpha) = 0$ for all α; in particular, $(\delta, \delta) = 0$. Note that if ϵ_i denotes the canonical i-th vector and $\delta \Sigma_i \delta_i \epsilon_i$, $\delta_i \in \mathbb{N}$, then $(\delta, \epsilon_i) = 0$ iff each $\delta_i = \frac{1}{2}\Sigma_j$ adjacent to $_i\delta_j$.

By the results proved at homework, any non-Dynkin diagram contains an extended Dynkin one and we use such δ, which satisfies $(\delta, \delta) \leq 0$.

(2) If $(\cdot, \cdot)$ is not positive definite, then there exists $\alpha \in \mathbb{Q}^{Q_0}$ such that $(\alpha, \alpha) \leq 0$: we can choose $\alpha \in \mathbb{Z}^{Q_0}$ simply multiplying by a common multiple of the denominators of the entries.

We write $\alpha = \alpha_+ - \alpha_-$, where $\alpha_+, \alpha_- \in \mathbb{N}^{Q_0}$ have disjoint support. As

$$(\alpha, \alpha) = (\alpha_+, \alpha_+) + (\alpha_-, \alpha_-) - 2(\alpha_+, \alpha_-) \leq 0$$

and $(\alpha_+, \alpha_-) \leq 0$, we conclude that $(\alpha_+, \alpha_+) \leq 0$ or $(\alpha_-, \alpha_-) \leq 0$.

$\square$

Definition 2.8. Such an i in an extended Dynkin diagram is called an *extending vertex.*

Theorem 2.9 (Gabriel)**.** *Let* $\mathbf{k}$ *be any field,and* Q *a quiver.*

(1) Q *satisfies* $\dim(\mathrm{Indec}_\alpha Q/PGL_\alpha) = 0$ *for all* α *if and only if* Q *is Dynkin.*
(2) *If* Q *is Dynkin,*

$$\mathrm{Indec}_\alpha Q/PGL_\alpha = \begin{cases} \textit{point} & \alpha \in \Delta_+, \\ \varnothing & \textit{otherwise.} \end{cases}$$

To prove this we shall use simple reflections: $s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i$.

Definition 2.10. The set of *real roots* is defined by

$$\Delta^{re} := \left\{ \alpha \in \mathbb{Z}^{Q_0} - : \exists i, j_1, ..., j_t \in \{1, \ldots, n\} / \alpha = s_{j_1} \cdots s_{j_t} \epsilon_i \right\}.$$

We have $\Delta^{re} = \Delta^{re}_+ \sqcup (-\Delta^{re}_+)$, where $\Delta^{re}_+ = \Delta^{re} \cap \mathbb{N}^{Q_0}$, and $\Delta = \Delta^{re}$ when Q is Dynkin.

It remains to prove (2). The idea is to use the action of the s_i's over Δ, in particular s_i acts over $\Delta_+ \backslash \{\epsilon_i\}$.

We want to categorify $\underline{\dim}\ \alpha \rightsquigarrow \mathrm{Rep}_\alpha Q$, and to define functors F_i in the way

$$V \in \mathrm{Rep}_\alpha Q \mapsto F_i V \in \mathrm{Rep}_{s_i \alpha} Q$$

in order to apply to any indecomposable enough times to get S_j for some $j \in Q_0$.

It turns out that this does not work as stated. However, it works if we change Q slightly before to apply F_i.

Definition 2.11.
- A vertex i is a *sink* if all the arrows incident to i go in.
- A vertex i is a *source* if all the arrows incident to i go out.

$$\begin{tikzcd} & \circ \arrow[d] & \\ \circ \arrow[r] & \circ^i & \circ \arrow[l] \\ & \circ \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} & \circ & \\ \circ & \circ^i \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \circ \\ & \circ & \end{tikzcd}$$

i a sink $\qquad$ i a source

Definition 2.12. If i is a sink, we define $F_i^+(Q)$ as the same quiver Q except that all the arrows meeting i are reversed (in this way, i is a source for $F_i^+(Q)$).

For each $V = (V_i, \rho_a) \in \mathrm{Rep}_\alpha Q$, we define $F_i^+ V = (V_i', \rho_a') \in \mathrm{Rep}_\alpha F_i^+ Q$ by

$$V_j' = \begin{cases} V_j & j \neq i, \\ \ker\left(\oplus_{a:l\to i} \rho_a : \oplus V_l \to V_i\right) & j = i; \end{cases} \qquad \rho_b' = \begin{cases} \rho_b & b \text{ not incident}, \\ \pi_j \circ \iota & b : i \to j. \end{cases}$$

where $\iota : V_i' \hookrightarrow \oplus_{a:l\to i} V_l$ and $\pi_j : \oplus_{a:l\to i} V_l \twoheadrightarrow V_j$ are the canonical morphisms.

Definition 2.13. If i is a source, we define $F_i^-(Q)$ as the same quiver Q except that all the arrows meeting i are reversed (in this way, i is a sink for $F_i^-(Q)$).

For each $V = (V_i, \rho_a) \in \mathrm{Rep}_\alpha Q$, we define $F_i^- V = (V_i', \rho_a') \in \mathrm{Rep}_\alpha F_i^- Q$ by

$$V_j' = \begin{cases} V_j & j \neq i, \\ \mathrm{coker}\left(\prod_{a:l\to i} \rho_a : V_i \to \oplus V_l\right) & j = i; \end{cases} \qquad \rho_b' = \begin{cases} \rho_b & b \text{ not incident}, \\ \pi \circ \iota_j & b : i \to j. \end{cases}$$

where $\iota_j : V_j \hookrightarrow \oplus_{a:i\to l} V_l$ and $\pi : \oplus_{a:i\to l} V_l \twoheadrightarrow V_i$ are the canonical morphisms.

Remark 2.14. $F_i^+ S_i = F_i^- S_i = 0$.

Proposition 2.15. (1) *Suppose that i is a sink of Q. The following are equivalent:*
- (a) *V has no S_i as sumands;*
- (b) *$F_i^- F_i^+ V \cong V$;*
- (c) *$\oplus_{a:l\to i} \rho_a : \oplus_{a:l\to i} V_l \to V_i$ is surjective.*

(2) *Suppose that i is a source of Q. The following are equivalent:*
- (a) *V has no S_i as sumands;*
- (b) *$F_i^+ F_i^- V \cong V$;*
- (c) *$\prod_{a:i\to l} \rho_a : V_i \to \oplus_{a:l\to i} V_l$ is injective.*

Proof. To be filled in. □

Corollary 2.16. *Let i be a sink of Q. the previous gives a bijection*

$$\left\{ \begin{array}{c} \textit{isoclasses of representations of } Q \\ \textit{without } S_i \textit{ as summand} \end{array} \right\} \quad \underset{F_i^-}{\overset{F_i^+}{\rightleftarrows}} \quad \left\{ \begin{array}{c} \textit{isoclasses of representations of } F_i^+ Q \\ \textit{without } S_i \textit{ as summand} \end{array} \right\},$$

which restricts to indecomposables.

Proof. (Gabriel's Theorem, (ii)) We prove it in several steps.

(i) To begin with, we need to prove that each Q Dynkin has a sink and a source. It follows because the underlying undirected graph is acyclic: if we suppose that Q has no source, we can go backwards infinitely, so we get a cycle because Q is finite, which is a contradiction; similar argument to prove we have a sink, going forwards this time.

(ii) If V does not have S_i as summand, then $\underline{\dim}\ F_i^+ V = s_i \underline{\dim}\ V$. It follows because

$$\dim \ker\left(\oplus_{a:l\to i} \rho_a\right) = \sum_{a:l\to i} \dim V_l - \dim V_i, \quad (s_i\alpha)_i = -\alpha_i + \sum_{a:l\to i} \alpha_l,$$

and the other components do not change.

Therefore, if we could show that there exists a sequence of vertices $i_1, ..., i_m$ such that $s_{i_m} \cdots s_{i_1} \alpha = \epsilon_j$ for some j and does not pass through any negative vector, any $V \in \mathrm{Indec}_{alpha}(Q)$ will map to $F_{i_m}^+ \cdots F_{i_1}^+ V \in \mathrm{Indec}_{\epsilon_j}(F_{i_m}^+ \cdots F_{i_1}^+ Q)$, provided we always have the need sink. It remains to show that this sequence always exists.

(iii) Since there always exists a sink, there exists an ordering $j_1, \ldots, j_n$ of Q_0 such that $F_{j_m}^+ \cdots F_{j_1}^+ Q$ is well defined, for all $m = 1, \ldots, n$.

If we pick a sink each time, we can avoid repeating a vertex until we have used all them. In this way,

$$F^+_{j_n} \cdots F^+_{j_1} Q = Q$$

because we have flipped all arrows twice. About the dimension vectors,

$$\begin{aligned} &V \to F^+_{j_n} \cdots F^+_{j_1} V \\ &\alpha \mapsto c(\alpha), \quad c := s_{j_n} \cdots s_{j_1}. \end{aligned}$$

Remark 2.17. c is called the *Coxeter element*. It is independent of the ordering $j_1, \dots, j_n$.

Now if V is indecomposable, to show that V reflects to S_j, it is enough to show that $\alpha = \underline{\dim}\, V$ reflects to ϵ_j by applying enough reflections.

Proposition 2.18. *There exists $m \geq 0$ such that $c^m\alpha \notin \mathbb{N}^{Q_0}$ for any $\alpha \in \mathbb{N}^{Q_0}$.*

Proof. The Weyl group $\mathcal{W}_Q$ is finite, because it is a subgroup of the group of permutations of Δ. Then c has finite order $k > 1$.

Suppose $c\beta = s_{j_n} \cdots s_{j_1}\beta = \beta$ for some β. The s_j are all different and each s_j changes only the j-th component. As each changes a different component, $s_j\beta = \beta$ for all j, so $\beta = 0$.

Therefore, c does not have 1 as an eigenvalue, so $1 + c + \ldots + c^{k-1} = 0$. For each $\alpha \in \mathbb{N}^{Q_0}$,

$$\alpha + c\alpha + \ldots + c^{k-1}\alpha = 0,$$

so at least one of the summands is not in $\mathbb{N}^{Q_0}$. ☐

This result implies that $F_c^m V = 0$ if $V \in \mathrm{Rep}_\alpha Q$; i.e. along the way we have had S_i as summand.

(iv) In this way, for all V indecomposable we have a sequence $F^+_{i_m} \cdots F^+_{i_1} V = 0$. We take the longest sequence of reflections F^+ not killing V, $W = F^+_{i_k} \cdots F^+_{i_1} V$. Then, $W = S_{i_{k+1}}$, and

$$V = F^-_{i_1} \cdots F^-_{i_k} S_{i_{k+1}}.$$

This implies that if $\mathrm{Indec}_\alpha\, Q \neq \emptyset$, then $\alpha \in \Delta_+$ is obtained from any of the ϵ_i's by a sequence of reflections.

Now $\mathcal{W}_Q$ acts transitively on Δ, so

$$\{\text{point}\} = \{\mathrm{Indec}_{\epsilon_j} / \sim\} \xrightarrow{F^+_{i_1}} \{\mathrm{Indec}_{s_{i_1}\epsilon_j} / \sim\} \xrightarrow{F^+_{i_2}} \cdots$$

and by the transitively, $\{\mathrm{Indec}_\alpha\, Q/\sim\} = \{\text{point}\}$ for all $\alpha \in \Delta_+$. ☐

Note 2.19. There was a technical detail: we have used the Coxeter element C.

In general, for Q non Dynkin, c still exists but does not have finite order, so we cannot use this proof. Nonetheless, the bijection between $\{\mathrm{Indec}_\alpha\, Q/\sim\}$ is still valid, and there exists a unique indecomposable of dimension $\alpha \in \Delta_+^{re}$ up to isomorphism.

Theorem 2.20 (Kac)**.** *Let Q be a quiver without loops. Then:*

(1) $\dim(\mathrm{Indec}_\alpha\, Q/PGL_\alpha) > 0 \iff \alpha \in \Delta_+^{im}$.
(2) $\mathrm{Indec}_\alpha\, Q/PGL_\alpha)$ *is a point* $\iff \alpha \in \Delta_+^{re}$.

The main idea is to define reflection functors as before, in order to conclude

$$\begin{array}{ccc} \mathrm{Indec}_\alpha\, Q & \underset{F_i^-}{\overset{F_i^+}{\rightleftarrows}} & \mathrm{Indec}_{s_i\alpha}\, F_i^+ Q \\ i \text{ a sink} & & i \text{ a source,} \end{array}$$

for any quiver Q, where i is a sink and $\alpha \neq \epsilon_i$.

Since we are changing of quiver, we cannot a priori do this arbitrarily. We want to have $F_c^+ = F_{i_n}^+ \cdots F_{i_1}^+$, where $i_1, ..., i_n$ are an ordering of Q such that this is defined. Showed that for all α, $c^m\alpha \notin \mathbb{Z}_{\geq 0}^{Q_0}$ fro some $m \geq 0$, it is possible to go from V indecomposable to a simple S_i of Q by applying some reflection functors, which only depends on $\underline{\dim}\ V$,

$$\therefore V \cong F_{i_1}^- \cdots F_{i_n}^- S_i, \quad \forall V \in \operatorname{Indec}_\alpha Q.$$

Note 2.21. In fact we need $\mathbf{k}$ algebraically closed: in this case, $\operatorname{Indec}_\alpha Q/\sim \cong \operatorname{Indec}_\alpha Q'/\sim$, of Q' is obtained from Q reversing an arrow. This will be used to prove Kac's Theorem.

Lemma 2.22. (1) *If $\alpha \in \Delta_+^{re}$, then $\operatorname{Indec}_\alpha Q/\sim$ is a point.*
(2) *If $\operatorname{Indec}_\alpha Q \neq \emptyset$, then $\alpha \in \Delta_+$.*

Proof. (1) Any $\alpha \in \Delta_+^{re}$ is of the form $\alpha = s_{i_m} \cdots s_{i_1}\epsilon_j$. Now,

$$\{\operatorname{Indec}_\alpha / \sim\} \xrightarrow{F_{i_m}^+} \{\operatorname{Indec}_{s_{i_m}\alpha} / \sim\} \xrightarrow{F_{i_{m-1}}^+} \cdots \xrightarrow{F_{i_1}^+} \{\operatorname{Indec}_{s_{i_1}\cdots s_{i_m}\alpha} / \sim\},$$

and $s_{i_1} \cdots s_{i_m}\alpha = \epsilon_j, \{\operatorname{Indec}_{\epsilon_j} / \sim\} = S_i$.

(2) Let α be such that $\operatorname{Indec}_\alpha Q \neq \emptyset$. If $(\alpha, \epsilon_i) > 0$, $\alpha \neq \epsilon_i$, we can apply s_i and then $s_i\alpha = \alpha - (\alpha, \epsilon_i)\epsilon_i < 0$, and $\operatorname{Indec}_\alpha / \sim \to \operatorname{Indec}_{s_i\alpha} / \sim$ is bijective. Eventually,

- we have a bijection $\operatorname{Indec}_\alpha / \sim \to \operatorname{Indec}_\beta / \sim$ for some β such that $(\beta, \epsilon_i) \leq 0$ for all i and β is obtained from α by a sequence of simple reflections, in which case β is in the fundamental region, and then $\alpha \in \Delta_+^{im}$,
  or
- we get $\beta = \epsilon_i$ after applying some reflections, whit a bijection as above, in which case $\alpha \in \Delta_+^{re}$.

□

Rather than to prove that $\operatorname{Indec}_\alpha / \sim$ does not change under arrow reversal, we prefer to generalize reflection functors not to require the vertex is a sink or a source.

The idea is to use the double quiver $\bar{Q}$ defined by: $\bar{Q}_0 = Q_0$ and $\bar{Q}_1 = Q_1 \sqcup Q_1^*$, where Q_1^* is the opposite quiver.

$$Q = \circ \longleftarrow \circ \longrightarrow \circ \qquad \Longrightarrow \qquad \bar{Q} = \circ \rightleftarrows \circ \rightleftarrows \circ$$

We shall prove that $\operatorname{Rep}_\alpha \bar{Q} = T^*\operatorname{Rep}_\alpha Q$, which is symplectic under the action of GL_α. By studying preprojective algebras we will give a proof of Kac's Theorem.

Example 2.23. For $Q = D_4$,
$$\begin{tikzcd} \circ \arrow[r, "a_1"] & \circ & \circ \arrow[l, "a_2"'] \\ & \circ \arrow[u, "a_4"] & \end{tikzcd} .$$

This quiver has 12 indecomposable modules up to isomorphism. Except for the external S_i's, the other are parameterized by subspaces V_1, V_2, V_3 of the vector space V corresponding to the central vertex. As it was proved in the exercises, it depends on the 9 parameters:

$$\dim V,\ \dim V_i,\ \dim(V_i \cap V_j),\ \dim(V_1 \cap V_2 \cap V_3),\ \dim(V_1 + V_2 + V_3).$$

3. Some results about representations of Q

Note that, for all $\alpha \in \mathbb{N}^{Q_0}$, $\operatorname{Rep}_\alpha Q = \operatorname{Rep}_\alpha \mathbf{k}Q$. We consider $\operatorname{Rep}_\alpha Q$ as a variety.

First fact: For any V representation of finite dimension of Q, $V \cong V_1 \oplus \ldots \oplus V_n$, for some V_i indecomposable.

Theorem 3.1 (Krull-Schmidt)**.** *Let $\mathcal{C}$ be an abelian category, $\mathcal{C} \subseteq FVec$. Then, a decomposition in indecomposables summands is unique up to isomorphism.*

Note 3.2. This is not true more generally; e.g. if ξ_1, ξ_2 are two non trivial indecomposables vector bundles over X (if $X = \operatorname{Spec} A$, they correspond to A-modules), it can happen

$$\xi_1 \oplus \xi_2 \cong \mathcal{O}^n = \mathcal{O} \oplus \cdots \oplus \mathcal{O}.$$

We will denote $\mathcal{O}_M$, $M \in \operatorname{Rep}_\alpha Q$, the PGL_α-orbit; i.e the isomorphism class of M. We can ask when we have $\overline{\mathcal{O}_M} \supseteq \mathcal{O}_N$ for $M, N \in \operatorname{Rep}_\alpha Q$.

Lemma 3.3. *Let A be a* $\mathbf{k}$*-algebra, $M, N \in \operatorname{Rep}_n A$. If $N = N_1 \oplus N_2$ and we have a short exact sequence $0 \to N_1 \to M \to N_2 \to 0$, then $\overline{\mathcal{O}_M} \supseteq \mathcal{O}_N$.*

Proof. We write $M = N_1 \oplus N'$ as vector spaces. A acts in M by $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} N_1 \\ N' \end{pmatrix}$. Then,

$$\phi_t = \begin{pmatrix} t\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \, a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \Longrightarrow \phi_t \cdot a \cdot \phi_t^{-1} = \begin{pmatrix} a_1 & ta_2 \\ 0 & a_3 \end{pmatrix}$$

$$\therefore \phi_t \cdot a \cdot \phi_t^{-1} \longrightarrow_{t \to 0} \begin{pmatrix} a_1 & 0 \\ 0 & a_3 \end{pmatrix} \Longrightarrow \lim_{t \to 0} \phi_t \cdot M \cong N_1 \oplus N_2.$$

Therefore $N \in \oplus\overline{\mathcal{O}_M}$, and $\overline{\mathcal{O}_M} \supseteq \mathcal{O}_N$.

□

The converse is not true in general. However,

Theorem 3.4 (Bongartz, others)**.** *Let Q Dynkin, $A = \mathbf{k}Q$. Then, $\overline{\mathcal{O}_M} \supseteq \mathcal{O}_N$ if and only if N can be obtained by iterating the above.*

We consider a composition series $0 = M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \ldots \hookrightarrow M_n = M$. the semisimple module

$$ss\, M := \oplus_{i=1}^n M_i / M_{i-1}$$

is unique up to isomorphism by Jordan-Hlder Theorem. By Lemma, $\overline{\mathcal{O}_M} \supseteq \mathcal{O}_{ssM}$, so any closed orbit contains the orbit of a semisimple module.

Theorem 3.5. *For each M, $\overline{\mathcal{O}_M}$ contains a unique orbit corresponding to a semisimple module; i.e. $\mathcal{O}_{ssM}$.*

Proof. Suppose there exists N semisimple such that $\overline{\mathcal{O}_M} \supseteq \mathcal{O}_N$

We shall use the characteristic polynomial. Given a $\mathbf{k}$-algebra A and M and A-module of finite dimension, $\chi_M : A \to \mathbf{k}[x]$ is the composition $A \to \operatorname{End}_k M \to^{char} \mathbf{k}[x]$. The polynomial function χ_M is constant in $\mathcal{O}_M$, so it is constant also in $\overline{\mathcal{O}_M}$

Claim 3.6. *Let N, N' semisimple polynomials such that $\chi_N = \chi_{N'}$. Then $N \cong N'$.*

If we prove this, we end the proof of the theorem, because $\chi_{ss\,M} = \chi_M = \chi_N$, so $ss\,M \cong N$.

So we prove the claim. In order to do that, replace A by $\bar{A} := A/ann(N \oplus N')$, so $\bar{A} \hookrightarrow \operatorname{End}_k(N \oplus N')$. By Weddenburg's Theorem,

$$\bar{A}/rad\bar{A} = \oplus_{i=1}^q \operatorname{Mat}_{n_i}(D_i), \quad D_i \text{ finite dimensional division algebra over } \mathbf{k}.$$

Then there exist exactly q simple modules $S_1, \ldots, S_q$. Now we can detect the S_i isotypical part of any semisimple $\bar{A}/rad\bar{A}$-module M by

$$\chi_M(0, \ldots, 0, \underbrace{\mathrm{Id}}_{i-th}, 0, \ldots, 0) = (x-1)^{d_i}, \quad d_i = \dim S_i.$$

So $N \cong N'$.

□

For Q Dynkin, we deduce that $\text{Rep}_\alpha Q$ contains a unique closed PGL_α-orbit, $\mathcal{O}_{\oplus_{i\in Q_0} S_i^{\alpha_i}}$. Therefore, there exists a unique closed point in $\text{Rep}_\alpha Q/PGL_\alpha$ (in fact, this is true for Q a tree).

What about the other orbits?

Claim 3.7. *There exists a unique open orbit.*

Basically, when Q is Dynkin there are finitely many orbits (finitely many indecomposables). Since $\text{Rep}_\alpha Q$ is an affine space (so irreducible), one of these orbits is dense, so open. More generally, for all Q there are not finitely many orbits in general but only finitely many ways to decompose $\alpha = \alpha^{(1)} + ... + \alpha^{(m)}$.

Lemma-Definition 3.8 (Generic decomposition)**.** *For all Q and all α, there exists a decomposition $\alpha = \alpha^{(1)} + ... + \alpha^{(m)}$ such that generically $M \in \text{Rep}_\alpha Q$ has the form*

$$M = M_1 \oplus \ldots \oplus M_m, \quad \underline{\dim}\ M_i = \alpha^{(i)}.$$

For Q extended Dynkin, remember that $\{\alpha \in \Delta_+ : \alpha_i \leq \delta_i, \forall i, \alpha \neq \delta\} \cong \widetilde{\Delta}_+$, where $\widetilde{\Delta}_+$ is the set of positive roots of the associated Dynkin diagram. Therefore, $\text{Rep}_\alpha Q$ contains a unique closed PGL_α-orbit as above for each $\alpha \in \widetilde{\Delta}_+$.

QUIVERS IN REPRESENTATION THEORY (18.735, SPRING 2009) LECTURES 5, 6, 7 AND 8

TRAVIS SCHEDLER, TYPED BY IVÁN ANGIONO

Recall that:

(1) If Q is Dynkin, there exists only one closed orbit in $\mathrm{Rep}_\alpha Q$ for each α: $\mathcal{O}_{\oplus_{i\in Q_0} S_i^{\alpha_i}}$.
   Furthermore, this is still true for all Q without directed cycles. In such case there exists a sink, and inductively any $V \in \mathrm{Rep}_\alpha Q$ is an iterated extension of simples of the form S_i.
(2) For all Q there exists a generic decomposition $\alpha = \alpha^{(1)} + \ldots + \alpha^{(m)}$ such that the generic orbit $\mathcal{O}_V$ satisfies
$$V \cong V_1 \oplus \ldots \oplus V_m,$$
for some V_i such that
$$\underline{\dim}\ V_i = \alpha_i.$$

In particular, if Q is Dynkin, there exists just one open orbit in $\mathrm{Rep}_\alpha Q$ for each α.

1. Finding generic decompositions (Lecture 5)

Claim 1.1. *Let Q be Dynkin. If $\alpha \in \Delta_+$, then the generic decomposition is simply $\alpha = \alpha$; i.e., $\mathcal{O}_V$ corresponds to V indecomposable.*

Note 1.2. One way to prove this is to show explicitly that any other decomposition is obtained by degenerating this; i.e., we can form extensions of summands to get indecomposables.

Simple argument: Counting dimensions
$\mathcal{O}_V$ is open if and only if $\dim \mathcal{O}_V = \dim \mathrm{Rep}_\alpha = \sum_{a:i\to j} \alpha_i\alpha_j$. Also,

$$\begin{aligned} \dim \mathcal{O}_V &= \dim GL_\alpha - \dim \mathrm{Isotropy}(V) \\ \mathrm{Isotropy}(V) &= \mathrm{Aut}(V) =: G < GL_\alpha \text{ such that } \mathrm{Ad}\, g \text{ preserves } V, \quad \forall g \in G. \end{aligned}$$

Note that $\dim\ \mathrm{Aut}\, V = \dim \mathrm{End} V$, so

$$\begin{aligned} \dim \mathcal{O}_V &= \sum_{i\in Q_0} \alpha_i^2 - \dim \mathrm{End} V \\ \therefore \dim \mathrm{Rep}_\alpha - \mathcal{O}_V &= \sum_{a:i\to j} \alpha_i\alpha_j - \big(\sum_{i\in Q_0} \alpha_i^2 - \dim \mathrm{End} V\big) \\ &= \dim \mathrm{End} V - \frac{1}{2}(\alpha,\alpha). \end{aligned}$$

Therefore, $\mathcal{O}_V$ is open iff $\dim \mathrm{End} V = \frac{1}{2}(\alpha,\alpha)$. But we know that $\frac{1}{2}(\alpha,\alpha) = 1$ for each $\alpha \in \Delta_+$. So, $\mathcal{O}_V$ is open iff $\mathrm{End} V = \mathbf{k}$.

Definition 1.3. If $\mathrm{End} V = \mathbf{k}$, then V is called a *brick*.

Note 1.4. In general, if $\mathbf{k} = \overline{\mathbf{k}}$, then any simple is automatically a brick. All the simples S_i are bricks, and hence every simple is a brick for every quiver without directed cycles. In general, we have simple $\Rightarrow$ brick $\Rightarrow$ indecomposable, but arrows don't go in the other direction.

For Dynkin quivers, when $\alpha \in \Delta_+$, we see that the open orbit must be of the form $\mathcal{O}_V$, where V is a brick. This implies that V is indecomposable, but there is a unique such V up to isomorphism.

We conclude that in the case that Q is Dynkin, V is a brick if and only if V is indecomposable, and in this case, $\mathcal{O}_V$ is open in $\mathrm{Rep}_{\underline{\dim}\ V}(Q)$.

When $\alpha \notin \Delta_+$, the generic decomposition is more complicated.

Note, though, that we do know the following about the generic decomposition:[1]

Claim 1.5. *(Refined generic decomposition) Generically in* $\mathrm{Rep}_\alpha Q$*, the orbits* $\mathcal{O}_{V_1 \oplus \cdots \oplus V_m}$ *have the property that* $\mathrm{Ext}^1(V_i, V_j) = 0$ *for* $i \neq j$*, and the dimension vectors* $(\alpha^{(1)}, \ldots, \alpha^{(m)})$ *of* $V_1, \ldots, V_m$ *are unique up to permutation. Moreover, generically,* $\underline{\dim}\ \mathrm{End}(V_1 \oplus \cdots \oplus V_m)$ *is minimal, so there exists a brick of dimension* α *if and only if* $m = 1$ *and generic orbits* $V = V_1$ *are bricks.*

Proof. We proved the last part last time: the reason why the $\alpha^{(i)}$ are unique up to permutation is that there are finitely many choices of $\alpha^{(1)}, \ldots, \alpha^{(m)}$ such that $\alpha^{(1)} + \cdots + \alpha^{(m)} = \alpha$, and since $\mathrm{Rep}_\alpha Q$ is irreducible (it is an affine space), only one choice up to permutation can form a dense subvariety. Then, the first statement follows from the lemma of last time saying that if $0 \to V_1 \to V_2 \to V_3 \to 0$ is a nonsplit short exact sequence, then $\overline{\mathcal{O}_{V_2}} \supseteq \mathcal{O}_{V_1 \oplus V_3}$.

We claim that in fact $\mathcal{O}_{V_2} \neq \mathcal{O}_{V_1 \oplus V_3}$, that is:

Lemma 1.6. *If* $0 \to V_1 \to V_2 \to V_3 \to 0$ *is a nonsplit short exact sequence, then* $V_2 \ncong V_1 \oplus V_3$*.*

Proof. We use the long exact sequence from the functor $\mathrm{Hom}(-, V_1)$. This yields $\mathrm{Hom}(V_1, V_1) \to \mathrm{Ext}^1(V_3, V_1) \to \mathrm{Ext}^1(V_2, V_1) \twoheadrightarrow \mathrm{Ext}^1(V_1, V_1)$, where the image of the identity under the first map is (more or less by definition) the nontrivial element corresponding to the nonsplit extension. If $V_2 \cong V_1 \oplus V_3$, then we get $\mathrm{Hom}(V_1, V_1) \to \mathrm{Ext}^1(V_3, V_1) \to \mathrm{Ext}^1(V_1, V_1) \oplus \mathrm{Ext}^1(V_3, V_1) \twoheadrightarrow \mathrm{Ext}^1(V_1, V_1)$, and exactness and dimension count would require that the second map is injective, which can't hold by the above. $\square$

Thus, if $0 \to V_j \to \widetilde{V_{ij}} \to V_i \to 0$ is a nontrivial extension, then $\mathcal{O}_{V_1 \oplus \cdots \hat{V_i} \cdots \hat{V_j} \cdots \oplus V_m \oplus V_{ij}}$ would be an orbit whose closure contains $\mathcal{O}_{V_1 \oplus \cdots \oplus V_m}$, and hence would also be in any dense open subvariety of $\mathrm{Rep}_\alpha Q$ containing $\mathcal{O}_{V_1 \oplus \cdots \oplus V_m}$. This contradicts the assertion that the $\alpha^{(1)}, \ldots, \alpha^{(m)}$ are unique up to permutation.

The fact that $\dim\ \mathrm{End}(V_1 \oplus \cdots \oplus V_m)$ is minimal generically is a consequence of the fact that this number is always upper semicontinuous (since, for each ℓ, the variety of $W \in \mathrm{Rep}_\alpha Q$ such that $\dim\ \mathrm{End}(W) \geq \ell$ is cut out by polynomials, i.e., it is closed in the Zariski topology). From this immediately follows the claim about bricks. $\square$

Definition 1.7. $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$ is called a *Schur root* if there exists a brick of dimension α, or equivalently, the generic orbit in $\mathrm{Rep}_\alpha Q$ is a brick.

For Dynkin quivers, the Schur roots are exactly the positive roots, and for extended Dynkin quivers, the Schur roots are the positive real roots together with δ.

Example 1.8.

$$Q = \circ^2 \longrightarrow \circ^1 \longrightarrow \circ^3 \longrightarrow \circ^2, \quad A_4 \text{ directed.}$$

The generic decomposition in the case of A_4 as directed above is as follows:

$$\circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}} \oplus \circ^{\mathbf{k}} \longrightarrow \circ^0 \longrightarrow \circ^0 \longrightarrow \circ^0 \oplus \circ^0 \longrightarrow \circ^0 \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}}$$

$$\oplus \circ^0 \longrightarrow \circ^0 \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^0$$

[1]The claim below and the following paragraph were not in the original lecture.

Proposition 1.9. *For Q of type A_n, the generic decomposition is determined inductively as follows: $\alpha = \alpha^{(1)} + \cdots + \alpha^{(m)}$, where each $\alpha^{(i)}$ is a maximal root such that $\alpha^{(i)} \leq (\alpha - (\alpha^{(1)} + \cdots + \alpha^{(i-1)}))$.*

The procedure for determining the generic decomposition above is called *the greedy algorithm* since we iteratively pick as large a positive root as we can in our decomposition.

To prove the proposition, we need to explain:

(1) Why open orbits correspond to decompositions $\alpha = \alpha^{(1)} + \cdots + \alpha^{(m)}$ such that there does not exist β such that, for some i, j, $\alpha^{(i)} + \alpha^{(j)} > \beta \in \Delta_+$ (i.e., $\alpha^{(i)} + \alpha^{(j)} - \beta \in \mathbb{Z}_{\geq 0}^{Q_0}$), and $\alpha^{(i)} < \beta$, $\alpha^{(j)} < \beta$;

(2) For the A_n case, the greedy algorithm has this property.

Example 1.10.

$$Q = \circ^1 \longleftarrow \circ^2 \longrightarrow \circ^1, \quad A_3 \text{ directed.} \tag{1.11}$$

In this case, note that

$$\circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}} \hookrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^0 \oplus \circ^0 \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^{\mathbf{k}} \twoheadrightarrow \circ^0 \longrightarrow \circ^{\mathbf{k}} \longrightarrow \circ^0.$$

Then by a previous lemma, $\overline{\mathcal{O}_{\mathbf{kk}0\oplus 0\mathbf{kk}}} \supset \mathcal{O}_{\mathbf{kkk}\oplus 0\mathbf{k}0}$, and $\mathcal{O}_{\mathbf{kk}0\oplus 0\mathbf{kk}}$ is open.

We can use reflection functors:

$$\begin{pmatrix} V \in \mathrm{Rep}_\alpha, \\ \text{no } S_i \text{ as summand, } i = \textit{sink} \end{pmatrix} \begin{matrix} \longrightarrow \\ \longleftarrow \end{matrix} \begin{pmatrix} V' \in \mathrm{Rep}_{s_i(\alpha)}, \\ \text{no } S_i \text{ as summand, } i = \textit{source} \end{pmatrix}$$

Since all indecomposables are obtained by reflecting a simple one, then $\mathrm{End}V = \mathbf{k}$ for each V indecomposable. Also, if we take an open orbit without S_i's as summands, it is reflected to an open orbit.

To proceed, it is useful to study $\mathrm{Ext}^*(V, W)$, for each pair of representations V, W.

Proposition 1.12. *Let Q be a quiver without directed cycles, and $V \in \mathrm{Rep}_\alpha Q$. Then,*

$$\dim\, \mathrm{End}V - \dim\, \mathrm{Ext}^1(V, V) = \frac{1}{2}(\alpha, \alpha). \tag{1.13}$$

We will use this also in the case where there are directed cycles: a proof appears in Homework 2 and also is given in Proposition 2.5.

Remark 1.14. In order to use this result, an orbit is open iff $\dim \mathrm{End}V = \frac{1}{2}(\alpha, \alpha)$ iff $\dim\, \mathrm{Ext}^1(V, V) = 0$.

When Q is Dynkin and V is indecomposable, $\dim\, \mathrm{Ext}^1(V, V) = 0$, because $2\alpha \notin \Delta_+$ for each $\alpha \in \Delta_+$.

Therefore, $V_1 \oplus \ldots \oplus V_n$ forms an open orbit iff $\mathrm{Ext}^1(V_i, V_j) = 0$ for all i, j.

Proof. To begin with, we define the *Euler form*:

$$\langle V, W \rangle := \dim\, \mathrm{Hom}(V, W) + \sum_{i \geq 1} (-1)^i \dim\, \mathrm{Ext}^i(V, W). \tag{1.15}$$

Note 1.16. In general, $\langle , \rangle : \mathrm{Rep} \times \mathrm{Rep} \to \mathbb{Z}$ descends to a $\mathbb{Z}$ pairing $\langle , \rangle : \mathcal{K}_0 \times \mathcal{K}_0 \to \mathbb{Z}$, where $\mathcal{K}_0$ denotes the Grothendieck group.

This is because for each short exact sequence $0 \to V_1 \to V \to V_2 \to 0$, we can take the associated long exact sequence:

$$0 \to \mathrm{Hom}(V_2, W) \to \mathrm{Hom}(V, W) \to \mathrm{Hom}(V_1, W) \to \mathrm{Ext}^1(V_2, W) \to \cdots$$

If eventually all Ext^i vanish at some i, we get

$$\langle V, W \rangle = \langle V_1, W \rangle + \langle V_2, W \rangle.$$

Also, we have the following result

Lemma 1.17. *If Q has no directed cycles, then $\mathcal{K}_0 = \mathbb{Z}^{Q_0}$.*

I.e., any representation is an extension of the S_i's. Therefore,

$$\langle V, V\rangle = \langle ssV, ssV\rangle.$$

For the S_i's we have

$$\mathrm{Ext}^*(S_i, S_j) = \begin{cases} \mathbf{k}[0] & i = j, \\ \mathbf{k}[1] & i \to j \text{ arrow}, \\ 0 & \textit{otherwise}. \end{cases} \tag{1.18}$$

Note that if $a : i \to j$ is an arrow, we have the short exact sequence:

$$S_i \hookrightarrow \circ_i^{\mathbf{k}} \longrightarrow \circ_j^{\mathbf{k}} \twoheadrightarrow S_j.$$

So if $\underline{\dim}\ V = \alpha$, $\underline{\dim}\ W = \beta$, we have

$$\begin{aligned} \langle V, W\rangle &= \langle ssV, ssW\rangle = \sum_{i\in Q_0} \alpha_i\beta_i - \sum_{a:i\to j} \alpha_i\beta_j, \\ \therefore (\alpha, \beta) &= \langle V, W\rangle + \langle W, V\rangle, \\ \therefore \langle V, V\rangle &= \frac{1}{2}(\alpha, \alpha). \end{aligned}$$

We conclude the proof with provided that $\mathrm{Ext}^2(V, W) = 0$ for all $l \geq 2$, which we shall prove in the next proposition. □

Proposition 1.19. $\mathrm{gl}\dim \mathcal{R}_Q = 1$*; i.e.,* $\mathrm{Ext}^l(V, W)$ *for all* $l \geq 2$ *and* $V, W \in \mathcal{R}_Q$*.*

Note that $\mathrm{gl}\dim A \leq \mathrm{Hoch}\dim A$ for all associative algebra A. To prove this, given $M \in A{-}Mod$, we consider an A-bimodule resolution

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to A \to 0,$$

and apply $- \otimes_A M$:

$$0 \to P_m \otimes_A M \to \cdots \to P_1 \otimes_A M \to P_0 \otimes_A M \to M \to 0.$$

This is a projective resolution of M, so $\mathrm{proj}\dim M \leq m$. Taking max between all M, we have $\mathrm{gl}\dim A \leq \mathrm{Hoch}\dim A$.

Therefore, Proposition 1.19 follows from the following fact:

Proposition 1.20. *Hochschild dimension of* $\mathbf{k}Q$ *is 1; i.e.,* $\mathbf{k}Q$ *has a length-1 projective* $\mathbf{k}Q$*-bimodule resolution:*

$$0 \to P_1 \to P_0 \to \mathbf{k}Q \to 0.$$

Proof. We consider $S := \langle Q_0\rangle \subseteq \mathbf{k}Q$, the span of length-0 paths: it is a semisimple $\mathbf{k}$-algebra, $S = \mathbf{k}^{Q_0}$. We have:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{k}Q \otimes_S \langle Q_1\rangle \otimes_S \mathbf{k}Q & \longrightarrow \mathbf{k}Q \otimes_S \mathbf{k}Q & \longrightarrow_{mult} & \mathbf{k}Q \longrightarrow 0 \\ & & f \otimes a \otimes g & \mapsto fa \otimes g - f \otimes ag. & & \end{array}$$

This is a left $\mathbf{k}Q$-split resolution:

$$\mathbf{k}Q \hookrightarrow \mathbf{k}Q \otimes_S \mathbf{k}Q,$$
$$a \mapsto a \otimes 1.$$

In order to prove that this is the projective resolution we are looking for, note that $\mathbf{k}Q \cong \oplus_{i\in Q_0}\mathbf{k}Qi$, where i denotes the idempotent of S, and similarly $\mathbf{k}Qi \otimes_S j\mathbf{k}Q$ are projective $\mathbf{k}Q$-bimodules for all $i,j \in Q_0$. Therefore,

$$\mathbf{k}Q \otimes_S \mathbf{k}Q \cong \oplus_{i\in Q_0}\mathbf{k}Qi \otimes_S j\mathbf{k}Q$$

is projective, and similarly,

$$\mathbf{k}Q \otimes_S \langle Q_1\rangle \otimes_S \mathbf{k}Q \cong \oplus_{a:i\to j}\mathbf{k}Qi \otimes_S \langle a\rangle \otimes_S j\mathbf{k}Q$$

is projective, so we conclude the proof.

□

Now we are able to give a proof of our greedy algorithm for A_n:

Proof. (Prop. 1.9) If we have non trivial extensions among $V_1 \oplus \ldots \oplus V_m$:

$$0 \to V_i \hookrightarrow V \twoheadrightarrow \oplus_{j\neq i}V_i \to 0,$$

in that case

$$\circ^0 \longrightarrow \circ^0 \ldots \longrightarrow \circ^0 \longrightarrow \circ^{\mathbf{k}}_i \cdots \longrightarrow \circ^{\mathbf{k}}_j \longrightarrow \circ^0_{j+1} \cdots \longrightarrow \circ^0 \quad \hookrightarrow \quad \left\{ \begin{array}{c} \text{some sum of} \\ \text{indecomposables} \end{array} \right\}.$$

means that there exists a summand on right hand side.

Therefore if we denote by V_{ij} the previous indecomposable module, $\underline{\dim}\ V_{ij} = \sum_{k=i}^{j}\alpha_k$, all nontrivial extensions are sums of $V_{ij} \hookrightarrow V_{i'j} \twoheadrightarrow V_{i'i}$, $\quad i' \leq i \leq j$, and $\underline{\dim}\ V = \underline{\dim}\ V_1 + \underline{\dim}\ V_2$ for each short exact sequence $0 \to V_1 \to V \to V_2 \to 0$.

But it is not possible to form any such extension as above among summands if we follow the greedy algorithm, since if we end up with indecomposables V_i, V_j, for $i < j$, such that $\underline{\dim}\ V_i + \underline{\dim}\ V_j \in \Delta_+$, the greedy algorithm would have demanded that V_i be replaced by an indecomposable of dimension vector at least $\underline{\dim}\ V_i + \underline{\dim}\ V_j$ (or else we would have already run into a problem with $V_{i'}$ for $i' < i$). □

2. Extended Dynkin case (Lectures 5–6)

By Kac's Theorem, $\mathrm{Rep}_\alpha Q$ has a unique indecomposable for each $\alpha \in \Delta_+^{re}$; that is, for each $\alpha < \delta$, where $\delta \in \mathbb{N}^{Q_0}$ is minimal with the property: $(\delta, \alpha) = 0$ for all α.

The case $\alpha = \delta$ is much more interesting. In order to study this case, we need some other results.

Lemma 2.1 (Fitting). *Let A be a $\mathbf{k}$-algebra, $\mathbf{k}$ any field. A finite dimensional A-module M is indecomposable iff for all $\varphi \in \mathrm{End}M$, φ is nilpotent or an isomorphism.*

Proof. ($\Rightarrow$) Assume M indecomposable. If $\varphi \in \mathrm{End}M$ is not nilpotent, $\exists l \gg 0$ such that $\operatorname{im}\varphi^l = \operatorname{im}\varphi^{l+1} \neq 0$. Then $\ker\varphi^l = \ker\varphi^{l+1}$ and $M \cong \ker\varphi^l \oplus \operatorname{im}\varphi^l$. As M is indecomposable, $\ker\varphi = 0$ and φ is an isomorphism.

($\Leftarrow$) Suppose M decomposable: $M = M_1 \oplus M_2$; then the projection $M \twoheadrightarrow M_1 \hookrightarrow M$ is neither nilpotent nor an isomorphism. □

Lemma 2.2 (Ringel). *Let A be a $\mathbf{k}$-algebra ($\mathbf{k}$ algebraically closed), such that $\mathrm{Ext}^2(V,W) = 0$ for all finite-dimensional V, W. Let M be a finite-dimensional indecomposable module. Then, either M is a brick or there exists $M_0 \subseteq M$ brick such that $\mathrm{Ext}^1(M_0, M_0) \neq 0$.*

In particular, the Ext^2 condition holds for a path algebra, so this applies to the case where $M \in \mathrm{Rep}_\alpha(Q)$ for any quiver Q and any dimension vector $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$.

Proof. Assume M indecomposable not a brick. By induction, it is sufficient to find an indecomposable $M_0 \subsetneq M$ such that $\mathrm{Ext}^1_A(M_0, M_0) \neq 0$; then either M_0 is a brick or we iterate for M_0.

Therefore $\exists \varphi \in \mathrm{End}M$, $\varphi \notin \mathbf{k}$. If we consider M' the maximal semisimple quotient of M, $\mathrm{End}M'$ is a division algebra over $\mathbf{k}$, but as $\mathbf{k} = \overline{\mathbf{k}}$ we have $\mathrm{End}M' = \mathbf{k}$. By Fitting Lemma, φ is nilpotent. We can assume that $\mathrm{rk}\varphi$ is minimal. We shall find an indecomposable $K \subsetneq M$ such that $\mathrm{Ext}^1_A(K, K) \neq 0$.

We have $0 \neq \ker\varphi \subsetneq M$; write $\ker\varphi = \oplus_i K_i$, K_i indecomposables. By the assumption $\mathrm{rk}\varphi \neq 0$ minimal, $\varphi^2 = 0$, so we have $\mathrm{im}\,\varphi \hookrightarrow \ker\varphi = \oplus_i K_i$.

Using again the the assumption $\mathrm{rk}\varphi \neq 0$ minimal, there exists i_0 such that

$$\iota : \mathrm{im}\,\varphi \hookrightarrow \ker\varphi \twoheadrightarrow K_{i_0}$$

is injective, so non zero.

Use the sequence associated to φ to obtain by push out:

$$\begin{array}{cccccl} 0 \to & \ker\varphi \to & M \to & \mathrm{im}\,\varphi \to & 0 & \rightsquigarrow \text{ nonsplit } M \text{ indecomposable,} \\ & \downarrow_{proj} & \downarrow \pi & \downarrow_{\mathrm{id}_{\mathrm{im}\,\varphi}} & & \\ 0 \to & K_{i_0} \to & M' \to & \mathrm{im}\,\varphi \to & 0 & \therefore \text{ nonsplit (see below)} \end{array}$$

Note that the second sequence must be nonsplit; otherwise, there is a retraction $h : M' \twoheadrightarrow K_{i_0}$ which splits the second sequence, and hence $h \circ \pi$ splits K_{i_0} as a summand of M, which is impossible since M is indecomposable. We get that $\mathrm{Ext}^1(\mathrm{im}\,\varphi, K_{i_0}) \neq 0$.

From the short exact sequence

$$0 \to \mathrm{im}\,\varphi \to K_{i_0} \to K_{i_0}/\,\mathrm{im}\,\varphi \to 0,$$

we consider the long exact sequence obtained from $\mathrm{Hom}(-, K_{i_0})$, where we have:

$$\mathrm{Ext}^1(K_{i_0}, K_{i_0}) \twoheadrightarrow \mathrm{Ext}^1(\mathrm{im}\,\varphi, K_{i_0}) \to \mathrm{Ext}^2(K_{i_0}/\,\mathrm{im}\,\varphi, K_{i_0}) = 0.$$

Then, $\mathrm{Ext}^1(K_{i_0}, K_{i_0}) \neq 0$. $\square$

Now we can prove:

Proposition 2.3. *The generic orbit in* $\mathrm{Rep}_\delta Q$ *is a brick of codimension 1.*

Proof. First of all the generic orbit is indecomposable because there are only finitely many other orbits, $\dim PGL_\delta = \dim \mathrm{Rep}_\delta - 1$, not open.

Therefore it is enough to prove that every indecomposable is a brick for dimension δ.

Suppose then that $V \in \mathrm{Rep}_\delta$ indecomposable is not a brick. By, Ringel's Lemma, there exists $W \subseteq V$ brick such that $\mathrm{Ext}^1(W, W) \neq 0$.

In that case $\underline{\dim}\ W < \delta$, so $\underline{\dim}\ W \in \Delta^{re}_+$. But, then $0 \geq \dim\mathrm{End}(W) - \dim(\mathrm{Ext}^1(W, W)) = \frac{1}{2}(\underline{\dim}\ W, \underline{\dim}\ W) = 1$, a contradiction. $\square$

Looking at $\mathrm{Rep}_\alpha Q$, we conclude that for δ,

$$\mathrm{Rep}_\delta \supseteq \{\text{bricks of codim 1 orbits}\} \bigsqcup \{\text{decomposable orbits}\}.$$

We want to know now what happens with the multiples of δ: $l\delta$, $l \in \mathbb{N}$.

Proposition 2.4. *The generic decomposition is* $\ell\delta = \delta + \cdots + \delta$. *Generic orbits are associated to*

$$V_1 \oplus \cdots \oplus V_\ell, \quad \underline{\dim}\ V_i = \delta.$$

Proof. Using the dimension formula below (proved for the case where Q has no directed cycles, but it is true in general by the homework or by Proposition 2.5 below), we have that the codimension of $\mathcal{O}_{V_1\oplus\cdots\oplus V_\ell}$ is dim $\mathrm{End}(V_1 \oplus \cdots \oplus V_\ell)$. Generically, we claim that $\mathrm{Hom}(V_i, V_j) = 0$ for $i \neq j$. This implies that dim $\mathrm{End}(V_1 \oplus \cdots \oplus V_\ell) = \ell$ generically, so the generic orbit above has codimension ℓ. Since there is an ℓ-dimensional family of such, the above orbits form a dense subvariety of $\mathrm{Rep}_{\ell\delta}(Q)$, and hence the generic orbit is indeed of the above form.

To prove the claim that, generically, $\mathrm{Hom}(V_i, V_j) = 0$ for $i \neq j$, note first that there are finitely many positive roots α such that $\alpha < \delta$. For each such positive root, let W_α be an indecomposable with $\underline{\dim}\, W_\alpha = \alpha$ (which is unique up to isomorphism by Kac's theorem). Then, generically, either $\mathrm{Hom}(V, W_\alpha) = 0$ or $\mathrm{Hom}(V, W_\alpha) \neq 0$ for bricks $V \in \mathrm{Rep}_\delta(Q)$[2] The same is true for $\mathrm{Hom}(W_\alpha, V)$. Now, if $\mathrm{Hom}(V_i, V_j) \neq 0$ generically, then it would follow that, for some α, then $\mathrm{Hom}(V_i, W_\alpha)$ and $\mathrm{Hom}(W_\alpha, V_j)$ are both nonzero for generic bricks $V_i, V_j \in \mathrm{Rep}_\delta Q$.

It remains to prove the following claim: there does not exist α such that $\mathrm{Hom}(W_\alpha, V)$ and $\mathrm{Hom}(V, W_\alpha)$ are both nonzero for generic bricks $V \in \mathrm{Rep}_\delta Q$. Otherwise, there would exist a brick $V \in \mathrm{Rep}_\delta Q$ such that $\mathrm{Hom}(V, V) \neq \mathbf{k}$, contradicting that it is a brick. □

We also have seen that for $\alpha \in \Delta_+^{re}$ there exists a unique indecomposable in $\mathrm{Rep}_\alpha Q$, and concluded that it is a brick using:

$$\left.\begin{array}{c}\dim \mathrm{End}M - \dim\ \mathrm{Ext}^1(M,M) = \frac{1}{2}(\alpha,\alpha) = 1,\\ \mathrm{Ext}^1(M,M) = 0 \text{ since } 2\alpha \notin \Delta_+\end{array}\right\} (*) \qquad \begin{array}{c}\text{proved it when } Q \text{ has no cycles,}\\ \text{true in general by the HW.}\end{array}$$

In fact we can prove a more general geometric statement: *(*)* says that

$$\dim\ \mathrm{Ext}^1(M,M) = \mathrm{codim}(\mathcal{O}_M).$$

Proposition 2.5. *There exists a canonical isomorphism*

$$\textit{normal space to } \mathcal{O}_M \textit{ in } \mathrm{Rep}_\alpha Q \textit{ at } M \xrightarrow{\sim} \mathrm{Ext}^1(M,M).$$

Proof. The problem is how to define an extension

$$0 \to M_1 \to \tilde{M} \to M_2 \to 0,$$

where we consider $M \cong M_1 \oplus M_2$, $M_i \cong M$ as vector spaces.

We look for an action $\tilde{\rho} : A \to \mathrm{End}(M_1 \oplus M_2)$, restricting to $\rho : A \to \mathrm{End}(M_1)$ an descending to $\rho : A \to \mathrm{End}(\tilde{M}/M_1)$, $\tilde{M}/M_1 \cong M$, so it has the form

$$a \mapsto \begin{pmatrix} \rho(a) & \rho'(a) \\ 0 & \rho(a) \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

We want $\tilde{\rho}$ to be an homomorphism: $\tilde{\rho}(ab) = \tilde{\rho}(a)\tilde{\rho}(b)$, so $\rho' : A \to \mathrm{Hom}(M_2, M_1)$ satisfies:

$$\rho'(ab) = \rho'(a)\rho(b) + \rho(a)\rho'(b).$$

That is, ρ' is a ρ-derivation $A \to \mathrm{Hom}(M_2, M_1)$.

Rewrite $\tilde{M} = M \oplus \epsilon M$, where $\epsilon^2 = 0$. Consider $\tilde{A} := A[\epsilon]/(\epsilon^2)$:

$$\tilde{\rho} \text{ as above} \Leftrightarrow \tilde{\rho} = \rho + \epsilon\rho' : \tilde{A} \to \mathrm{End}(\tilde{M}) \text{ is an homomorphism.}$$

This produces a map

$$\left\{\begin{array}{c}\text{infinitesimal changes of}\\ A - \quad \text{mod structures of } M\\ (T_M \mathrm{Rep}_\alpha Q)\end{array}\right\} \twoheadrightarrow \mathrm{Ext}^1(M,M). \tag{2.6}$$

[2] in the latter case, in fact $\mathrm{Hom}(V, W_\alpha) \neq 0$ for all representations V, by upper semicontinuity of the dimension of $\mathrm{Hom}(-, W_\alpha)$.

For sanity, we can see this has the right dimension: bimodule derivations ρ' as above are equivalent to linear maps $Q_1 \to \mathrm{Hom}_{\mathbf{k}}(M_1, M_2)$ such that, for each $a : i \to j$, $a \mapsto \mathrm{Hom}((M_2)_i, (M_1)_j))$.

The question now is when we get the trivial extension. This means that there exists a change of basis $\mathrm{id} + \epsilon\Psi$ of $M + \epsilon M$ such that

$$\rho = (\mathrm{id} + \epsilon\Psi)\tilde{\rho}(\mathrm{id} + \epsilon\Psi)^{-1} = (\mathrm{id} + \epsilon\Psi)\tilde{\rho}(\mathrm{id} - \epsilon\Psi), \quad \rho, \tilde{\rho} : \tilde{A} \to \mathrm{End}_{\mathbf{k}[\epsilon]/(\epsilon^2)}(\tilde{M}).$$

Therefore, the kernel of the map 2.6 is the set of infinitesimal changes of basis of M, $T_M\mathcal{O}_M$. $\square$

Remark 2.7. In general we have

(2.8)
$$\mathrm{Ext}^1(M, N) \cong HH^1(A, \mathrm{Hom}_{\mathbf{k}}(M, N)) \cong \mathrm{Der}(A, \mathrm{Hom}_{\mathbf{k}}(M, N)) \big/ \mathrm{InnerDer}(A, \mathrm{Hom}_{\mathbf{k}}(M, N))$$

(in general, an inner derivation $A \to P$, P an A-bimodule, is $a \mapsto [a, p]$, for some $p \in P$). Note that $\mathrm{Hom}_{\mathbf{k}}(M, N)$ is an A-bimodule.

The first part follows from the general homological result:

$$\mathrm{Ext}^i_A(M, N) \cong HH^i(A, \mathrm{Hom}_{\mathbf{k}}(M, N)), \quad i \geq 0. \tag{2.9}$$

Note that, for $i = 0$, this says that $Hom_A(M, N) = Hom_{A-Bimod}(A, \mathrm{Hom}_{\mathbf{k}}(M, N))$. This isomorphism is actually the canonical adjunction $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, \mathrm{Hom}_{\mathbf{k}}(M, N)) \cong \mathrm{Hom}_A(A \otimes_{A^{\mathrm{op}}} M, N) = \mathrm{Hom}_A(M, N)$, and by naturality of the adjunction, we get (2.9) by passing to the derived version (replacing modules by projective resolutions).

3. Preprojective algebras (Lecture 6)

Definition 3.1. A *Poisson algebra* B is a commutative algebra equipped with a Lie bracket $\{\cdot, \cdot\}$ satisfying

$$\{ab, c\} = a\{b, c\} + b\{a, c\} \qquad \text{(Leibniz identity)}. \tag{3.2}$$

An *affine Poisson variety* is $X = \mathrm{Spec}\, B$.

Note 3.3. $\{\cdot, \cdot\} = 0$ is a Poisson structure, not interesting.

Definition 3.4. An *infinitesimal action* of a Lie algebra $\mathfrak{g}$ on $X = \mathrm{Spec}\, B$ is a map $* : \mathfrak{g} \otimes B \to B$ satisfying

$$g * (ab) = (g * a)b + a(g * b), \qquad a, b \in B, g \in \mathfrak{g}. \tag{3.5}$$

If B is Poisson, we also require

$$g * \{a, b\} = \{g * a, b\} + \{a, g * b\}, \qquad a, b \in B, g \in \mathfrak{g}. \tag{3.6}$$

Definition 3.7. A moment map μ for an action of $\mathfrak{g}$ on X is a map $\mu : X \to \mathfrak{g}$ such that the pullback $\mu^* : \mathbf{k}[\mathfrak{g}^*] \to B = \mathbf{k}[X]$ satisfies:

(a) $\{\mu^*(g), b\cdot\} = g * b$, $\forall b \in B, g \in \mathfrak{g}$ (note that $\mathfrak{g} \subseteq \mathbf{k}[\mathfrak{g}^*]$). I.e. $\mathfrak{g}$ acts identically on B with $\mathrm{ad}_{\{\cdot,\cdot\}}$ or μ^*.

(b) μ is a Poisson morphism. Equivalently, $\mu^*|_{\mathfrak{g}}$ is a Lie morphism.

If the action of $\mathfrak{g}$ comes from an action of G on X, then we require in addition

(b') μ is G-equivariant.

Note that, using (a), (b) is equivalent to μ being $\mathfrak{g}$-equivariant, since $\mu^*(g)$ is the action of g for $g \in \mathfrak{g}$. Thus, assuming (a), then (b') implies (b), and conversely in the case that G is connected.

Note that the Poisson bracket on $\mathfrak{g}^*$ comes from $\mathbf{k}[\mathfrak{g}^*] = \mathrm{Sym}\, \mathfrak{g}$, extending $[,]$ by Leibniz rule. Or $\pi \in \Gamma(\Lambda^2 T\mathfrak{g}^*)$ is $\pi|_f(g_1 \wedge g_2) = f([g_1, g_2])$.

The fact that μ is Poisson means that the $\mathfrak{g}$-action on $\mathfrak{g}^*$ pulls back to $\{\mu^*(g), -\}$, a $\mathfrak{g}$-action on X; i.e. $\mu : X \to \mathfrak{g}^*$ is $\mathfrak{g}$-equivariant.

Also, if G is connected, then (b') implies (b).

In general, μ is not unique. So we need (b) to ensure the uniqueness of μ.

In particular, if μ satisfies (a) and $f \in \mathfrak{g}^*$, then $\mu + f$ also satisfies (a); in fact, $(\mu+f)(g) = \mu(g) + f(g)$, with $f(g) \in \mathbf{k}$, and $\{1, b\} = 0$ for all $b \in \mathbf{k}[X]$.

More generally, let $Z(\mathbf{k}[X]) = \{b \in \mathbf{k}[X] : \{b, c\} = 0, \forall c \in \mathbf{k}[X]\}$. If $\phi \in \mathrm{Hom}_{\mathbf{k}}(\mathfrak{g}, Z(\mathbf{k}[X]))$, then $\mu + \phi$ still satisfies (a), because

$$\{\mu(g) + \phi(g), -\} = \{\mu(g), -\}, \forall g \in \mathfrak{g}.$$

In case that $f \in \mathfrak{g}^*$, (b) says that f is a character; i.e. $f|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

Definition 3.8. A *symplectic structure* is a non degenerate Poisson structure such that

$$\forall x \in X, \forall f \in \mathbf{k}[X] \text{ such that } df|_x \neq 0 \text{ gets } \{f, g\}|_x \neq 0 \text{ for some } g \in \mathbf{k}[X].$$

I.e., we have a Poisson bivector $\pi \in \Gamma(\Lambda^2 TX)$ such that $\{f, g\} = \pi(df \otimes dg)$, which exists by Leibniz identity and skew symmetry, and defines an isomorphism $i_\pi(-) : T^*X \to TX$.

In this case, the inverse $\omega : TX \to T^*X$ is a non degenerate 2-form $\omega \in \Gamma(\Lambda^2 T^*X)$.

Remember that the *Schouten bracket* $[\cdot, \cdot] : \Lambda^{\cdot} TX \oplus \Lambda^{\cdot} TX \to \Lambda^{\cdot} TX$ is given by

$$\begin{aligned} [\xi, \eta] &= \text{Lie bracket}, \quad \xi, \eta \in \Gamma(TX), \\ [\xi, f] &= \xi(f), \quad \xi \in \Gamma(TX), f \in \mathbf{k}[X], \\ [\xi_1 \wedge \xi_2, \eta] &= \xi_1 \wedge [\xi_2, \eta] + (-1)|\xi_1|\,|\xi_2| \xi_2 \wedge [\xi_1, \eta], \\ [\xi, \xi] &= 0. \end{aligned}$$

We have the following result:

Proposition 3.9. *Jacobi identity holds* $\Longleftrightarrow d\omega = 0 \Longleftrightarrow [\pi, \pi] = 0$.

This gives an equivalence

$$\begin{array}{lcr} \text{Poisson structures on } X & \longleftrightarrow & \pi \text{ bivector such that } [\pi, \pi] = 0, \\ \text{Symplectic structures on } X & \longleftrightarrow & \omega \text{ nondegerate such that } d\omega = 0. \end{array}$$

Example 3.10. This is the main example. T^*X (total space) is a symplectic variety for all X. The symplectic form $\omega = d\eta$, $\eta \in \Gamma(T^*(T^*X))$ given by

$$\begin{array}{ll} i_V \eta|_{x,P} = (pr_* V, P) \in \mathbf{k} & pr : T^*X \to X, \\ V \in \Gamma(T(T^*X)), pr_* V \in T_x X, P \in T^*_x X, & pr_* : T(T^*X) \to TX. \end{array}$$

ω is non degenerate because in, local coordinates x_i near $x \in X$ and p_i in the vertical direction of T^*X, we have

$$\eta = \sum_i p_i dx_i \quad \Longrightarrow \quad \omega = d\eta = \sum_i dp_i \wedge dx_i.$$

We have $\pi = \omega^{-1}$. If $X = \mathrm{Spec}\, A$ this gets a Poisson bracket $\{\cdot, \cdot\}$ on $B = \mathrm{Sym}_A \mathrm{Der}\, A$, which is uniquely defined extending

$$\{a, b\} = 0, \qquad a, b \in A, \tag{3.11}$$

$$\{\xi, a\} = 0, \qquad \xi \in \mathrm{Der}\, A = \Gamma(TX), \tag{3.12}$$

$$\{\xi, \eta\} = [\xi, \eta]_{Lie}, \qquad \xi, \eta \in \mathrm{Der}\, A, \tag{3.13}$$

by using Leibniz identity and skew symmetry.

The easiest way to prove this is using local coordinates: p_i is identified with $\partial/\partial x_i$, and we have

$$\begin{aligned}\{p_i, x_j\} &= \delta_{ij} = \frac{\partial}{\partial x_i}(x_j),\\ \{x_i, x_j\} &= 0 \qquad x_i\text{'s generate } A \subset B,\\ \{p_i, p_j\} &= 0.\end{aligned}$$

Remark 3.14. This bracket makes sense when X is singular.

Note that the moment map μ is bilinear, but is not unique and does not always exist; e.g, if $\{,\} = 0$ and $\mathfrak{g}$ acts non trivially.

Note 3.15. Assume that X is symplectic, $\mathfrak{g}$ is semisimple (that is, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$) and there exists a moment map μ. Then μ is unique.

Proof. Let μ' another moment map. As X is symplectic, $Z(\mathbf{k}[X]) = \mathbf{k}$, since $\{f, -\}|_x = 0$ implies $df|_x = 0$ for all $x \in X$, so $f \in \mathbf{k}$. In this way, as $(\mu - \mu')(g) \in Z(\mathbf{k}[X])$ for all $g \in \mathfrak{g}$, we have that $\mu - \mu'$ is a character. But as $\mathfrak{g}$ is semisimple, this character is 0. □

When $\mathfrak{g}$ acts on X, so we have as before a symplectic action of $\mathfrak{g}$ on T^*X, note that our formula is $\mathfrak{g}$-equivariant: $\mu^*(g) = \mathrm{act}(g) \in \Gamma(TX) \subseteq \mathbf{k}[T^*X]$. Therefore, μ is Poisson.

Claim 3.16. *Let X be an affine algebraic variety over $\mathbf{k}$, and $\mathfrak{g}$ acting over X: T^*X has a bracket as above. Then there exists a moment map $\mu((x,p)) = (g \mapsto (act_x(g), p))$.*

Proof. Note that the action of $\mathfrak{g}$ on X extends to an action of $\mathfrak{g}$ to T^*X; on B acts by Lie derivative $L_{\mathrm{act}(g)}$. Then,

$$\{\mu^*(g), \xi\} = \{\mathrm{act}(g), \xi\} = L_{\mathrm{act}(g)}(\xi) = g * \xi.$$

Therefore μ as above is a moment map. □

Definition 3.17. Let X be an affine algebraic variety and G an algebraic group acting on X. G is called a *Hamiltonian* if the action of $\mathfrak{g} := Lie(G)$ has a moment map μ. In such case, μ is also called a moment map for the action of G on X.

Example 3.18. Let G be a group action on X freely. Note that an action of G on X gives place to an action of G on T^*X. We can consider X/G, and $(T^*X)/G$. The latter is a bundle over X/G of rank dim X, whose sections over $u \in X/G$ are G-equivariant sections of T^*X on $G \cdot u \subseteq X$.

This variety is not symplectic in general. Indeed, any symplectic variety has even dimension, and by the above, $\dim(T^*X)/G = 2\dim X - \dim G$.

Definition 3.19.
- Let B a Poisson algebra. An ideal $I \subseteq B$ is *Poisson* if $\{I, B\} \subseteq I$. In such case, B/I has a natural structure of Poisson algebra.
- Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g} \otimes B \to B$ a Poisson action, and $\mu : Y = \operatorname{Spec} B \to \mathfrak{g}$. We define

$$J := \{\mu^*(g) : g \in \mathfrak{g}\}.$$

Note that J is not always a Poisson ideal. Remember that $B^{\mathfrak{g}} := \{b \in B : g * b = 0, \forall g \in \mathfrak{g}\}$.

Claim 3.20. $\{J, B^{\mathfrak{g}}\} \subseteq J$.

Proof. $\{\mu^*(g), b\} = g * b = 0, \quad \forall g \in \mathfrak{g}, B^{\mathfrak{g}}$. □

Definition 3.21. A *Hamiltonian reduction* of Y by $\mathfrak{g}$ is $B^{\mathfrak{g}}/J^{\mathfrak{g}}$.

If we have G acting on X, we can use $B^G/J^G =:$ *Hamiltonian reduction* of Y by G

Notation 3.22. $Y//_hG := \operatorname{Spec} B^G/J^G$.

Proposition 3.23. *If Y is symplectic and G is Hamiltonian and acts freely on Y, then $Y//_hG$ is symplectic.*

Proof. Let $\pi = \omega^{-1}$ be the Poisson bivector: π is non degenerate. If $\{f, B^G\}|_x = 0$, then $i_\pi df$ is tangent to the G-action at x.

If $(i_\pi df - \mathrm{act}(g))|_x = 0$ for some g, then $i_\pi(df - d\mu^*(g))|_x = 0$. By the non degeneracy of π.

Then in B^G/J^G, $\{f, -\}_x = 0 \Rightarrow df|_x = 0$. $\square$

Our main example to be considered is $Y = T^*X$, where $X = \mathrm{Rep}_\alpha Q$.

Remember that for Q quiver, its double quiver $\bar{Q}$ is defined by $\bar{Q}_0 = Q_0$ and $\bar{Q}_1 = Q_1 \sqcup Q_1^*$, where Q_1^* is the opposite quiver.

Proposition 3.24. *We have a canonical isomorphism:* $Y \cong \mathrm{Rep}_\alpha \bar{Q}$.

Proof. This is just a several edges version of the following case:

$$Q = \circ \longrightarrow \circ \qquad \Longrightarrow \qquad \bar{Q} = \circ \rightleftarrows \circ$$

In this case we have

$$\begin{aligned} \mathrm{Rep}_\alpha Q &= \mathrm{Hom}(\mathbf{k}^{\alpha_1}, \mathbf{k}^{\alpha_2}) =: U, \\ \mathrm{Rep}_\alpha \bar{Q} &= \mathrm{Hom}(\mathbf{k}^{\alpha_1}, \mathbf{k}^{\alpha_2}) \oplus \mathrm{Hom}(\mathbf{k}^{\alpha_2}, \mathbf{k}^{\alpha_1}) =: U \oplus U^*. \end{aligned}$$

For any vector space U, $T_u U \cong U$ for all $u \in U$, and $T_u^* U \cong U^*$ for all $u \in U$. So $T^*U \cong U^* \times U$ canonically.

Now explicitly, $\mathbf{k}[\mathrm{Rep}_\alpha Q] = \mathbf{k}[M_a : a \in Q_1]$ is a polynomial algebra in $\dim \mathrm{Rep}_\alpha Q = \sum_{a:i\to j \in Q_1} \alpha_i \alpha_j$ variables, where M_a denotes the matrix-valued function in $\alpha_i\alpha_j$ variables (coordinates functions): $\rho \mapsto \rho_a \in \mathrm{Hom}(\mathbf{k}^{\alpha_1}, \mathbf{k}^{\alpha_2})$.Therefore,

$$M_a = \Big((M_a)_{lp}\Big)_{l=1,\dots,\alpha_i,\, p=1,\dots,\alpha_j}, \quad M_a = \left(\frac{\partial}{\partial\,(M_a)_{lp}}\right)_{l=1,\dots,\alpha_i,\, p=1,\dots,\alpha_j}$$

and we have

$$\mathbf{k}[\mathrm{Rep}_\alpha \overline{Q}] = \mathbf{k}[M_a, M_{a^*} : a \in Q_1] \cong \mathrm{Sym}_{\,\mathbf{k}[\mathrm{Rep}_\alpha Q]}\, \mathrm{Der}\,(\mathbf{k}[\mathrm{Rep}_\alpha Q]) = \mathbf{k}[T^*\mathrm{Rep}_\alpha Q],$$

where the isomorphism is canonical. $\square$

Consider GL_α acting on $\mathrm{Rep}_\alpha Q$. It induces an action of GL_α on $\mathrm{Rep}_\alpha \bar{Q} \cong T^*\mathrm{Rep}_\alpha Q$: with this action, GL_α is Hamiltonian, and gets $\mu : \mathrm{Rep}_\alpha \bar{Q} \to \mathfrak{gl}_\alpha^*$.

Claim 3.25. $\mu\left((\rho_a, \rho_{a^*})\right) = \sum_{a\in Q} [\rho_a, \rho_{a^*}] \in \oplus_{i\in Q_0} \mathrm{End}(V_i) \cong \mathfrak{gl}_\alpha \cong \mathfrak{gl}_\alpha^*$.

Proof. Note that

$$\{(M_a)_{lp}, (M_b)_{l'p'}\} = -\delta_{a^*,b}\delta_{l,l'}\delta_{p,p'} + \delta_{a,b^*}\delta_{l,l'}\delta_{p,p'}.$$

By definition, $\mu\left((V_i, \rho_a, \rho_{a^*})\right) = (g \mapsto ((\mathrm{act}(g)|_{\rho_a}, \rho_{a^*}))$, where we have

$$\mathrm{act}(g)|_{\rho_a} = \mathrm{ad}g|_{\rho_a} = [g, \rho_a] \in T_{\rho_a}\mathrm{Rep}_\alpha Q$$

(because $\mathrm{Ad}(\mathrm{Id} + \epsilon g)(\rho_a) = \rho_a + \epsilon[g, \rho_a]$).

As $g \mapsto \sum_a \mathrm{Tr}\left([g, \rho_a]\rho_{a^*}\right) = \sum_a \mathrm{Tr}\left(g[\rho_a, \rho_{a^*}]\right)$, we have

$$\mu\left((V_i, \rho_a, \rho_{a^*})\right) = \sum_a [\rho_a, \rho_{a^*}] \in \mathfrak{gl}_\alpha.$$

$\square$

Under $eval_\alpha : \mathbf{k}\bar{Q} \to \mathbf{k}[\mathrm{Rep}_\alpha Q] \otimes \mathfrak{gl}_\alpha$,

$$a \mapsto M_a, \quad a^* \mapsto M_{a^*},$$

we have $J = \mathfrak{gl}_\alpha^*(eval_\alpha(\sum_a [\rho_a, \rho_{a^*}]))$.

Corollary 3.26. $(\mu^*(g) | g \in \mathfrak{gl}_\alpha) = \left(\sum_{a \in Q_1} [M_a, M_{a^*}]\right) \subseteq \mathbf{k}[\mathrm{Rep}_\alpha \bar{Q}]$.

Upshot: $\mu^{-1}(0) = \mathbf{k}[\mathrm{Rep}_\alpha \bar{Q}]/J = \mathrm{Rep}_\alpha \Pi_Q$, where $\Pi_Q = \mathbf{k}\bar{Q}/(\sum_a [\rho_a, \rho_{a^*}])$. Note that $\mathrm{Rep}_\alpha \Pi_Q$ corresponds to the representations of $\bar{Q}$ such that $[M_a, M_{a^*}] = 0$.

Definition 3.27. (1) Π_Q is called the *preprojective algebra* of Q.
(2) For each $\lambda \in \mathbf{k}Q_0$, we also define:

$$\Pi_Q^\lambda := k\bar{Q}/(\sum_a [\rho_a, \rho_{a^*} - \lambda]).$$

In the same way, $\mathrm{Rep}_\alpha \Pi_Q^\lambda = \mu^{-1}(\lambda \mathrm{Id}) = \mu^{-1}(\sum_{i \in Q_0} \lambda_i \mathrm{Id}_{\mathrm{End}(\mathbf{k}^{\alpha_i})})$.

Note that $T^*\mathrm{Rep}_\alpha Q //_h GL_\alpha = \mathrm{Rep}_\alpha \Pi_Q // GL_\alpha$. We will next see that the same is true for Π_Q^λ, if we allow Hamiltonian reduction along orbits.

4. Hamiltonian reduction along orbits

Given a Hamiltonian action of G on Y, it makes sense to consider not just $\mu^{-1}(0)//G$, but more generally, if $\mathcal{O} \subset \mathfrak{g}^*$ is a closed coadjoint orbit (i.e., an orbit of the $\mathrm{Ad}\, G$ action on $\mathfrak{g}^*$), we may consider $\mu^{-1}(\mathcal{O})//G$. (Recall that we require μ to be G-equivariant, so that $\mu^{-1}(\mathcal{O})$ is indeed G-stable.)

Definition 4.1. $R(X, G, \mathcal{O}) := \mu^{-1}(\mathcal{O})//G$.

This can be advantageous, for instance, when G does not act freely on $\mu^{-1}(0)$, but does act freely on some other $\mu^{-1}(\mathcal{O})$. In particular, if Y is symplectic, then in this case we will obtain a symplectic manifold by taking the quotient, of dimension $\mathrm{dim} Y + \mathrm{dim} \mathcal{O} - 2\mathrm{dim}\ G$. The reason we obtain a symplectic manifold is that, for any $y \in \mu^{-1}(z) \subset \mu^{-1}(\mathcal{O})$, the conormal space to y in $\mu^{-1}(\mathcal{O})$ is identified with the vanishing of $i_{\mathfrak{g}_z}\omega$, where ω is the symplectic form, and $\mathfrak{g}_z$ is the annihilator of $z \in \mathfrak{g}^*$ under the coadjoint action, i.e., the conormal space to $\mathcal{O}$ at z. This is exactly the annihilator of the tangent space $T_z\mathcal{O}$, and hence ω descends to a nondegenerate two-form at $Gy \in \mu^{-1}(\mathcal{O})//G$. By freeness of the action and because the dimension of $i_{\mathfrak{g}_\mathfrak{z}}\omega$ is $\dim G - \dim \mathcal{O}$, we get that $\dim R(X, G\mathcal{O}) = \mathrm{dim} \mu^{-1}(\mathcal{O}) - 2\mathrm{dim}\ G$.

If $\mathcal{O}$ is a coadjoint orbit which is not closed, we can similarly define $R(X, G, \overline{\mathcal{O}}) = \mu^{-1}(\overline{\mathcal{O}})//G$.

Let us now consider the case $Y = T^*X$ where X is a smooth affine variety. For any smooth subvariety $Z \subset X$, let $N_Z \subset T^*X$ denote the conormal bundle to Z, i.e., the subbundle of $T^*X|_Z$ whose fiber at $z \in Z$ is $(T_z Z)^\perp$.

Claim 4.2. $\mu^{-1}(0) = \bigsqcup_{Gx \subset X} N_{Gy}$.

Proof. Note that $\mu^{-1}(0)$ is the vanishing locus of the functions $\mathrm{act}(g) \in \Gamma(TX) \subset \mathbf{k}[T^*X]$, and hence is the subbundle over X of T^*X, whose fibers at $x \in X$ are the annihilator of the $\mathfrak{g}$-action. The claim follows. □

Now if we let $\mathcal{O} \subset \mathfrak{g}^*$ be any closed orbit, then $\mu^{-1}(\mathcal{O})$ is the union of the vanishing loci of $\mathrm{act}(g) - f(g)$, for all $f \in \mathcal{O}$. In particular, for a nonzero orbit, we will not get any point in the zero section of T^*X, and the picture looks more complicated. However, the tangent space to $\mu^{-1}(\mathcal{O})$ at (x, p) can be described: in the X-direction, this is the $\mathfrak{g}$-action, and in the vertical (T_x^*X) direction, this is the conormal space to $\mathrm{act}(\mathfrak{g}_z)$, where $z = \mu(x, p) = (g \mapsto \langle \mathrm{act}(g)_x, p \rangle) \in \mathfrak{g}^*$. In particular, the intersection $T_x^*X \cap \mu^{-1}(\mathcal{O})$ is the G_x-orbit of the affine space $p + (\mathfrak{g}_z)^\perp$, where $G_x < G$ is the isotropy at $x \in X$ of the G-action on X.

5. Π_Q^λ and Hamiltonian reduction

For any $\lambda \in \mathbf{k}^{Q_0}$, and any dimension vector $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$, consider $\lambda \cdot \mathrm{Id} := \sum_{i \in Q_0} \lambda_i \mathrm{Id}_{\mathbf{k}^{\alpha_i}} \in \mathfrak{gl}_\alpha$. This is evidently invariant under GL_α, i.e., it is an orbit (consisting of a single point). We may consider the inverse image $\mu^{-1}(\lambda \cdot \mathrm{Id})$. As in the case $\lambda = 0$, we get

$$\mu^{-1}(\lambda \cdot \mathrm{Id}) = \mathrm{Rep}_\alpha \Pi_Q^\lambda, \quad R(T^*\mathrm{Rep}_\alpha Q, GL_\alpha, \{\lambda \cdot \mathrm{Id}\}) = \mathrm{Rep}_\alpha \Pi_Q^\lambda // GL_\alpha. \tag{5.1}$$

6. Representations of Π_Q^λ

Theorem 6.1. *Let $\mathbf{k}$ be algebraically closed. A representation V of $\mathbf{k}Q$ extends to a representation of Π_Q^λ iff for all summand $W \leq V$ (i.e. exists W' submodule of V such that $V \cong W \oplus W'$), $\lambda \cdot \underline{\dim}\ W = 0$.*

Note that this is equivalent to the following: a indecomposable module V of $\mathbf{k}Q$ extends to a representation of Π_Q^λ iff $\lambda \cdot \underline{\dim}\ V = 0$. Before to prove this result, we give a simple consequence.

Corollary 6.2. *Let $\mathbf{k}$ be algebraically closed. Every representation V of $\mathbf{k}Q$ extends to a representation of Π_Q.*

Proof. (Theorem): Necessity. This is easy. Let $(\rho_a) \in \mathrm{Rep}_\alpha Q$ which extends to $(\rho_a, \rho_{a^*}) \in Rep_\alpha \Pi_Q^\lambda$: $\sum_{a \in Q_1} [\rho_a, \rho_{a^*}] = \lambda \mathrm{Id}$, and

$$0 = \sum_{a \in Q_1} \mathrm{tr}([\rho_a, \rho_{a^*}]) = \mathrm{tr}\left(\sum_{a \in Q_1} [\rho_a, \rho_{a^*}]\right) = \mathrm{tr}(\lambda \mathrm{Id}) = \lambda \cdot \alpha.$$

Sufficiency. It is enough to take V indecomposable and proves it can be extended. We want to look up ρ_{a^*} such that $\sum_{a \in Q_1} [\rho_a, \rho_{a^*}] = \lambda \mathrm{Id}$. It would be enough to find an exact sequence

$$\begin{array}{ccccccc}
\mathrm{Rep}_\alpha Q^* & \xrightarrow{\Psi} & \mathfrak{gl}_\alpha & \longrightarrow & \mathbf{k} & \longrightarrow & 0 \\
(\rho_{a^*}) \mapsto & & \sum_{a \in Q_1} [\rho_a, \rho_{a^*}] & & & & \\
& & A \mapsto & & \mathrm{tr} A & & .
\end{array}$$

We want Ψ surjective onto to the set of λ such that $\lambda \cdot \underline{\dim}\ W = 0$, but this is not true in general. We have to change $\mathbf{k}$ for a "bigger" term. We can take:

$$\begin{tikzcd}
& \mathfrak{gl}_\alpha^* \arrow[dr, "restr."] & & \\
\mathrm{Rep}_\alpha Q^* \arrow[r, "\Psi"] & \mathfrak{gl}_\alpha \arrow[u, "\wr"] \arrow[r, "\xi"] & \mathrm{End}_{\mathbf{k}Q}(V)^* \arrow[r] & 0
\end{tikzcd} \tag{6.3}$$

$$A \cong (B \mapsto \mathrm{tr}(AB)) \mapsto \qquad \text{restr. to } B \in \mathrm{End}_{\mathbf{k}Q}(V) \qquad .$$

Note this will be sufficient provided that $\lambda \mathrm{Id} \mapsto 0$ under ξ. This follows from Fitting Lemma (here we use that $\bar{\mathbf{k}} = \mathbf{k}$): $\mathrm{End} V = \mathbf{k} \oplus Nilp$, so

$$\xi(\lambda \mathrm{Id}) = \left(\begin{array}{c} \mathrm{Id} \mapsto \mathrm{tr}(\lambda \mathrm{Id}) = 0, \\ N \in Nilp \mapsto \mathrm{tr}(\lambda N) = 0 \end{array} \right) = 0.$$

So we have to prove (6.3). Remember the projective resolution:

$$0 \longrightarrow \mathbf{k}Q \otimes_{\mathbf{k}Q_0} \langle Q_1 \rangle \otimes_{\mathbf{k}Q_0} \mathbf{k}Q \longrightarrow \mathbf{k}Q \otimes_{\mathbf{k}Q_0} \mathbf{k}Q \longrightarrow \mathbf{k}Q \longrightarrow 0.$$

We apply $- \otimes_{\mathbf{k}Q} V$ in order to obtain a resolution of V:

$$0 \longrightarrow \mathbf{k}Q \otimes_{\mathbf{k}Q_0} \langle Q_1 \rangle \otimes_{\mathbf{k}Q_0} V \longrightarrow \mathbf{k}Q \otimes_{\mathbf{k}Q_0} V \longrightarrow V \longrightarrow 0.$$

Now, we apply $\mathrm{Hom}_{\mathbf{k}Q}(-, V)$, and consider the long exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{End}_{\mathbf{k}Q}(V) &\longrightarrow \mathrm{End}_{\mathbf{k}Q}(\mathbf{k}Q \otimes_{\mathbf{k}Q_0} V, V) \longrightarrow \mathrm{End}_{\mathbf{k}Q}(\mathbf{k}Q \otimes_{\mathbf{k}Q_0} \langle Q_1 \rangle \otimes_{\mathbf{k}Q_0} V, V) \\ &\longrightarrow \mathrm{Ext}^1(V, V) \longrightarrow \mathrm{Ext}^1(\mathbf{k}Q \otimes_{\mathbf{k}Q_0} V, V) \cdots . \end{aligned}$$

Now, we change that a bit. First,

$$\mathrm{End}_{\mathbf{k}Q}(\mathbf{k}Q \otimes_{\mathbf{k}Q_0} V, V) = \mathrm{End}_{\mathbf{k}Q_0}(V) = \mathfrak{gl}_\alpha.$$

Also, $\mathrm{Ext}^1(\mathbf{k}Q \otimes_{\mathbf{k}Q_0} V, V) = 0$, because the first module is projective, and

$$\mathrm{End}_{\mathbf{k}Q}(\mathbf{k}Q \otimes_{\mathbf{k}Q_0} \langle Q_1 \rangle \otimes_{\mathbf{k}Q_0} V, V) = \mathrm{End}_{\mathbf{k}Q}(\mathbf{k}Q_1 \otimes_{\mathbf{k}Q_0} V, V) \xrightarrow{\sim} \mathrm{Rep}_\alpha Q,$$

where the last isomorphism is given by $a \otimes V \mapsto \theta_a \in \mathrm{Hom}(V_i, V_j)$, $a : i \to j$. Then,

$$0 \to \mathrm{End} V \to \mathfrak{gl}_\alpha \to \mathrm{Rep}_\alpha Q \to \mathrm{Ext}^1(V, V) \to 0.$$

We can dualize:

$$0 \longrightarrow \mathrm{Ext}^1(V, V)^* \longrightarrow \mathrm{Rep}_\alpha Q^* \xrightarrow{\Omega} \mathfrak{gl}_\alpha^* \xrightarrow{\Phi} \mathrm{End}(V)^* \longrightarrow 0,$$

where we use that $(\mathrm{Rep}_\alpha Q)^* \cong \mathrm{Rep}_\alpha Q^*$ canonically, Φ is the restriction (because is the dual map of the morphism induced by the multiplication) and

$$\begin{array}{c} \Theta : \mathfrak{gl}_\alpha \to \mathrm{Rep}_\alpha Q \\ A \mapsto \sum [\rho_a, A] \end{array} \quad \Rightarrow \quad \begin{array}{c} \Psi = \Theta^* : \mathrm{Rep}_\alpha Q^* \to \mathfrak{gl}_\alpha \cong \mathfrak{gl}_\alpha^* \\ (\rho_{a^*}) \mapsto \sum [\rho_a, \rho_{a^*}]. \end{array}$$

So we obtain the desired exact sequence. $\square$

Remark 6.4. Note that if $(\rho_{a^*}), (\rho'_{a^*})$ are two liftings, then $(\rho_{a^*}) - (\rho'_{a^*}) \in \mathrm{Ext}^1(V, V)^*$ in a natural way. $\mathrm{Ext}^1(V, V)^*$ corresponds to the conormal space to $\mathcal{O}_V$ at V, and the whole cotangent fiber are all the (ρ_{a^*})'s.

QUIVERS IN REPRESENTATION THEORY (18.735, SPRING 2009) LECTURES 9 TO 14

TRAVIS SCHEDLER, TYPED BY IVÁN ANGIONO

1. Proof of Kac's Theorem - Part I

Construction of reflection functors for Π_Q^λ:

We fix $i \in Q_0$. Assume that $\lambda_i \neq 0$ and i is loop free. Anyway, we allow Q to have loops. So we have to redefine notions of root systems:

$$\begin{aligned}
\Delta^{re} &:= \{\alpha : \exists j, i_1, ..., i_l \in Q_0 \text{ loop free} \,/ \alpha = s_{i_l} \cdots s_{i_1} \epsilon_j\}, \\
\Delta^{im} &:= \{\alpha : \exists i_1, ..., i_l \in Q_0 \text{ loop free}, \delta \in F \,/ \alpha = s_{i_l} \cdots s_{i_1} \delta\}, \\
\Delta &:= \Delta^{re} \sqcup \Delta^{im}
\end{aligned}$$

Proposition 1.1.
- $\Delta^{re} = \Delta_+^{re} \sqcup (-\Delta_+^{re})$,
- $\Delta^{im} = \Delta_+^{im} \sqcup (-\Delta_+^{im})$,
- $\Delta = \Delta_+ \sqcup \Delta_-$.

We use the following notation: given $i, j \in \bar(Q)_0$, $j \leftrightarrow i$ means that there exists an arrow $j \to i$, or an arrow $j \leftarrow i$ (i.e. i and j are connected by an edge if we consider the underlying graph corresponding to Q).

$\bar{Q}$ looks like $\cdots \circ_j \rightleftarrows \circ_i \rightleftarrows \circ_k \cdots$ near i. Consider

$$\theta : V_i \to \oplus_{j \in Q_0 : j \leftrightarrow i} V_j, \qquad \theta = \sum_{a \in Q_1 / a: i \to j} \rho_a + \sum_{a \in Q_1 / a : j \to i} \rho_{a^*}, \tag{1.2}$$

$$\phi : \oplus_{j \in Q_0 : j \leftrightarrow i} V_j \to V_i, \qquad \phi = \sum_{a \in Q_1 / a : j \to i} \rho_a - \sum_{a \in Q_1 / a : i \to j} \rho_{a^*}. \tag{1.3}$$

If we denote by $\mathrm{inc}_k : V_k \hookrightarrow V$, $\mathrm{pr}_k : V \twoheadrightarrow V_k$ the canonical morphisms for each $k \in Q_0$, then

$$\phi \circ \theta = \mathrm{pr}_i \circ \left(\sum_{a \in Q_1} [\rho_a, \rho_{a^*}] \right) \circ \mathrm{inc}_i = \lambda_i \mathrm{Id}_i = \lambda_i \, \mathrm{pr}_i \circ \mathrm{inc}_i \,. \tag{1.4}$$

In this way, if $\lambda_i \neq 0$, V_i is realized as a summand of $\oplus_{j \in Q_0 : j \leftrightarrow i} V_j$ using $\lambda_i^{-1} \theta$. Let $V_i' := \ker \phi$ be the complementary summand and denote the projection and inclusions by $\mathrm{inc}_{V_i}, \mathrm{pr}_{V_i}, \mathrm{inc}_{V_i'}, \mathrm{pr}_{V_i'}$.

We replace V_i with V_i': note that $\mathrm{dim} V_i' = \sum_{j \leftrightarrow i} \mathrm{dim} V_j - \mathrm{dim} V_i$, as desired. For all j adjacent to i, let the arrow $V_i' \to V_j$ be $\mu \, \mathrm{pr}_j \circ \mathrm{inc}_{V_i'}$, and let the arrow $V_j \to V_i'$ be $\mathrm{pr} \circ \mathrm{inc}_j$, where $\mathrm{pr}_{V_i'}$ is similarly the projection onto $\mathrm{coker}\, \theta$.

We want to know what happens with $\sum_{a \in Q_1} [\rho_a, \rho_{a^*}]$. We have, for all j adjacent to i,

$$\sum_{a \in Q_1 : a : i \leftrightarrow j} [\rho_a, \rho_{a^*}]|_{V_j} = \mathrm{pr}_j \circ (\theta \circ \phi) \circ \mathrm{inc}_j = \sum_{a : i \leftrightarrow j} -\lambda_i \, \mathrm{pr}_j \circ (\mathrm{inc}_{V_i} \circ \mathrm{pr}_{V_i}) \circ \mathrm{inc}_j \,. \tag{1.5}$$

This will change to

$$\sum_{a \in Q_1 : a : i \leftrightarrow j} [\rho_a', \rho_{a^*}']|_{V_j} = -\mu \sum_{a : i \leftrightarrow j} \mathrm{pr}_j \circ (\mathrm{inc}_{V_i'} \circ \mathrm{pr}_{V_i'}) \circ \mathrm{inc}_j \,. \tag{1.6}$$

Since $\mathrm{inc}_{V_i} \circ \mathrm{pr}_{V_i} + \mathrm{inc}_{V_i'} \circ \mathrm{pr}_{V_i'} = \mathrm{Id}_{\bigoplus_{i \leftrightarrow j} V_j}$, if we set $\mu := -\lambda$, then we will obtain that we add $\lambda_i \cdot \mathrm{Id}_j$ to the sum of commutators:

$$\sum_{a \in Q_1} [\rho_a', \rho_{a^*}']|_{V_j} = \sum_{a \in Q_1} [\rho_a, \rho_{a^*}]|_{V_j} + \sum_{a \in Q_1 : a : i \leftrightarrow j} \lambda_i \mathrm{Id}_j. \tag{1.7}$$

Also, it is clear that at the vertex i itself, setting $\mu = -\lambda$ will yield

$$\sum_{a \in Q_1} [\rho_a', \rho_{a^*}']|_{V_i} = -\lambda_i \mathrm{Id}_i. \tag{1.8}$$

The total result is

$$\sum_{a \in Q_1} [\rho_a', \rho_{a^*}'] = \lambda' \mathrm{Id}, \quad \lambda_j = \begin{cases} -\lambda_i & j = i \\ \lambda_j + \sum_{a : i \to j,\, a : j \to i} \lambda_i & j \neq i. \end{cases}$$

Definition 1.9. We define $r_i : \mathbb{R}^{Q_0} \to \mathbb{R}^{Q_0}$ the transformation $r_i(\lambda) = \lambda'$ as before.

Note 1.10. If we do this process twice at i, we get back to the same module. Also, r_i verifies

$$\alpha \cdot r_i(\lambda) = s_i(\alpha) \cdot \lambda.$$

Therefore, we have the following result:

Proposition 1.11. *Let i be loop free such that $\lambda_i \neq 0$. There exists an equivalence*

$$\Pi_Q^{\lambda} \longrightarrow \Pi_Q^{\lambda'},$$
$$V \mapsto F_i V = \textit{reflection at } i.$$

Theorem 1.12 (Kac, part I)**.** *Let k be an algebraically closed field of characteristic 0.*

1. *If $\mathrm{Indec}_\alpha Q \neq \emptyset$, then $\alpha \in \Delta_+$.*
2. *If $\alpha \in \Delta_+^{re}$, then there exists a unique indecomposable V, $\underline{\dim}\ V = \alpha$, up to isomorphism.*

In this way, it remains to show that $\dim(\mathrm{Indec}_\alpha Q / \sim) > 0$ if $\alpha \in \Delta_+^{im}$.

The procedure to prove the first part is the following: given $M \in \mathrm{Indec}_\alpha Q$, pick λ such that $\lambda \cdot \alpha = 0$, and $\lambda \cdot \gamma \neq 0$ for all $\gamma < \alpha$, $\gamma \in \Delta_+$, $\gamma \notin \mathbb{Q}\alpha$.

Extend M to a representation of Π_Q^λ. we will see that there are no submodules which are $\mathbf{k}Q$-modules of dimension γ, $\lambda \cdot \gamma \neq 0$.

If α is indivisible (i.e the gcd of the α_i's is 1), then M is a simple Π_Q^λ-module.

Claim 1.13. *Either $\alpha = \epsilon_i$, or $\alpha \in F$, or there exists i such that $\lambda_i \neq 0$, $(\alpha, \epsilon_i) > 0$.*

Proof. Assume that $\alpha \neq \epsilon_i$ and $\alpha \notin F$. In such case, $(\alpha, \epsilon_i) > 0$ for some $i \in Q_0$. Therefore i must be loop free; otherwise, if $t := \#\text{loops at } i \geq 1$, and

$$(\alpha, \epsilon_i) = 2(1-t)\alpha_i - s \sum \alpha_i \alpha_j \leq 0.$$

Suppose now $\lambda_i = 0$, so $\phi \circ \theta = 0$. Since M is simple, $\ker \theta = 0 = \mathrm{coker}\, \phi$, so

$$\sum_{i \to j} \dim V_j \geq V_i \quad \Longrightarrow \quad (\alpha, \epsilon_i) \leq 0.$$

which is a contradiction. Then $\lambda_i \neq 0$. ☐

For the last case considered in the claim, we can apply the reflection functor, so $\alpha \rightsquigarrow s_i \alpha < \alpha$, and using the invariance of the bilinear form with respect the action of the Weyl group,

$$(s_i \alpha, \epsilon_i) = (\alpha, s_i \epsilon_i) = -(\alpha, \epsilon_i) < 0.$$

Iterating, we eventually get:

- $\alpha' = \epsilon_i$, so $\alpha \in \Delta_+$, or
- $(\alpha', \epsilon_i) \leq 0$ for all i. Since M is simple, α' has connected support, so $\alpha' \in F$ and $\alpha \in \Delta_+$.

Proof. (Kac, part I) (i) Write $\alpha = l\beta$, where $\beta \in \mathbb{N}^{Q_0}$ is indivisible and $l \geq 1$. Choose λ as above.

Extend $M \in \mathrm{Indec}_\alpha Q$ to a Π_Q^λ-module. If M is simple, use previous claim and we are done. Otherwise, there exists a submodule $N \subseteq M$ simple, $\underline{\dim}\ N = m\beta$, for some $0 < m < l$ (by choice of λ, any indecomposable $\mathbf{k}Q$-submodule N which is a Π_Q^λ-submodule verifies $\underline{\dim}\ N \in \mathbb{Q}\alpha$).

Apply reflections to $m\beta, \lambda$ as we have described: since N is simple, this gets eventually $\beta \rightsquigarrow \beta' \in F$, or $\beta \rightsquigarrow \beta' = \epsilon_i$ for some i. If $\beta' = \epsilon_i$, then N is simple (i loop free), so $m = 1$, M indecomposable ($l = 1$) and $\beta \in \Delta_+$. Otherwise, $\beta' \in F$, so $m\beta' \in \Delta_+^{im}$ and also we conclude $\alpha \in \Delta_+$.

(ii) Let $\alpha \in \Delta_+^{re}$. Write $\alpha = s_{i_m} \cdots s_{i_1}\epsilon_j$, with j loop free, n minimal. We want to construct

$$\begin{array}{lr} \Pi_Q^{\nu^1} \to \Pi_Q^{\nu^2} \to \cdots & \to \Pi_Q^\lambda, \\ \epsilon_j \mapsto s_{i_1}\epsilon_j \mapsto \cdots & \mapsto \alpha. \end{array}$$

If we do this, we get that any indecomposable $\mathbf{k}Q$-module extends to one in the image of S_j under these transformations.

Remark 1.14. Real roots are always indivisible (it can be proved by induction on the minimal number of s_i's when we write the real root as a combination of them applied to a simple root). Therefore indecomposables of this dimension always extends to a simple Π_Q^λ-module, for some generic λ.

Pick $\nu = \nu_1$ such that $\nu_j = \left\{ \begin{array}{ll} 0, & j = i; \\ 1, & j \neq i \end{array} \right.$.

Claim 1.15. $(r_{i_k} \cdots r_{i_1}(\nu))_{i_{k+1}} \neq 0$.

Proof. Note that

$$(r_{i_k} \cdots r_{i_1}(\nu))_{i_{k+1}} = (r_{i_k} \cdots r_{i_1}(\nu)) \cdot \epsilon_{i_{k+1}} = \nu \cdot \left(s_{i_1} \cdots s_{i_k}(\epsilon_{i_{k+1}})\right).$$

In this is 0, $s_{i_1} \cdots s_{i_k}(\epsilon_{i_{k+1}}) \in \Delta^{re}$ must be $\pm\epsilon_j$; then $\pm\epsilon_{i_{k+1}} = s_{i_k} \cdots s_{i_1}\epsilon_j$, which contradicts the minimality of m. □

In this way, we can apply $F_{i_{k+1}} F_{i_k} \cdots F_{i_1}(S_j)$. We get that there exists a unique simple in Π_Q^λ of dimension α, where $\lambda = r_{i_1} \cdots r_{i_m}\nu$. The result follows because we have noted that any indecomposable of $\mathbf{k}Q$ extends to a simple one of Π_Q^λ. □

The main tool using here was reflection functors. This gives equivalences of abelian categories:

$$\Pi_Q^\lambda - mod \xrightarrow{\sim} \Pi_Q^{\lambda'} - mod$$

For non commutative geometry, the notion of Morita equivalence of rings replaces isomorphism of commutative rings. In fact:

Proposition 1.16. *Let A, B be commutative rings such that $A \simeq_{Morita} B$. Then, $A \cong B$.*

Proof. First, show that $Z(A) \cong \mathrm{End}(Id_{A-mod})$. □

2. Proof of Kac's Theorem - Part II

We have to show that $\dim(\mathrm{Indec}_\alpha Q/\sim) > 0$ if $\alpha \in \Delta_+^{im}$.

We know that if $\alpha \in F$, then $(\alpha, \epsilon_i) \leq 0$ for all i.

If $q(\alpha) = 0$, then $\alpha \in \mathbb{N}^{Q_0}$ is in the kernel of the Cartan form on suppα, so α is a multiple of δ, where δ corresponds to the extended Dynkin diagram on suppα.

To prove this, note that if A is the symmetric adjacency matrix, the largest eigenvalue is achieved (by Frobenius-Perron Theorem) to a unique eigenvector $\alpha \in \mathbb{N}^{Q_0}$, up to scaling. If it is in $\ker(2\mathrm{Id} - A)$, the largest eigenvalue of A is 2, it corresponds to an Extended Dynkin diagram. Such eigenvector has no zero coordinates, so the previous case corresponds to α a multiple of δ, when we restrict the graph to suppα, which is an extended Dynkin one.

For the case $\alpha = \delta$ on the extended Dynkin diagram corresponding to suppα, we showed that there exists a one-dimensional space of non isomorphic bricks.

We will show it for the case $q(\alpha) < 0$, based on the following Lemmas:

Lemma 2.1 (Kac). *If $\alpha \in F$ is such that $q(\alpha) < 0$, then $\mathrm{Rep}_\alpha Q$ has a generic orbit which corresponds to a brick.*

Lemma 2.2 (Kac). *$\dim(\mathrm{Indec}_\alpha Q//GL_\alpha)$ is independent of the orientation of Q.*

Lemma 2.1 had a nice proof, we will outline it. Lemma 2.2 had a complicated proof involving counting over $\mathbb{F}_q$. We will give Crawley-Boevey's proof for α indivisible or satisfying $q(\alpha) < 0$.

Using Lemma 2.2, we can apply reflection functors and change of orientations to reduce $\alpha \in \Delta_+^{im}$ to $\alpha \in F$. By Lemma 2.1,

$$\begin{aligned}\dim(\mathrm{Indec}_\alpha Q//GL_\alpha) \geq & \dim(\text{generic union of bricks})//GL_\alpha \\ & = \dim\mathrm{Rep}_\alpha Q - (\dim GL_\alpha - 1) = 1 - q(\alpha) \geq 1.\end{aligned}$$

I don't understand this inequality, I think I forgot some of your explanations. Today there are some 'holes' in my notes because I was slow to take notes, I'm sorry

So we conclude the proof of part II of Kac's Theorem, except that it only leaves case $\alpha = l\delta$ for $l \geq 1$, and δ as before.

2.1. Proof of Lemma 2.1.

Lemma 2.3. *Let $\alpha = \alpha^{(1)} + \ldots + \alpha^{(n)}$ be the generic decomposition of α. Then,*

$$q(\alpha) \geq q(\alpha^{(1)}) + \ldots + q(\alpha^{(n)}).$$

Proof. Consider the generic representation $V = V^{(1)} \oplus \ldots \oplus V^{(n)}$, where the $V^{(i)}$'s are indecomposable, $\underline{\dim}\, V^{(i)} = \alpha_i$. We saw that $\mathrm{Ext}^1(V^{(i)}, V^{(j)}) = 0$ generically (otherwise, there exists an orbit whose closure contains this, so it would also have to be generic). Using this we have

$$\begin{aligned} q(\alpha) &= \dim\mathrm{End}(V) - \dim\mathrm{Ext}^1(V,V) \\ &= \sum_i \left(\dim\mathrm{End}(V^{(i)}) - \dim\mathrm{Ext}^1(V^{(i)}, V^{(i)})\right) + \sum_{i,j} \dim\mathrm{Hom}(V^{(i)}, V^{(j)}) \\ &\geq \sum_i q(\alpha_i). \end{aligned}$$

$\square$

We will explain first why a generic representation is indecomposable. Then we will explain that this implies that a generic representation is always a brick.

In order to prove that a generic decomposition is indecomposable, it is enough to prove:

Lemma 2.4. *If $\alpha \in F$, $\alpha = \alpha^{(1)} + \ldots + \alpha^{(n)}$, $\alpha^{(i)} \in \mathbb{N}^{Q_0} \setminus 0$, then $\operatorname{supp}\alpha$ corresponds to an extended Dynkin diagram, and each $\alpha^{(i)}$ is a multiple of δ.*

Proof. In fact, given $\alpha^{(i)} \in \mathbb{N}^{Q_0} \setminus 0$, restrict to $\operatorname{supp}\alpha$.

Define a new bilinear form determined by $\widetilde{(\epsilon_i, \epsilon_j)} := \alpha_i\alpha_j(\epsilon_i, \epsilon_j)$. If we consider $f(\beta) = \sum_i \frac{\beta_i}{\alpha_i}\epsilon_i$, then

$$f(\alpha) = \mathbf{1} = (1, ..., 1), \quad (\beta, \gamma) = \widetilde{(f(\beta), f(\gamma))}.$$

Claim 2.5. *Let β, γ be such that $\mathbf{1} = \beta + \gamma$ and $q(\mathbf{1}) \geq q(\beta) + q(\gamma)$. Then, $\mathbf{1}, \beta, \gamma$ are proportional and $q(\mathbf{1}) = 0$.*

Note that if we prove the claim, proof of Lemma is completed. Also, the condition about q is equivalent to $\widetilde{(\beta, \gamma)} \geq 0$. Now,

$$\begin{aligned} 0 \leq \widetilde{(\beta, \gamma)} &= \sum_{i,j} \beta_i\gamma_j \widetilde{(\epsilon_i, \epsilon_j)} \\ &= \sum_i \widetilde{(\beta_i\epsilon_i, \gamma_i\epsilon_j)} + \frac{1}{2}\sum_{i,j}(\beta_i - \beta_j)(\gamma_j - \gamma_i)\widetilde{(\epsilon_i, \epsilon_j)}. \end{aligned} \tag{2.6}$$

As $\widetilde{(\mathbf{1}, \epsilon_j)} \leq 0$, the first summand of (2.6) is ≤ 0. Also, $\beta_i = 1 - \gamma_i$, so $\gamma_j - \gamma_i = \beta_i - \beta_j$ and also the second summand is ≤ 0. In this way, both summands in (2.6) are 0, and $\widetilde{(\beta, \gamma)} = 0$. Therefore $\beta_i = \beta_j$ when i, j are adjacent vertices, so there exist $b, c > 0$ such that $\beta = b\mathbf{1}, \gamma = c\mathbf{1}$, $b + c = 1$. To finish,

$$q(\mathbf{1}) = q(\beta) + q(\gamma) = (b^2 + c^2)q(\mathbf{1}) \implies bc\, q(\mathbf{1}) = 0 \implies q(\mathbf{1}) = 0.$$

□

Definition 2.7. Let $V \in \operatorname{Rep}_\alpha Q$. We say that V is *stably indecomposable* if there exists a neighborhood $\mathcal{U}$ of V in $Rep_\alpha Q$ such that $\mathcal{U} \subseteq \operatorname{Indec}_\alpha Q$.

Theorem 2.8. *To be a brick is equivalent to be stably indecomposable; i.e. if $\alpha = \alpha^{(1)} + \ldots + \alpha^{(n)}$ is the generic decomposition, with generic $V = V^{(1)} \oplus \ldots \oplus V^{(n)}$, then each $V^{(i)}$ is a brick.*

I don't understand the second conclusion of that Theorem

Note 2.9. We saw that $\operatorname{Ext}^1(V^{(i)}, V^{(j)}) = 0$. The new data is $\operatorname{End}(V^{(i)}) = \mathbf{k}$.

Proof. If V is a brick, it must be stably indecomposable since $\dim\operatorname{End}(W) : \operatorname{Rep}_\alpha \to \mathbb{Z}$ is upper semicontinuous. Dimension can only jump, not decrease, and the generic part has minimal dimension.

On the other direction, let V be indecomposable, not a brick: there exists $g_0 \in \operatorname{Aut}(V), g_0 \notin \mathbf{k}^\times$. Set

$$S := \{g \in GL_\alpha : \dim\operatorname{Rep}_\alpha^g = \dim\operatorname{Rep}_\alpha^g = \dim\operatorname{Rep}_\alpha^{g_0}\}.$$

Also, for $\mathcal{U}$ an open neighborhood of V, let $E := \cup_{W \in \mathcal{U}}\operatorname{End}W$.

We want to show that U contains a decomposable. By Fitting's Lemma, it is enough to show that E contains a non zero semisimple element.

Claim 2.10. (i) *$E \cap S \subseteq S$ is open, dense.*
(ii) *Semisimple elements are dense in S.*

Proof. (i) Consider $L = \{(V, g) : g(V) \subseteq V\} \subseteq \operatorname{Rep}_\alpha \times S$ the vector bundle of rank $\dim\operatorname{Rep}_\alpha^{g_0}$ over S. Now, $S \cap E$ is the image by the canonical $\pi : \operatorname{Rep}_\alpha \times S \to S$ of an open inside L containing (V, g_0), and π is open. That open set inside L is $L \cup (S \times \mathcal{U})$.

(ii) Typically COMPLETAR!!!
Generically speaking we have:

Definition 2.11. Let $G \subseteq GL(W)$ be a reductive group. A *sheet* $\mathcal{S}_d$, $d \geq 1$, is a component of the set $\{g \in G : \dim W^g = d\}$.

Lemma 2.12. *Semisimple elements are dense in* $\mathcal{S}_d$

Note that S is a union of sheets so the claim follows. □

With this claim we complete the proof of Lemma 2.1. □

2.2. **Proof of Lemma 2.2.** In what follows, we will denote ${}_{GL_\alpha}Y := Y//GL_\alpha$.

Remember that we shall prove cases α indivisible or $q(\alpha) < 0$, and we leave case $q(\alpha) = 0$, α divisible.

First we prove it for α indivisible.

Lemma 2.13 (Crawley-Boewey). *Consider*

$$\begin{tikzcd} \mathrm{Rep}_\alpha \Pi_Q^\lambda \arrow[r, hook] \arrow[d, "\pi"] & \mathrm{Rep}_\alpha \overline{Q} = T^*\mathrm{Rep}_\alpha Q \arrow[dl] \\ Rep_\alpha Q. & \end{tikzcd}$$

Let $Y \subseteq \operatorname{im}\pi$ *be* GL_α *stable. Then,*

$$\dim_{GL_\alpha} Y = \dim(\pi^{-1}Y) - \dim \mathrm{Rep}_\alpha Q. \tag{2.14}$$

Proof. When $\lambda = 0$, note that:

$$\pi^{-1}Y = \bigsqcup_{\mathcal{O} \subseteq Y\ GL_\alpha-orbit} \text{conormal bundle to } \mathcal{O} \text{ in } T^*\mathrm{Rep}_\alpha Q$$

$$\therefore \quad \dim(\pi^{-1}Y) = \dim_{GL_\alpha} Y + \dim \mathrm{Rep}_\alpha Q.$$

For a general λ, $\pi^{-1}(\mathcal{O})$ satisfies:

- it is either empty (when the representations do not lift to Π_Q^λ-representations), or
- $\dim(\pi^{-1}\mathcal{O}) = \dim\mathcal{O} + \dim(fibers) = \dim\mathcal{O} + \dim \mathrm{Ext}^1(V, V)$ (two representations of Π_Q^λ in the same fiber of π differ by representations of Π_Q^0 in this fiber).

□

We can complete the proof for α indivisible, under the hypothesis that $\mathbf{k}$ is algebraically closed of characteristic 0. In such case, there exists λ such that $\lambda \cdot \alpha = 0$, and $\lambda \cdot \beta \neq 0$ for any $\beta < \alpha$, $\beta \in \Delta_+$. Now $V \in \mathrm{Rep}_\alpha Q$ extends to $\mathrm{Rep}_\alpha \Pi_Q^\lambda$ if and only if V is indecomposable.

If we consider $Y = \mathrm{Indec}_\alpha Q$, equation (2.14) says that

$$\dim_{GL_\alpha} \mathrm{Indec}_\alpha Q = \dim(\mathrm{Rep}_\alpha \Pi_Q^\lambda) - \dim \mathrm{Rep}_\alpha Q,$$

and the right hand side is independent of the orientation of Q.

Now let $\alpha \in \Delta_+$ arbitrary. Pick λ such that $\lambda \cdot \alpha = 0$, and $\lambda \cdot \beta \neq 0$ for any $\beta < \alpha$, $\beta \in \Delta_+ \setminus \mathbb{Q}\alpha$. Now $V \in \mathrm{Rep}_\alpha Q$ extends to $\mathrm{Rep}_\alpha \Pi_Q^\lambda$ if and only if all summands of V have as dimension an element of $\mathbb{Q}\alpha$.

Call $E_\alpha Q$ the set of elements $V \in \mathrm{Rep}_\alpha Q$ such that $\dim V \in \mathbb{Q}\alpha$:

$$\dim_{GL_\alpha} E_\alpha Q = \dim(\mathrm{Rep}_\alpha \Pi_Q^\lambda) - \dim \mathrm{Rep}_\alpha Q,$$

so it is independent of the orientation of Q.

Now suppose that $q(\alpha) < 0$, $\alpha \in \Delta_+^{im}$. Therefore $\alpha = s_{i_m} \cdots s_{i_1} \alpha'$ for some $\alpha' \in F$ and $1 \le i_j \le n$. Using the independence of orientations, the reflections functors and Lemma 2.1,

$$\begin{aligned} \dim_{GL_\alpha} E_\alpha Q &= {}_{GL_{\alpha'}} E_{\alpha'} Q' = \dim_{GL_{\alpha'}} Brick_{\alpha'} Q' = \dim \mathrm{Rep}_{\alpha'} Q' - (\dim GL_{\alpha'} - 1) \\ &= 1 - q(\alpha') = 1 - q(\alpha) > 1 \end{aligned}$$

(note also that $Brick_{\alpha'}Q'$ is dense in $\mathrm{Rep}_{\alpha'}Q'$, so also in $E_{\alpha'}Q'$).

To conclude, it suffices to show that $\mathrm{Indec}_\alpha Q \subseteq E_\alpha Q$ is dense (or $\mathrm{Indec}_\alpha Q//GL_\alpha \subsetneq E_\alpha Q//GL_\alpha$ is dense). Note that we have a surjective map

$$\bigcup_{\beta \in \mathbb{Q}\alpha,\, 0<\beta<\alpha} E_\beta \times E_{\alpha-\beta} \twoheadrightarrow E_\alpha \setminus \mathrm{Indec}_\alpha .$$

So it suffices to show that

$$\dim_{GL_\alpha} E_\beta + \dim_{GL_\alpha} E_{\alpha-\beta} < \dim_{GL_\alpha} E_\alpha. \tag{2.15}$$

This would imply that $\dim_{GL_\alpha} \mathrm{Indec}_\alpha Q = \dim_{GL_\alpha} E_\alpha Q = 1 - q(\alpha) > 1$, which is independent of the orientation.

To to this, consider $\gamma \in \mathbb{N}^{Q_0}$, $m, n \ge 1$ such that

$$\alpha - \beta = n\gamma, \quad \beta = m\gamma \qquad \Longrightarrow \qquad \alpha = (m+n)\gamma$$

(for example, choose γ indivisible). Therefore (2.15) simply says that

$$(1 - m^2 q(\gamma)) + (1 - n^2 q(\gamma)) < (1 - (m+n)^2 q(\gamma))$$

i.e. $1 < -2mn\, q(\gamma)$, which is true by hypothesis of α ($q(\gamma) = (m+n)^{-2} q(\alpha)$ is an integer).

Note 2.16. For $\alpha \in \Delta_+$ we have

- $q(\alpha) = 1$: there exists only one indecomposable of dimension α up to isomorphism;
- $q(\alpha) = 0$ α indivisible: 1-dimensional space of indecomposables of dimension α;
- $q(\alpha) = 1$: $(1 - q(\alpha)$-dimensional space of indecomposables of dimension α.

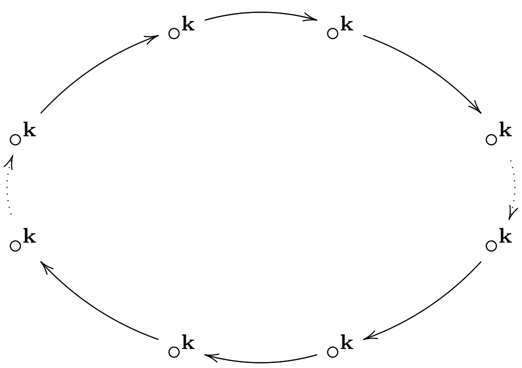
Remark 2.17. Δ defined as before coincides with the roots corresponding to the Kac-Moody Lie algebra.

Remark 2.18. In the cases we proved, the generic orbit was indecomposable (a brick.

In case $\alpha = l\delta$ with $l > 1$, the generic decomposition is $\alpha = \delta + \ldots + \delta$, so $\mathrm{Indec}_{l\delta} Q$ is peripheral in $\mathrm{Rep}_{l\delta} Q$. Showing $\dim({}_{GL_\alpha} \mathrm{Indec}_\alpha Q) = 1$ (or > 1) is quite different geometrically.

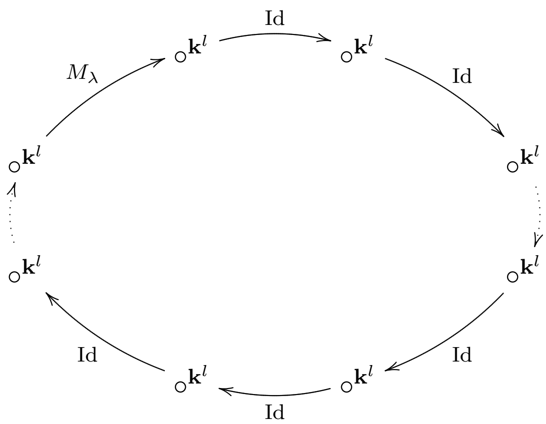
For Q an extended Dynkin diagram and $\alpha = l\delta$, one can find the indecomposables explicitly.

Example 2.19. Consider $\tilde{A}_n$ and the indecomposables of dimension $\delta = (1, ..., 1)$. Generically all the arrows are isomorphisms. Consider the cycle



obtained by taking each arrow or its inverse. This gives an element $\lambda \in \mathbf{k}^\times$ in some arrow, and by a change of basis in each vertex, any representation is of the form λ in some arrow, and the identity on the others. Conversely, if $\lambda \neq lambda'$, the corresponding representations are not isomorphic. This gets $\mathbf{k}^\times \hookrightarrow \mathrm{Indec}_\delta\, Q/GL_\delta$.

Now we can define for each $\lambda \in \mathbf{k}^\times$ and each $l \geq 1$ an indecomposable $V_\lambda^{(l)}$, where $\underline{\dim}\ V_\lambda^{(l)} = l\delta$, by "Jordan block" extension of the case in dimension 1. For this, consider bases $(e_{i,j})_{j=1,\ldots,l}$ of V_i, for each $i \in Q_0$: each arrow will be an isomorphism. We can consider:



for $M_\lambda = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$, the corresponding indecomposable Jordan matrix associated to λ.

Exercise 2.20. Consider $\tilde{D}_4$, and the typical indecomposable module of dimension δ

$$\begin{tikzcd}
\langle(1,0)\rangle \arrow[dr, hook] & & \langle(0,1)\rangle \arrow[dl, hook] \\
& \mathbf{k}^2 & \\
\langle(1,\lambda)\rangle \arrow[ur, hook] & & \langle(1,1)\rangle \arrow[ul, hook]
\end{tikzcd}$$

for $\lambda \neq 0, 1$. Try to cook up an indecomposable extension of dimension $\alpha = l\delta$, for all $l \geq 1$. These should be non isomorphic for distinct λ's.

3. More on Morita equivalence

As we said above, Morita equivalence is the non commutative replacement for isomorphism of rings in the commutative case.

Proposition 3.1. *Let A be an associative algebra. Then,* $\mathrm{End}(\mathrm{Id}_{A-Mod}) = Z(A)$.

From this result it is immediate that if B, C are commutative, $B \simeq_{Morita} C$, then $B \cong C$.

Proof. We show first that $\mathrm{End}(\mathrm{Id}_{A-Mod}) \supseteq Z(A)$. For each $z \in Z(A)$ define $\psi_M \in \mathrm{End}_A(M)$ for each A-module M by $\psi_M(m) = z \cdot m$. For each $\varphi : M \to N$ morphism of A-modules,

$$\begin{tikzcd} M \arrow[r, "\psi_M"] \arrow[d, "\varphi"] & M \arrow[d] \\ N \arrow[r, "\psi_N"] & N^{\varphi} \end{tikzcd} \tag{3.2}$$

commutes, so $(\psi_M)_{M \in A-Mod} \in \mathrm{End}(\mathrm{Id}_{A-Mod})$.

On the other hand, given $(\psi_M)_{M \in A-Mod} \in \mathrm{End}(\mathrm{Id}_{A-Mod})$, take $M = A$: $\mathrm{End}_A(A) = A^{op}$, so ψ_A is the right multiplication by some $z \in A$. Also, for each $N \in A - Mod$, we consider for each $n \in N$ $\varphi_n : A \to N$ defined by $1 \mapsto n$. Considering (3.2) for $M = A$ and this morphism:

$$\psi_N(n) = \psi_N \circ \varphi_n(1) = \varphi_n \circ \psi_A(1) = a \cdot n.$$

Therefore the family $(\psi_M)_{M \in A-Mod}$ is given as above, by left multiplication by z. From the fact that ψ_A commutes with each φ_a, $a \in A$, it follows $z \in Z(A)$.

So we have $\mathrm{End}(\mathrm{Id}_{A-Mod}) \subseteq Z(A)$, and we end the proof. □

3.1. **Examples.**

(1) $\mathrm{Rep}\Pi_Q^{\lambda} \simeq_{Morita} \Pi_Q^{r_i \lambda}$ using the F_i (reflection functor), when i is loop free, $\lambda_i \neq 0$.

(2) For each A associative $\mathbf{k}$-algebra, $Mat_{\mathbf{k}} \simeq_{Morita} A$:

$$\mathbf{k}^n \otimes_A V \longleftarrow\!\shortmid V$$

$$W \longmapsto (1, 0, ..., 0) \otimes_{Mat_n A} W.$$

(3) **Brower Groups:** For each field $\mathbf{k}$ it is defined as

$$Br(\mathbf{k}) := \left\{ \begin{array}{c} \mathbf{k}\text{-algebras such that} \\ A \times_{\mathbf{k}} \overline{\mathbf{k}} \cong Mat_n(\overline{\mathbf{k}}),\ \text{some } n \end{array} \right\} /\text{Morita equivalence}.$$

The group structure is $\otimes_{\mathbf{k}}$; all the $A \times_{\mathbf{k}} \overline{\mathbf{k}}$ are representable by division algebras over $\mathbf{k}$.

If $\mathbf{k} = \mathbb{R}$, $Br(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\} = \mathbb{Z}/2$:

$$\mathbb{H} \cong \mathbb{H}^{op} \quad \Longrightarrow \quad \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{op} \cong \mathrm{End}_{\mathbb{R}}(\mathbb{H}) \cong Mat_4(\mathbb{R}).$$

If $\mathbf{k}$ is a local archimedian field $\mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$: $Br(\mathbf{k}) = \mathbb{Q}/\mathbb{Z}$.

In general, if $\mathbf{k}^{sep}$ denotes the maximal separable extension of $\mathbf{k}$,

$$Br(\mathbf{k}) = H^2(Gal(\mathbf{k}^{sep}/\mathbf{k}), (\mathbf{k}^{sep})^*).$$

(4) Let G be a finite group:

$$\mathbb{C}[G] \cong \oplus_{\rho irrep} Mat_{\dim \rho}(\mathbb{C}) \simeq_{Morita} \mathbb{C}^{\#\text{ irreps of } G}.$$

We can prove the last equivalence. One construction is the following. Let e_i be a full collection of inequivalent primitive idempotents; i.e. $\mathbb{C}[G]e_i \cong \rho_i$ for all i: one e_i for each irreducible representation.

Get $\mathbb{C}[G]e_i\mathbb{C}[G] = \mathbb{C}[G]e_i \otimes_{\mathbb{C}[G]} \mathbb{C}[G] = \mathrm{End}_{\mathbb{C}}(\rho_i)$ as a factor of $\mathbb{C}[G]$ (in general, given V, W representations of G,

$$W \otimes_{\mathbb{C}[G]} V = (W \otimes V)^G \cong \mathrm{Hom}_G(\mathbb{C}, W \otimes V) \cong \mathrm{Hom}_G(V^*, W)\,).$$

We have the following general result:

Proposition 3.3. *Let A be an associative algebra and $e \in A$ an idempotent such that $AeA = A$. Then,*

$$\begin{array}{ccc} A & \cong_{Morita} & eAe \\ V \longmapsto & & eA \otimes_A V \\ Ae \otimes_{eAe} W & \longleftarrow & VW \end{array}$$

Proof. (sketch) Verify that

$$\begin{aligned} Ae \otimes_{eAe} eA \otimes_A V &= AeA \otimes_A V = A \otimes_A V = V, \\ eA \otimes_A Ae \otimes_{eAe} W &= eAe \otimes_{eAe} W = W, \end{aligned}$$

so the two functors are (quasi) inverse to each other. □

In our case, consider $e = \sum_i e_i$, which satisfies $\mathbb{C}[G]e\mathbb{C}[G] = \mathbb{C}[G]$. Also,

$$e\mathbb{C}[G]e = e\mathbb{C}[G] \otimes \mathbb{C}[G]e$$

. As $1 = \sum f_i$ primitive idempotents, $\dim \rho_i$ of them for each i, we have

$$\oplus e\mathbb{C}[G]f_i \otimes f_j\mathbb{C}[G]e = \oplus e_i\mathbb{C}[G]e_i = \mathbb{C}^{\# irreps\, of\, G}.$$

In this way, the equivalence is obtained

$$\begin{array}{c} \mathbb{C}[G] - mod \longrightarrow \mathbb{C}^{\# irreps\, of\, G} \\ V \longmapsto e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} V. \end{array}$$

Note 3.4. We can also use e for algebras over $\mathbb{C}[G]$.

Do you want to add some division here? e.g. section, subsection...

Suppose G is a finite group acting over a vector space V: $V//G$ is singular. There is a notion of stack quotient $[V/G]$ (also for algebraic groups), which is smooth.

By our philosophy, the geometry of X is captured by coherent (quasicoherent) sheaves on X (modules over B when $X = \operatorname{Spec} B$).

Rather than consider $Coh(V//G)$-modules over $\mathbb{C}[V]^G$, it is much better to consider G-equivariant $\mathbb{C}[V]$-modules: such modules M with an action of G verifying $g \cdot (v \cdot m) = (g * v)(g \cdot m)$.

We can define $A := \mathbb{C}[V]\#G$. As a vector space, $A = \mathbb{C}[V] \otimes \mathbb{C}[G]$, and the multiplication is defined by

$$g \cdot v = (g * v)g.$$

(we use here the identifications $\mathbb{C}[V] \hookrightarrow \mathbb{C}[V] \otimes 1$, $\mathbb{C}[G] \hookrightarrow 1 \otimes \mathbb{C}[G]$).

Then the G-equivariant $\mathbb{C}[V]$-modules correspond with modules over A.

Definition 3.5. The set of G-equivariant modules over $\mathbb{C}[V]$ is $Coh([V/G]_{Stack})$, the set of A modules, which is smooth as an stack.

($Coh([X/G]_{Stack})$ = set of equivariant coherent sheaves on X)

There is a formalism of algebraic stacks: gluing together X/G (X a sheaf, G an algebraic group), sheaves, smoothness, etc.

For our purposes, G finite, the fact that A is smooth says: A has a finite A-bimodule resolution (to get it, take a Koszul resolution of $\mathbb{C}[V]$ and tensor by $\mathbb{C}[G]$). The length of the resolution is equal to the dimension of V, which says that A is smooth of such dimension.

We have that:

- points of $V//G$ correspond to the $SpecMax(\mathbb{C}[V]^G)$: G-orbits of V, singular at 0;

- (certain) G-equivariant modules over $\mathbb{C}[V]$ correspond to points of Spec A (simple A-modules); we have π : G-equivariant modules over $\mathbb{C}[V] \twoheadrightarrow V//G$, and π^{-1} includes the simple G-modules M with trivial action of V.

I don't understand this part

4. McKay correspondance

Consider $V = \mathbb{C}^2$, G a subgroup of $SL_2\mathbb{C}$, and " Spec A" $\twoheadrightarrow \mathbb{C}^2/G$, $A = \mathbb{C}[x,y]\#G$. We also have $e \in \mathbb{C}[G]$, $e\mathbb{C}[G]e = \mathbb{C}^{\#irreps}$ and $\mathbb{C}[G]e\mathbb{C}[G] = \mathbb{C}[G]$. Therefore $AeA = A$, and by the previous considerations, $A \simeq_{Morita} eAe$, which is an algebra over $\mathbb{C}^{\#irreps}$.

Proposition 4.1. *$eAe \cong \Pi_Q^0$, where Q is an extended Dynkin diagram such that its double $\bar{Q}$ satisfies:*

(1) *its vertices are labeled by the irreducible representations,*
(2) *given $i, j \in Q_0$, the number of arrows $i \to j$ is equal to*

$$\dim \mathrm{Hom}_G(\rho_i, V \otimes \rho_j) = \#\ \textit{copies of } \rho_i \textit{ in } V \otimes \rho_j.$$

4.1. Classification of finite subgroups of $SL_2\mathbb{C}$.

We have that $SU_2\mathbb{C}$ is a maximal component of $SL_2\mathbb{C}$, and any other component is conjugate to this. Therefore, if G is a finite subgroup of $SL_2\mathbb{C}$, then G is contained in a maximal component and is in consequence conjugate to a finite subgroup of $SU_2\mathbb{C}$:

$$SU_2\mathbb{C} = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a| + |b| = 1 \right\} \cong S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4.$$

We define $\pi : SU(2) = S^3 \twoheadrightarrow SO_3\mathbb{R}$ as follows. Set $S^3 = B^3/\partial B^3$, where $B^3 \subseteq \mathbb{R}^3$ is the unit ball. Given $x \in B^3$, map it to the rotation in $SO_3\mathbb{R}$ orbit about $\overrightarrow{OX}$ at angle $2\pi\|x\|$. this gets $B^3/\partial B^3 \twoheadrightarrow SO_3\mathbb{R}$, in such way that I don't understand my notes here and can't deduce them, I'm sorry.

Classification: All the finite groups of $SO_2\mathbb{C} \subseteq SL_2\mathbb{C}$ are of the form $\pi^{-1}(G)$, where G is a finite subgroup of $SO_3\mathbb{R}$, except the odd cycles $\left\{ \begin{pmatrix} \phi^a & 0 \\ 0 & \phi^{-a} \end{pmatrix} \right\}_{a=0,1,\dots,m-1}$, $\phi = e^{2\pi i/m}$, m odd.

Finite subgroups of $SO_3\mathbb{R}$ are:

- cyclic groups,
- dihedral groups,
- rotational groups of symmetries of Platonic solids (three of these: tetrahedral, cube-octahedral, icosahedral-dodecahedral).

This gets that the finite subgroups of $SL_2\mathbb{C}$ are

- cyclic groups $(\mathbb{Z}/m)$,
- $\pi^{-1}(D_{2m}) = \tilde{D}_{2m}$,
- $\tilde{A}_4, \tilde{S}_4, \tilde{A}_5$ Another part where I can't understand my notes.

Example 4.2. For the cyclic group $\mathbb{Z}/m = \left\{ \begin{pmatrix} \phi^a & 0 \\ 0 & \phi^{-a} \end{pmatrix} \right\}_{a=0,1,\dots,m-1}$, $\phi = e^{2\pi i/m}$, the irreducible representations are ρ_i of dimension 1, $i \in \mathbb{Z}/m$, which satisfy:

$$V = \mathbb{C}^2 \cong \rho_1 \oplus \rho_{-1}, \qquad V \otimes \rho_i \cong \rho_{i-1} \oplus \rho_{i+1}.$$

The corresponding Mc Kay diagram is the double of $\widetilde{A_m}$:

$$
\begin{tikzcd}
& \circ^{\rho_0} \arrow[r, shift left] \arrow[dl, shift left] & \circ^{\rho_1} \arrow[l, shift left] \arrow[dr, shift left] & \\
\circ^{\rho_{m-1}} \arrow[ur, shift left] \arrow[d, dotted, shift left] & & & \circ^{\rho_2} \arrow[ul, shift left] \arrow[d, dotted, shift left] \\
\cdot \arrow[u, dotted, shift left] \arrow[dr, shift left] & & & \cdot \arrow[u, dotted, shift left] \arrow[dl, shift left] \\
& \circ^{\rho_{k+1}} \arrow[ul, shift left] \arrow[r, shift left] & \circ^{\rho_k} \arrow[l, shift left] \arrow[ur, shift left] &
\end{tikzcd}
.
$$

Example 4.3. Consider the binary dihedral group

$$\tilde{D}_{2n} = \left\langle \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \varphi = e^{2\pi i/m}.$$

Consider first one-dimensional representations. Note that $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}$ is conjugate of $\begin{pmatrix} \varphi^{-1} & 0 \\ 0 & \varphi \end{pmatrix}$ so under a character it acts by ± 1, and determines the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. As $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, we have two choices for the scalar under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts. This gets four one-dimensional representations between the irreducible representations of $\tilde{D}_{2n}$.

We also know that in $\tilde{D}_{2n}$ there are $m+3$ conjugacy classes, and the order of $\tilde{D}_{2n}$ is $4m = \sum_{i=1}^{m+3}(\dim\rho_i)^2$. This gets that all the other irreducible representations are 2-dimensional:

$$V_i : \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \varphi^i & 0 \\ 0 & \varphi^{-i} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & (-1)^{-1} \\ 1 & 0 \end{pmatrix}, \qquad i = 1, ..., m.$$

We get $V = V_1 \cong V_1^*$ and $V_1 \otimes V_1$ contains the trivial representation, V_2 and another 1-dimensional representation. Also, $V \otimes V_i \cong V_{i-1} \oplus V_{i+1}$, unless $i = m$. In consequence, the Mc Kay diagram for this group is the double of $\widetilde{D_{m+2}}$:

$$
\begin{tikzcd}
\circ^{\rho_1} \arrow[dr, shift left] & & & & & & \circ^{\rho_2} \arrow[dl, shift left] \\
& \circ^{V_1} \arrow[ul, shift left] \arrow[dl, shift left] \arrow[r, shift left] & \circ^{V_2} \arrow[l, shift left] \arrow[r, shift left] & \circ^{V_3} \arrow[l, shift left] \arrow[r, dotted, shift left] & \cdot \arrow[l, dotted, shift left] \arrow[r, shift left] & \circ^{V_m} \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & \\
\circ^{\rho_0} \arrow[ur, shift left] & & & & & & \circ^{\rho_3} \arrow[ul, shift left]
\end{tikzcd}
.
$$

Note that $\delta_i = (\dim\rho_i)$, where δ is the indivisible root in the kernel of the Cartan form.

Remark 4.4. Except for $\mathbb{Z}/m$, m odd, these diagrams are all bipartite: that is, there exists $S \subset Q_0$ such that each arrow has one extreme in S and the other in S^c (no arrows with both extremes in S, or both in S^c). Look at the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ($\mid G \mid$ is even if and only if $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in G$ since G does not contain any conjugate of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$): is must be ± 1, and tensoring by V switches this.

We deduce that

Proposition 4.5. *Let G be a finite subgroup of $SL_2\mathbb{C}$. The following statements are equivalent:*

- $|G|$ *is even;*
- $G = \pi^{-1}(G')$ *for some subgroup G' of $SO_3\mathbb{R}$;*
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in G$ *(note that* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \ker \pi$*;*
- *the Mc Kay graph of G is bipartite.*

Note 4.6. We will see right now that $\delta := (\mathrm{dim}\rho_i)$ is the indivisible in the kernel of the Cartan form. If the graph is bipartite, then $Q_0 = S \sqcup (Q_0 \setminus S)$. This means for the Mc Kay graph that $Irreps(G) = S \sqcup S^c$,

- $\rho \in S \Longrightarrow V \otimes \rho = \sum \rho_i$ for some $\rho_i \in S^c$,
- $\rho \in S^c \Longrightarrow V \otimes \rho = \sum \rho_j$ for some $\rho_j \in S$.

As δ is in the kernel of the Cartan form, $2\delta_i = \sum_{j\,adjacent\,i} \delta_j$, so we want to conclude $\sum \delta_i$ even, so $|G| = \sum \delta_i^2$ is also even.

Example 4.7. Consider $G = \widetilde{\mathbb{A}_4}$. The irreducible representations with $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto 1$ correspond to the irreducible representations of $\mathbb{A}_4$:

- three 1-dimensional representations $(123) \mapsto e^{2k\pi i/3}$, $(12)(34) \mapsto 1$;
- the 3-dimensional representation $W = \ker \begin{pmatrix} \mathbb{C}^4 \to \mathbb{C} \\ e_1 \mapsto 1 \end{pmatrix}$, the standard representation of $\mathbb{S}_4 \supset \mathbb{A}_4$.

We also have three 2-dimensional representations: $V = V^* = \mathbb{C}^2$, and $V \otimes V$ contains the trivial representation and W. Therefore the Mc Kay diagram is the double quiver of $\widetilde{E_6}$:

$$\begin{tikzcd}
 & & \circ \arrow[d, shift left] & & \\
 & & \circ \arrow[u, shift left] \arrow[d, shift left] & & \\
\circ^{\rho_0} \arrow[r, shift left] & \circ^{V} \arrow[l, shift left] \arrow[r, shift left] & \circ^{W} \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left]
\end{tikzcd}$$

Remark 4.8. We know that $\widetilde{\mathbb{S}_4}$ and $\widetilde{\mathbb{A}_5}$ have to map to E-type diagrams, since they have irreducible representations of dimension > 2, and only $\tilde{E}_n$ have some $\delta_i > 2$. It follows that

- the Mc Kay diagram of $\widetilde{\mathbb{A}_4}$ is $\tilde{E}_6$,
- the Mc Kay diagram of $\widetilde{\mathbb{S}_4}$ is $\tilde{E}_7$,
- the Mc Kay diagram of $\widetilde{\mathbb{A}_5}$ is $\tilde{E}_8$,

for example looking at $|G| = \sum_i \delta_i^2$.

Proposition 4.9. *Let G be a finite subgroup of $SL_2\mathbb{C}$. The Mc Kay graph of G is the double of an extended Dynkin diagram, and $\delta = (\mathrm{dim}\rho_i)$ ($\delta_i = 1$ for some i, namely ρ_i the trivial).*

Proof. Consider $\mathbb{C}G$ the regular representation of G, $\mathbb{C}G \otimes \rho_i \cong \mathbb{C}G^{\oplus \mathrm{dim}\rho_i}$ for all i (if e_j is a basis of ρ_i, then $\mathbb{C}G \otimes \rho_i = \oplus \mathbb{C}G(1 \otimes e_j)$).

Now, $\mathbb{C}G \otimes V \cong \mathbb{C}G \oplus \mathbb{C}G$. Write $\mathbb{C}G = \oplus \rho_i^{\oplus \mathrm{dim}\rho_i}$, and denote A the adjacency matrix of Mc Kay quiver. As $V \otimes \rho_i = \sum_{j \sim i} V_j$,

$$V \otimes (\oplus \rho_i^{\oplus \mathrm{dim}\rho_i}) \cong \oplus \rho_i^{\oplus 2\mathrm{dim}\rho_i} \quad \Longrightarrow \quad A(\mathrm{dim}\rho_i) = 2(\mathrm{dim}\rho_i).$$

Remark 4.10. We have a G-isomorphism $vol : V^* \to V$ (using $G < SL_2G$), so

$$\begin{aligned}\#(i \to j) &= \dim \mathrm{Hom}(\rho_j, V \otimes \rho_i) = \dim \mathrm{Hom}(V \otimes \rho_i, \rho_j)\\ &= \dim \mathrm{Hom}(\rho_i, \rho_j \otimes V^*) = \dim \mathrm{Hom}(\rho_i, \rho_j \otimes V)\\ &= \dim \mathrm{Hom}(\rho_i, V \otimes \rho_j) = \#(j \to i).\end{aligned}$$

This says that A is symmetric.

Remark 4.11. If G is finite and V is faithful, the Mc Kay diagram is strongly connected.

We get A a symmetric matrix, which entries are integers ≥ 0 and δ is an eigenvector of eigenvalue 2. We know that for some orthogonal matrix O, OAO^{-1} is diagonal, whit its eigenvalues in the diagonal (we can assume that 2 is the first). From Perron-Frobenius Theorem (which we will prove after this), we can conclude that all the eigenvalues λ of A verify $|\lambda| < 2$ if $\lambda \neq 2$, and that $\lambda = 2$ has multiplicity one (thus δ is the unique eigenvector of eigenvalue 2 up to scaling).

Note that $2\mathrm{Id} - A$ is a Cartan matrix. By definition, the extended Dynkin diagrams are those with Cartan positive-semidefinite matrix, not positive definite. □

Theorem 4.12 (Perron-Frobenius)**.** *Let A be an $I \times I$ matrix with non-negative entries, strongly connected (i.e. the graph obtained by $i \to j$-arrow added in whenever $a_{ij} \neq 0$ is strongly connected). Then there exists a unique eigenvector v up to scaling such that $v \in \mathbb{R}^I_+$ and that the corresponding eigenvalue λ is positive, of multiplicity one, and $\lambda > |\lambda'|$ for any other eigenvalue λ'.*

Proof. A acts on $R^I_{\geq 0} \setminus 0/\mathbb{R}_+$ continuously (it cannot kill anything because the entries are non-negative and the matrix is strongly connected). By Brouwer's Fixed Point Theorem, there exists $v \in R^I_{\geq 0}$ such that $Av = \lambda v$ for some $\lambda > 0$: by strong connectivity we conclude that $v \in R^I_+$.

We can change of basis by a diagonal matrix with positive entries in order to have $v = \mathbf{1} = (1, ..., 1)$. Define $\|w\| = \max |w_i|$, $w \in \mathbb{R}^I$.

Claim 4.13. *$\|A\| = \lambda$ (operator norm), achieved exactly at multiple of v*

Note that this claim ends the proof of the Theorem. Note that if $\|w\| = 1$ and w has all its entries non-negative, $\|Aw\| \leq \|A\mathbf{1}\|$ (it is derived from the strong connectivity). Also, for a general w, $\|Aw\| \leq \|Aw'\|$, where w' is defined by $w'_i = |w_i|$. Therefore $\|w\| = 1$ implies $\|Aw\| \leq \|A\mathbf{1}\|$ in general. Moreover, the inequality is strict if $w \neq \pm\mathbf{1}$.

Now if μ is an eigenvalue of A with eigenvector w, $\|A^n w\| = |\mu|^n \|w\|$, so $|\mu| \leq \lambda$. Also, if $|\mu| = \lambda$, then $\|A^n w\| = \lambda^n \|w\|$, so w is a multiple of $\mathbf{1}$. □

From the previous result we conclude that $\{G < SL_2\mathbb{C} \text{ finite}\} \longrightarrow$ extended Dynkin diagram.

We have also $\mathbb{C}[G] \simeq_{Morita} \mathbb{C}^{\#irreps}$,

$$M \mapsto e\mathbb{C}[G] \otimes M, \quad e = \sum_{i=1}^{\#irreps} e_i,\ \mathbb{C}[G]e_i \cong \rho_i.$$

Claim 4.14. *$e(\mathbb{C}[x,y]\#G)e \cong \Pi^0_Q$, where Q is any orientation of the extended Dynkin diagram such that $\bar{Q}$ is the corresponding Mc Kay diagram of G.*

We have $\mathbb{C}[G] \simeq \mathbb{C}\bar{Q}_0$, defining the functor

$$F:\ \mathbb{C}[G] - bimod \xrightarrow{\sim} \mathbb{C}\bar{Q}_0 - bimod$$

$$M \longmapsto e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e.$$

Note that:

$$\begin{aligned} F(M \otimes_{\mathbb{C}[G]} N &= e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} N \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e \\ &= e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} N \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e \\ &= (e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e) \otimes_{e\mathbb{C}[G]e} e\mathbb{C}[G](\otimes_{\mathbb{C}[G]} N \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e), \end{aligned}$$

which says that the functor is monoidal.

Also, $F(\mathbb{C}[G]) = \mathbb{C}\bar{Q}_0$, $F(V) = \mathbb{C}\bar{Q}_1$ and

$$F(\rho_i \otimes V \otimes \rho_j = \mathrm{Hom}(\rho_i, V \otimes \rho_j) = i\mathbb{C}\bar{Q}_1.$$

So we conclude that

$$\mathbb{C}[G]\#TV = T_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes V) \to T_{\mathbb{C}\bar{Q}_0}\mathbb{C}\bar{Q}_1 = \mathbb{C}\bar{Q} = \text{path algebra of } \bar{Q},$$

$$\omega_\lambda = \text{sympl. form of } V^* - \sum_{i=1}^{\#irreps} \lambda_i \mathrm{Id}_i \mapsto \sum_i [e_i, e_i^*] - \lambda$$

$$\therefore TV\#G/([x,y] - \lambda) \to \Pi_Q^\lambda.$$

To explain the previous Morita equivalence in Claim 4.14, we need some other considerations. We want to know if given $\mathbf{k}$-algebras A, B such that $A \cong_{Morita} B$, this implies $A^{op} \cong_{Morita} B^{op}$. This is true and follows from the next Theorem. Remember that a *projective generator* P for a $\mathbf{k}$-algebra A is a projective A-module such that for any other A module M there exists some index I and a surjective morphism $P^{\oplus I} \twoheadrightarrow M$.

Theorem 4.15 (Morita). *The following are equivalent:*

(1) $A - mod \simeq B - mod$;
(2) *for a projective generator* $P \in B - mod$, $\mathrm{End}_B P \cong A^{op}$;
(3) $P \otimes_A - : A - mod \to B - mod$ *is an equivalence.*

Proof. (Sketch)(1) $\Rightarrow$ (2). If $\phi : A - mod \to B - mod$ gives such equivalence, consider $P := \phi(A)$.

(2) $\Rightarrow$ (3). In this case, $P \otimes_A -$ and $\mathrm{Hom}_B(P, -)$ are (quasi) inverse functors. We can show that $M \cong \mathrm{Hom}_B(P, P \otimes_A M)$ for any $M \in A - mod$ in several steps: first for $M = A$, then for $M = A^{\oplus I}$ for any index I, and finally for all M using the exactness of this functor and the (trivial) existence of $A^{\oplus I} \twoheadrightarrow M$ for some I. Similarly we prove that $N \cong P \otimes_A \mathrm{Hom}_B(P, M)$ for any $N \in A - mod$.

(3) $\Rightarrow$ (1). This is immediate. $\square$

Corollary 4.16. *If* $A \cong_{Morita} B$, *then* $B^{op} \cong_{Morita} A^{op}$

Proof. Use (2) of previous Theorem and the fact that ${}_B P_A$ gives place to ${}_{A^{op}} P_{B^{op}}$. $\square$

Corollary 4.17. $A \cong_{Morita} B$ *if and only if there exists modules* ${}_B P_A, {}_A Q_B$ *such that* $Q \otimes_B P \cong A$ *and* $P \otimes_A Q \cong B$.

Corollary 4.18. *If* $A \cong_{Morita} B$, *then there exists a monoidal equivalence* $F : A - bimod \to B - bimod$

Proof. Note that $A - bimod = A^\rho - mod$, where $A^\rho := A \otimes_{\mathbf{k}} A^{op}$. If $A \cong_{Morita} B$, then by the existence of P, Q as in (2) of the Theorem,

$$B \otimes_{\mathbf{k}} B^{op} \cong (P \otimes Q) \otimes_{A \otimes_{\mathbf{k}} A^{op}} (Q \otimes P), \quad A \otimes_{\mathbf{k}} A^{op} \cong (Q \otimes P) \otimes_{B \otimes_{\mathbf{k}} B^{op}} (P \otimes Q).$$

For the monoidal property, observe that

$$\begin{aligned} F(M \otimes_A N) &= P \otimes_B M \otimes_A N \otimes_A Q = P \otimes_B M \otimes_A A \otimes_A N \otimes_A Q \\ &= P \otimes_B M \otimes_A (Q \otimes_B P) \otimes_A N \otimes_A Q \\ &= (P \otimes_B M \otimes_A Q) \otimes_B (P \otimes_A N \otimes_A Q) = F(M) \otimes_B F(N). \end{aligned}$$

$\square$

In our situation, $B = eAe$, where $e^2 = e$ and $AeA = A$. Consider ${}_BP_A := eA$, ${}_AQ_B := Ae$. Then
$$F(M) = eA \otimes_A M \otimes_A Ae = \text{"}eMe\text{"} \quad \Longrightarrow \quad eMe \otimes_{eAe} eNe = e(M \otimes_B N)e.$$

Back to case $\mathbb{C}[G]$, we have $e = \sum_{i=1}^n e_i$, where e_i are the idempotents, $\mathbb{C}[G]e_i$ are de different irreducible representations of G, and
$$\mathbb{C}[G]e\mathbb{C}[G] = \mathbb{C}[G] \qquad \left(\mathbb{C}[G] = \sum_i \mathrm{End}_{\mathbb{C}}(\rho_i),\ e_i \in \mathrm{End}_{\mathbb{C}}(\rho_i)\right).$$
As $\mathbb{C}[G] \simeq_{Morita} \mathbb{C}Q_0$, where $\bar{Q}$ is the McKay quiver of $G < SL_2\mathbb{C}$, we have
$$F : \mathbb{C}[G] - bimod \xrightarrow{\sim} \mathbb{C}Q_0 - bimod.$$
What we want is to view $\mathbb{C}[x,y]\#G = \mathbb{C}[\mathbb{C}^2]\#G$ as an algebra in $\mathbb{C}[G]$-bimod, its image under F will be Π_Q^0: $e\mathbb{C}[\mathbb{C}^2]\#Ge = \Pi_Q^0$, so we will obtain the desired Morita equivalence.
Note that $(\mathbb{C}^2)^* \subset \mathbb{C}[\mathbb{C}^2]$. Also, as bimodule, $(\mathbb{C}^2)^*\#\mathbb{C}[G]$ has the following G-actions:
$$g(v \otimes h) = g * v \otimes gh, \qquad (v \otimes h)g = v \otimes hg.$$
Any $\mathbb{C}[G]$-bimodule is a direct sum of irreducible bimodules, and $\mathbb{C}[G] - bimod = G \times G^{op} - mod$. In consequence, the irreducible bimodules are $\rho_i \boxtimes \rho_j^*$, so we compute their image under F.

Claim 4.19. *$F(\rho_i \boxtimes \rho_j^*)$ is the 1-dimensional vector space $M = \langle m \rangle$ with $\mathbb{C}Q_0$-bimodule structure $jm = m = mi$; i.e., $M = \ \circ^i \longrightarrow \circ^j$.*

Proof. Note that $\rho_i \boxtimes \rho_j^* = \mathbb{C}[G]e_i \otimes e_j\mathbb{C}[G]$ (use here that $e_j\mathbb{C}[G]$ is a $G^{op} \cong G$-module isomorphic to ρ_j^*), because
$$\begin{aligned} e_i\mathbb{C}[G]e_j &= e_i\mathbb{C}[G] \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e_j = \begin{cases} \mathbb{C}, & i = j; \\ 0, & i \neq j; \end{cases} \\ e\mathbb{C}[G]e_j &= \langle e_j \rangle, \qquad e_i\mathbb{C}[G]e = \langle e_j \rangle. \end{aligned}$$
Then, $F(\rho_i \boxtimes \rho_j^*) = \langle e_i \otimes e_j \rangle$, and $m = e_i \otimes e_j$ gives the desired property. $\square$

Claim 4.20. $(\mathbb{C}^2)^*\#\mathbb{C}[G] \xmapsto{F} \mathbb{C}\bar{Q}_1$.

Proof. Note that
$$\begin{aligned} F((\mathbb{C}^2)^* \otimes \mathbb{C}[G]) &= \oplus((\mathbb{C}^2)^* \otimes \rho_i) \boxtimes \rho_j \\ &= \oplus \rho_j \otimes \mathrm{Hom}_G(\rho_j, (\mathbb{C}^2)^* \otimes \rho_i) = \oplus \rho_j \otimes \mathbb{C}^{\#(i \to j)}, \end{aligned}$$
(I don't understand this at all) the last equality by definition of $\bar{Q}$. Then it follows from our result above. $\square$

Therefore $T_{\mathbb{C}[G]}(\mathbb{C}^2)^* \otimes \mathbb{C}[G] \xmapsto{F} T_{\mathbb{C}\bar{Q}_0}\mathbb{C}\bar{Q}_1 = \mathbb{C}\bar{Q}$, and we have
$$\begin{tikzcd} T_{\mathbb{C}[G]}(\mathbb{C}^2)^*\#G \arrow[r, mapsto, "F"] \arrow[d, "\varphi_1"] & \mathbb{C}\bar{Q} \arrow[d, "\varphi_2"] \\ \mathbb{C}[\mathbb{C}^2]\#G \arrow[r, mapsto, "?"] & \Pi_Q. \end{tikzcd}$$
It remains to show that $F(\ker \varphi_1) = \ker \varphi_2$.

QUIVERS IN REPRESENTATION THEORY (18.735, SPRING 2009) LECTURES 15 TO PRESENT

TRAVIS SCHEDLER, TYPED BY IVÁN ANGIONO

1. McKay correspondance

Consider $V = \mathbb{C}^2$, G a subgroup of $SL_2\mathbb{C}$, and " Spec A" $\twoheadrightarrow \mathbb{C}^2/G$, $A = \mathbb{C}[x, y]\#G$. We also have $e \in \mathbb{C}[G]$, $e\mathbb{C}[G]e = \mathbb{C}^{\#irreps}$ and $\mathbb{C}[G]e\mathbb{C}[G] = \mathbb{C}[G]$. Therefore $AeA = A$, and by the previous considerations, $A \simeq_{Morita} eAe$, which is an algebra over $\mathbb{C}^{\#irreps}$.

Proposition 1.1. *$eAe \cong \Pi_Q^0$, where Q is an extended Dynkin diagram such that its double $\bar{Q}$ satisfies:*

(1) *its vertices are labeled by the irreducible representations,*
(2) *given $i, j \in Q_0$, the number of arrows $i \to j$ is equal to*

$$\dim \mathrm{Hom}_G(\rho_i, V \otimes \rho_j) = \#\ \textit{copies of } \rho_i \textit{ in } V \otimes \rho_j.$$

1.1. **Classification of finite subgroups of $SL_2\mathbb{C}$.** We have that $SU_2\mathbb{C}$ is a maximal component of $SL_2\mathbb{C}$, and any other component is conjugate to this. Therefore, if G is a finite subgroup of $SL_2\mathbb{C}$, then G is contained in a maximal component and is in consequence conjugate to a finite subgroup of $SU_2\mathbb{C}$:

$$SU_2\mathbb{C} = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a| + |b| = 1 \right\} \cong S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4.$$

We define $\pi : SU(2) = S^3 \twoheadrightarrow SO_3\mathbb{R}$ as follows. Set $S^3 = B^3/\partial B^3$, where $B^3 \subseteq \mathbb{R}^3$ is the unit ball. Given $x \in B^3$, map it to the rotation in $SO_3\mathbb{R}$ about the axis $\overrightarrow{OX}$ at angle $2\pi\|x\|$. This gets $B^3/\partial B^3 \twoheadrightarrow SO_3\mathbb{R}$, which is two-to-one: for each point in the image, there are exactly two points, on the same axis, that map to it, one on each side of the origin. More precisely, x and $(|x| - 1) \cdot x$ map to the same point, and these are exactly the fibers of points of $SO_3\mathbb{R}$ as x varies.

Classification: All the finite groups of $SO_2\mathbb{C} \subseteq SL_2\mathbb{C}$ are of the form $\pi^{-1}(G)$, where G is a finite subgroup of $SO_3\mathbb{R}$, except the odd cycles $\left\{ \begin{pmatrix} \phi^a & 0 \\ 0 & \phi^{-a} \end{pmatrix} \right\}_{a=0,1,\dots,m-1}$, $\phi = e^{2\pi i/m}$, m odd.

Finite subgroups of $SO_3\mathbb{R}$ are:

- cyclic groups,
- dihedral groups,
- rotational groups of symmetries of Platonic solids (three of these: tetrahedral, cube-octahedral, icosahedral-dodecahedral), otherwise known as A_4, S_4, and A_5, respectively.

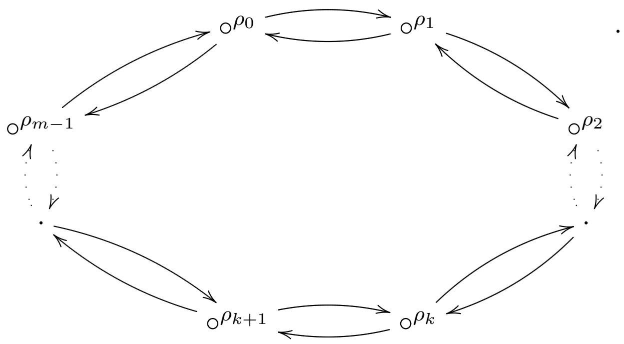
This gets that the finite subgroups of $SL_2\mathbb{C}$ are

- cyclic groups ($\mathbb{Z}/m$),
- $\pi^{-1}(D_{2m}) = \tilde{D}_{2m}$,
- $\tilde{A}_4, \tilde{S}_4, \tilde{A}_5$: these are the preimages of the rotational symmetry groups A_4, S_4, A_5 above under the two-to-one cover $SU_2\mathbb{C} \twoheadrightarrow SO_3\mathbb{R}$.

Example 1.2. For the cyclic group $\mathbb{Z}/m = \left\{ \begin{pmatrix} \phi^a & 0 \\ 0 & \phi^{-a} \end{pmatrix} \right\}_{a=0,1,\dots,m-1}$, $\phi = e^{2\pi i/m}$, the irreducible representations are ρ_i of dimension 1, $i \in \mathbb{Z}/m$, which satisfy:

$$V = \mathbb{C}^2 \cong \rho_1 \oplus \rho_{-1}, \qquad V \otimes \rho_i \cong \rho_{i-1} \oplus \rho_{i+1}.$$

The corresponding McKay diagram is the double of $\widetilde{A_m}$:



Example 1.3. Consider the binary dihedral group

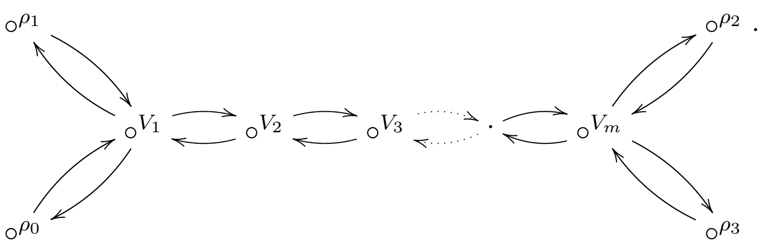
$$\tilde{D}_{2n} = \left\langle \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \varphi = e^{2\pi i/m}.$$

Consider first one-dimensional representations. Note that $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}$ is conjugate of $\begin{pmatrix} \varphi^{-1} & 0 \\ 0 & \varphi \end{pmatrix}$ so under a character it acts by ± 1, and determines the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. As $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, we have two choices for the scalar under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts. This gets four one-dimensional representations between the irreducible representations of $\tilde{D}_{2n}$.

We also know that in $\tilde{D}_{2n}$ there are $m+3$ conjugacy classes, and the order of $\tilde{D}_{2n}$ is $4m = \sum_{i=1}^{m+3} (\mathrm{dim}\rho_i)^2$. This gets that all the other irreducible representations are 2-dimensional:

$$V_i : \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \varphi^i & 0 \\ 0 & \varphi^{-i} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & (-1)^{-1} \\ 1 & 0 \end{pmatrix}, \qquad i = 1, \dots, m.$$

We get $V = V_1 \cong V_1^*$ and $V_1 \otimes V_1$ contains the trivial representation, V_2 and another 1-dimensional representation. Also, $V \otimes V_i \cong V_{i-1} \oplus V_{i+1}$, unless $i = m$. In consequence, the McKay diagram for this group is the double of $\widetilde{D_{m+2}}$:



Note that $\delta_i = (\mathrm{dim}\rho_i)$, where δ is the indivisible root in the kernel of the Cartan form.

Remark 1.4. Except for $\mathbb{Z}/m$, m odd, these diagrams are all bipartite: that is, there exists $S \subset Q_0$ such that each arrow has one extreme in S and the other in S^c (no arrows with both extremes in

S, or both in S^c). Look at the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ($|\, G\,|$ is even if and only if $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in G$ since G does not contain any conjugate of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$): is must be ± 1, and tensoring by V switches this.

We deduce that

Proposition 1.5. *Let G be a finite subgroup of $SL_2\mathbb{C}$. The following statements are equivalent:*

- $|G|$ *is even;*
- $G = \pi^{-1}(G')$ *for some subgroup* G' *of* $SO_3\mathbb{R}$*;*
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in G$ *(note that* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \ker \pi$*;*
- *the McKay graph of G is bipartite.*

Note 1.6. We will see right now that $\delta := (\mathrm{dim}\rho_i)$ is the indivisible in the kernel of the Cartan form. If the graph is bipartite, then $Q_0 = S \sqcup (Q_0 \setminus S)$. This means for the McKay graph that $Irreps(G) = S \sqcup S^c$,

- $\rho \in S \Longrightarrow V \otimes \rho = \sum \rho_i$ for some $\rho_i \in S^c$,
- $\rho \in S^c \Longrightarrow V \otimes \rho = \sum \rho_j$ for some $\rho_j \in S$.

As δ is in the kernel of the Cartan form, $2\delta_i = \sum_{j\, adjacent\, i} \delta_j$, so we want to conclude $\sum \delta_i$ even, so $|G| = \sum \delta_i^2$ is also even.

Example 1.7. Consider $G = \widetilde{\mathbb{A}_4}$. The irreducible representations with $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto 1$ correspond to the irreducible representations of $\mathbb{A}_4$:

- three 1-dimensional representations $(123) \mapsto e^{2k\pi i/3}$, $(12)(34) \mapsto 1$;
- the 3-dimensional representation $W = \ker \begin{pmatrix} \mathbb{C}^4 \to \mathbb{C} \\ e_1 \mapsto 1 \end{pmatrix}$, the standard representation of $\mathbb{S}_4 \supset \mathbb{A}_4$.

We also have three 2-dimensional representations: $V = V^* = \mathbb{C}^2$, and $V \otimes V$ contains the trivial representation and W. Therefore the McKay diagram is the double quiver of $\widetilde{E_6}$:

$$\begin{tikzcd}
 & & \circ \arrow[d, shift left] & & \\
 & & \circ \arrow[u, shift left] \arrow[d, shift left] & & \\
\circ^{\rho_0} \arrow[r, shift left] & \circ^{V} \arrow[l, shift left] \arrow[r, shift left] & \circ^{W} \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left]
\end{tikzcd}.$$

Remark 1.8. We know that $\widetilde{\mathbb{S}_4}$ and $\widetilde{\mathbb{A}_5}$ have to map to E-type diagrams, since they have irreducible representations of dimension > 2, and only $\tilde{E}_n$ have some $\delta_i > 2$. It follows that

- the McKay diagram of $\widetilde{\mathbb{A}_4}$ is $\tilde{E}_6$,
- the McKay diagram of $\widetilde{\mathbb{S}_4}$ is $\tilde{E}_7$,
- the McKay diagram of $\widetilde{\mathbb{A}_5}$ is $\tilde{E}_8$,

for example looking at $|G| = \sum_i \delta_i^2$.

Proposition 1.9. *Let G be a finite subgroup of $SL_2\mathbb{C}$. The McKay graph of G is the double of an extended Dynkin diagram, and $\delta = (\mathrm{dim}\rho_i)$ ($\delta_i = 1$ for some i, namely ρ_i the trivial).*

Proof. Consider $\mathbb{C}G$ the regular representation of G, $\mathbb{C}G \otimes \rho_i \cong \mathbb{C}G^{\oplus \dim \rho_i}$ for all i (if e_j is a basis of ρ_i, then $\mathbb{C}G \otimes \rho_i = \oplus \mathbb{C}G(1 \otimes e_j)$).

Now, $\mathbb{C}G \otimes V \cong \mathbb{C}G \oplus \mathbb{C}G$. Write $\mathbb{C}G = \oplus \rho_i^{\oplus \dim \rho_i}$, and denote A the adjacency matrix of McKay quiver. As $V \otimes \rho_i = \sum_{j \sim i} V_j$,

$$V \otimes (\oplus \rho_i^{\oplus \dim \rho_i}) \cong \oplus \rho_i^{\oplus 2 \dim \rho_i} \quad \Longrightarrow \quad A(\dim \rho_i) = 2(\dim \rho_i).$$

Remark 1.10. We have a G-isomorphism $vol : V^* \to V$ (using $G < SL_2 G$), so

$$\begin{aligned} \#(i \to j) &= \dim \operatorname{Hom}(\rho_j, V \otimes \rho_i) = \dim \operatorname{Hom}(V \otimes \rho_i, \rho_j) \\ &= \dim \operatorname{Hom}(\rho_i, \rho_j \otimes V^*) = \dim \operatorname{Hom}(\rho_i, \rho_j \otimes V) \\ &= \dim \operatorname{Hom}(\rho_i, V \otimes \rho_j) = \#(j \to i). \end{aligned}$$

This says that A is symmetric.

Remark 1.11. If G is finite and V is faithful, the McKay diagram of G and V is strongly connected. One way to prove this is as follows: It suffices to show that, if W is any representation, then $(W \otimes V^{\otimes N})^G \neq 0$ for some $N \geq 1$. In other words, $\sum_{g \in G} \operatorname{tr}(g|_W) \operatorname{tr}(g|_V)^N \neq 0$ for some $N \geq 1$. For large N, it suffices to restrict our attention to g such that $\operatorname{tr}(g|_V)$ has maximum absolute value, i.e., the set $S \subseteq G$ of elements such that $g|_V = \lambda_g \mathrm{Id}_V$, for some scalars λ_g. By faithfulness, all the λ_g are distinct. Now, the vectors $T_g := (1, \lambda_g, \lambda_g^2, \ldots, \lambda_g^{|S|-1})$ are linearly independent for all $g \in S$ (by, for example, the Vandermonde determinant). So, there must exist a nonnegative integer $n \leq |S| - 1$ such that $\sum_{g \in S} \operatorname{tr}(g|_W) \operatorname{tr}(g|_V)^n \neq 0$. Now, for sufficiently large m, we find that $\sum_{g \in G} \operatorname{tr}(g|_W) \operatorname{tr}(g|_V)^{n+m|G|} \approx \sum_{g \in S} \operatorname{tr}(g|_W) \operatorname{tr}(g|_V)^{n+m|G|} = (\dim V)^{m|G|} \sum_{g \in S} \operatorname{tr}(g|_W) \operatorname{tr}(g|_V) \neq 0$. This completes the proof.

We get that A a symmetric matrix, which entries are integers ≥ 0 and δ is an eigenvector of eigenvalue 2 with positive entries. We know that for some orthogonal matrix O, OAO^{-1} is diagonal, with its eigenvalues in the diagonal (we can assume that 2 is the first). Since $\delta_i > 0$ for all i, from the Perron-Frobenius Theorem (which we will prove after this), we can conclude that all the eigenvalues λ of A verify $|\lambda| < 2$ if $\lambda \neq 2$, and that $\lambda = 2$ has multiplicity one (thus δ is the unique eigenvector of eigenvalue 2 up to scaling). Also, note that all the eigenvalues must be real since A is symmetric. Hence, $\lambda < 2$ for all eigenvalues other than 2.

Note that $2\mathrm{Id} - A$ is a Cartan matrix. By the above, there exists an orthogonal matrix O such that $O(2\mathrm{Id} - A)O^t$ is diagonal with one diagonal entry zero and the other ones positive, where $O^t = O^{-1}$ is the transpose of O, and O is orthogonal. So $2\mathrm{Id} - A$ is positive-semidefinite but not positive-definite. More or less by definition, this means that A is the adjacency diagram of an extended Dynkin diagram. □

Theorem 1.12 (Perron-Frobenius). *Let A be an $I \times I$ matrix with non-negative entries, strongly connected (i.e. the graph obtained by $i \to j$-arrow added in whenever $a_{ij} \neq 0$ is strongly connected). Then there exists a unique eigenvector v up to scaling such that $v \in \mathbb{R}_+^I$ and that the corresponding eigenvalue λ is positive, of multiplicity one, and $\lambda > |\lambda'|$ for any other eigenvalue λ'.*

Proof. A acts on $R_{\geq 0}^I \setminus 0/\mathbb{R}_+$ continuously (it cannot kill anything because the entries are non-negative and the matrix is strongly connected). By Brouwer's Fixed Point Theorem, there exists $v \in R_{\geq 0}^I$ such that $Av = \lambda v$ for some $\lambda > 0$: by strong connectivity we conclude that $v \in R_+^I$.

We can change of basis by a diagonal matrix with positive entries in order to have $v = \mathbf{1} = (1, ..., 1)$. Define $\|w\| = \max |w_i|$, $w \in \mathbb{R}^I$.

Claim 1.13. $\|A\| = \lambda$ *(operator norm), achieved exactly at multiple of v*

Note that this claim ends the proof of the Theorem. Note that if $\|w\| = 1$ and w has all its entries non-negative, $\|Aw\| \leq \|A\mathbf{1}\|$ (it is derived from the strong connectivity). Also, for a general w, $\|Aw\| \leq \|Aw'\|$, where w' is defined by $w'_i = |w_i|$. Therefore $\|w\| = 1$ implies $\|Aw\| \leq \|A\mathbf{1}\|$ in general. Moreover, the inequality is strict if $w \neq \pm\mathbf{1}$.

Now if μ is an eigenvalue of A with eigenvector w, $\|A^n w\| = |\mu|^n \|w\|$, so $|\mu| \leq \lambda$. Also, if $|\mu| = \lambda$, then $\|A^n w\| = \lambda^n \|w\|$, so w is a multiple of $\mathbf{1}$. $\square$

From the previous result we conclude that $\{G < SL_2\mathbb{C} \text{ finite}\} \longrightarrow$ extended Dynkin diagram. We have also $\mathbb{C}[G] \simeq_{Morita} \mathbb{C}^{\#irreps}$,

$$M \mapsto e\mathbb{C}[G] \otimes M, \quad e = \sum_{i=1}^{\#irreps} e_i,\ \mathbb{C}[G]e_i \cong \rho_i.$$

Claim 1.14. *$e(\mathbb{C}[x,y]\#G)e \cong \Pi^0_Q$, where Q is any orientation of the extended Dynkin diagram such that $\bar{Q}$ is the corresponding McKay diagram of G.*

We have $\mathbb{C}[G] \simeq \mathbb{C}\bar{Q}_0$, defining the functor

$$F:\ \mathbb{C}[G] - bimod \xrightarrow{\sim} \mathbb{C}\bar{Q}_0 - bimod$$

$$M \longmapsto e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e.$$

Note that:

$$\begin{aligned} F(M \otimes_{\mathbb{C}[G]} N &= e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} N \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e \\ &= e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} N \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e \\ &= (e\mathbb{C}[G] \otimes_{\mathbb{C}[G]} M \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e) \otimes_{e\mathbb{C}[G]e} e\mathbb{C}[G](\otimes_{\mathbb{C}[G]} N \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e), \end{aligned}$$

which says that the functor is monoidal.

Also, $F(\mathbb{C}[G]) = \mathbb{C}\bar{Q}_0$, $F(V) = \mathbb{C}\bar{Q}_1$ and

$$F(\rho_i \otimes V \otimes \rho_j = \mathrm{Hom}(\rho_i, V \otimes \rho_j) = i\mathbb{C}\bar{Q}_1.$$

So we conclude that

$$\mathbb{C}[G]\#TV = T_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes V) \to T_{\mathbb{C}\bar{Q}_0}\mathbb{C}\bar{Q}_1 = \mathbb{C}\bar{Q} = \text{path algebra of } \bar{Q},$$

$$\omega_\lambda = \text{sympl. form of } V^* - \sum_{i=1}^{\#irreps} \lambda_i \mathrm{Id}_i \mapsto \sum_i [e_i, e_i^*] - \lambda$$

$$\therefore TV\#G/([x,y] - \lambda) \to \Pi^\lambda_Q.$$

To explain the previous Morita equivalence in Claim **??**, we need some other considerations. We want to know if given $\mathbf{k}$-algebras A, B such that $A \cong_{Morita} B$, this implies $A^{op} \cong_{Morita} B^{op}$. This is true and follows from the next Theorem. Remember that a *projective generator* P for a $\mathbf{k}$-algebra A is a projective A-module such that for any other A module M there exists some index I and a surjective morphism $P^{\oplus I} \twoheadrightarrow M$.

Theorem 1.15 (Morita)**.** *The following are equivalent:*

(1) *$A - mod \simeq B - mod$;*
(2) *for a projective generator $P \in B - mod$, $\mathrm{End}_B P \cong A^{op}$;*
(3) *$P \otimes_A - : A - mod \to B - mod$ is an equivalence.*

Proof. (Sketch)(1) $\Rightarrow$ (2). If $\phi : A-mod \to B-mod$ gives such equivalence, consider $P := \phi(A)$.

(2) $\Rightarrow$ (3). In this case, $P \otimes_A -$ and $\mathrm{Hom}_B(P,-)$ are (quasi) inverse functors. We can show that $M \cong \mathrm{Hom}_B(P, P\otimes_A M)$ for any $M \in A-mod$ in several steps: first for $M = A$, then for $M = A^{\oplus I}$ for any index I, and finally for all M using the exactness of this functor and the (trivial) existence of $A^{\oplus I} \twoheadrightarrow M$ for some I. Similarly we prove that $N \cong P \otimes_A \mathrm{Hom}_B(P, M)$ for any $N \in A-mod$.

(3) $\Rightarrow$ (1). This is immediate. □

Corollary 1.16. *If $A \cong_{Morita} B$, then $B^{op} \cong_{Morita} A^{op}$*

Proof. Use (2) of previous Theorem and the fact that ${}_BP_A$ gives place to ${}_{A^{op}}P_{B^{op}}$. □

Corollary 1.17. *$A \cong_{Morita} B$ if and only if there exists modules ${}_BP_A, {}_AQ_B$ such that $Q\otimes_B P \cong A$ and $P \otimes_A Q \cong B$.*

Corollary 1.18. *If $A \cong_{Morita} B$, then there exists a monoidal equivalence $F : A-bimod \to B-bimod$*

Proof. Note that $A-bimod = A^\rho - mod$, where $A^\rho := A\otimes_{\mathbf{k}} A^{op}$. If $A \cong_{Morita} B$, then by the existence of P, Q as in (2) of the Theorem,

$$B\otimes_{\mathbf{k}} B^{op} \cong (P\otimes Q)\otimes_{A\otimes_{\mathbf{k}}A^{op}} (Q\otimes P), \quad A\otimes_{\mathbf{k}} A^{op} \cong (Q\otimes P)\otimes_{B\otimes_{\mathbf{k}}B^{op}} (P\otimes Q).$$

For the monoidal property, observe that

$$\begin{aligned} F(M\otimes_A N) &= P\otimes_B M\otimes_A N\otimes_A Q = P\otimes_B M\otimes_A A\otimes_A N\otimes_A Q \\ &= P\otimes_B M\otimes_A (Q\otimes_B P)\otimes_A N\otimes_A Q \\ &= (P\otimes_B M\otimes_A Q)\otimes_B (P\otimes_A N\otimes_A Q) = F(M)\otimes_B F(N). \end{aligned}$$

□

In our situation, $B = eAe$, where $e^2 = e$ and $AeA = A$. Consider ${}_BP_A := eA$, ${}_AQ_B := Ae$. Then

$$F(M) = eA\otimes_A M\otimes_A Ae = "eMe" \quad\Longrightarrow\quad eMe\otimes_{eAe} eNe = e(M\otimes_B N)e.$$

Back to case $\mathbb{C}[G]$, we have $e = \sum_{i=1}^n e_i$, where e_i are the idempotents, $\mathbb{C}[G]e_i$ are de different irreducible representations of G, and

$$\mathbb{C}[G]e\mathbb{C}[G] = \mathbb{C}[G] \qquad \left(\mathbb{C}[G] = \sum_i \mathrm{End}_{\mathbb{C}}(\rho_i),\ e_i \in \mathrm{End}_{\mathbb{C}}(\rho_i)\right).$$

As $\mathbb{C}[G] \simeq_{Morita} \mathbb{C}Q_0$, where $\bar{Q}$ is the McKay quiver of $G < SL_2\mathbb{C}$, we have

$$F : \mathbb{C}[G]-bimod \xrightarrow{\sim} \mathbb{C}Q_0 - bimod.$$

What we want is to view $\mathbb{C}[x,y]\#G = \mathbb{C}[\mathbb{C}^2]\#G$ as an algebra in $\mathbb{C}[G]$-bimod, its image under F will be Π_Q^0: $e\mathbb{C}[\mathbb{C}^2]\#Ge = \Pi_Q^0$, so we will obtain the desired Morita equivalence.

Note that $(\mathbb{C}^2)^* \subset \mathbb{C}[\mathbb{C}^2]$. Also, as bimodule, $(\mathbb{C}^2)^*\#\mathbb{C}[G]$ has the following G-actions:

$$g(v\otimes h) = g*v\otimes gh, \qquad (v\otimes h)g = v\otimes hg.$$

Any $\mathbb{C}[G]$-bimodule is a direct sum of irreducible bimodules, and $\mathbb{C}[G]-bimod = G\times G^{op}-mod$. In consequence, the irreducible bimodules are $\rho_i \boxtimes \rho_j^*$, so we compute their image under F.

Claim 1.19. *$F(\rho_i\boxtimes\rho_j^*)$ is the 1-dimensional vector space $M = \langle m\rangle$ with $\mathbb{C}Q_0$-bimodule structure $jm = m = mi$; i.e., $M = \circ^i \longrightarrow \circ^j$.*

Proof. Note that $\rho_i \boxtimes \rho_j^* = \mathbb{C}[G]e_i \otimes e_j\mathbb{C}[G]$ (use here that $e_j\mathbb{C}[G]$ is a $G^{op} \cong G$-module isomorphic to ρ_j^*), because

$$\begin{aligned} e_i\mathbb{C}[G]e_j &= e_i\mathbb{C}[G] \otimes_{\mathbb{C}[G]} \mathbb{C}[G]e_j = \begin{cases} \mathbb{C}, & i = j; \\ 0, & i \neq j; \end{cases} \\ e\mathbb{C}[G]e_j &= \langle e_j \rangle, \qquad e_i\mathbb{C}[G]e = \langle e_j \rangle. \end{aligned}$$

Then, $F(\rho_i \boxtimes \rho_j^*) = \langle e_i \otimes e_j \rangle$, and $m = e_i \otimes e_j$ gives the desired property. $\square$

Claim 1.20. $(\mathbb{C}^2)^* \# \mathbb{C}[G] \xmapsto{F} \mathbb{C}\bar{Q}_1$.

Proof. Note that

$$\begin{aligned} (\mathbb{C}^2)^* \otimes \mathbb{C}[G] &= \oplus_i ((\mathbb{C}^2)^* \otimes \rho_i) \boxtimes \rho_i^* \\ &= \oplus_{i,j} (\rho_i \boxtimes \rho_j^*) \otimes \mathrm{Hom}_G(\rho_j, (\mathbb{C}^2)^* \otimes \rho_i) = \oplus (\rho_i \boxtimes \rho_j^*) \otimes \mathbb{C}^{\#(i \to j)}, \end{aligned}$$

the last equality by definition of $\bar{Q}$. Then it follows from our result above. $\square$

Therefore $T_{\mathbb{C}[G]}(\mathbb{C}^2)^* \otimes \mathbb{C}[G] \xmapsto{F} T_{\mathbb{C}\bar{Q}_0}\mathbb{C}\bar{Q}_1 = \mathbb{C}\bar{Q}$, and we have

$$\begin{tikzcd} T_{\mathbb{C}[G]}(\mathbb{C}^2)^* \# G \arrow[r, mapsto, "F"] \arrow[d, "\varphi_1"] & \mathbb{C}\bar{Q} \arrow[d, "\varphi_2"] \\ \mathbb{C}[\mathbb{C}^2] \# G \arrow[r, mapsto, "?"] & \Pi_Q. \end{tikzcd}$$

It remains to show that $F(\ker \varphi_1) = \ker \varphi_2$.

First of all, $\ker \varphi_1 = (\tilde{\omega})$, where $\tilde{\omega} = [x, y]$ is also a symplectic form on $\mathbb{C}^2 \otimes \mathbb{C}[G]$. If $\omega = y \otimes x - x \otimes y$ is the symplectic structure on $\mathbb{C}^2$, we have

$$\tilde{\omega}((v, g), (w, h)) = \omega(v, w)\delta_{g,h^{-1}}, \qquad g, h \in G.$$

As element of $T^2_{\mathbb{C}[G]}(\mathbb{C}^2) \otimes \mathbb{C}[G]$ just get $[y, x]$.

To compute $F([y, x]) = e[y, x]e$, we compute $e\tilde{\omega}e$ as symplectic structure on $F((\mathbb{C}^2)^* \otimes \mathbb{C}[G]) = \mathbb{C}\bar{Q}_1^* \cong \mathbb{C}\bar{Q}_1$.

Between two distinct vertices i, j, $F(\tilde{\omega})$ is the perfect pairing $i\bar{Q}_1 j \otimes j\bar{Q}_1 i \to \mathbb{C}$ because $F(\tilde{\omega})$ is symplectic on $i\bar{Q}_1 j$.

So in some bases of $i\bar{Q}_1 j$ (a_l, a_l^*), for all i, j, get

$$F(\tilde{\omega}) = \sum a_l \otimes a_l^* - a_l^* \otimes a_l.$$

This follows from the statement that any symplectic structure on V is $\sum a_l \otimes a_l^* - a_l^* \otimes a_l$ for some basis of V^*; i.e. $\omega(a_l^\vee, a_{l'}^\vee) = \omega((a_l^*)^\vee, (a_{l'}^*)^\vee) = 0$ and $\omega(a_l^\vee, (a_l^*)^\vee) = 1$.

Therefore if we define $Q = \{a_l\}$, we get $F(\tilde{\omega}) = \sum_{a \in Q_1} [a, a^*]$.

Remark 1.21. For $i \neq j$, we can simply let a_l a basis of arrows $i \to j$.

λ-deformed version:

We have $\mathbb{C}[G] = \sum_i \mathrm{End}_{\mathbb{C}}(\rho_i)$, and for $Id_i \in \mathrm{End}_{\mathbb{C}}(\rho_i)$, $F(\mathrm{Id}_i) = \epsilon_i$ is the elementary vector at $i \in Q_0$. If $\lambda \in \mathbb{C}Q_0$, $\lambda \cdot \mathrm{Id} = \sum_i \lambda_i \mathrm{Id}_i$, so

$$(\tilde{\omega} - \lambda \cdot \mathrm{Id}) \xmapsto{F} \sum_{a \in Q_1} [a, a^*] - \lambda$$

$$\therefore D_G^\lambda := \mathbb{C}[x, y] \# G / [x, y] - \lambda \cdot \mathrm{Id} \simeq \Pi_Q^\lambda.$$

For $\lambda = \mathbf{1}$, $\mathbf{1} \cdot \mathrm{Id} = \mathrm{Id}$, so $D_G^{\mathbf{1}} \cong D(\mathbb{C}^2)\#G$. Here, $D(\mathbb{C}^2)$ is the ring of differential operators on polynomial functions of $\mathbb{C}^2$, i.e.,

$$D(\mathbb{C}^2) = \mathbb{C}\langle x, \frac{\partial}{\partial x}\rangle/([\frac{\partial}{\partial x}, x] - 1), \tag{1.22}$$

which is also known as the Weyl algebra of the symplectic vector space $\mathbb{C}^2$ (equipped with the standard symplectic form, i.e., the determinant).

The objects $\Pi_Q^\lambda, D_G^\lambda, B_Q^\lambda := e_0 D_G^\lambda e_0 = e_0 \Pi_Q^\lambda e_0$ (in particular, $B_Q^0 = \mathbb{C}[x,y]^G = \mathbb{C}[\mathbb{C}^2/G]$) were studied in *W. Crawley-Boevey and M. P. Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), no. 3, 605635.* This gives a way to resolve the singularity $\mathbb{C}^2/G$, and produces its deformation.

Think of $\mathbb{C}[x,y]\#G$ (D_G^0-mod=G-equivariant $\mathbb{C}[x,y]$-modules) as a non commutative resolution of singularities of C^2/G ($B_G^0 = \mathbb{C}[x,y]$-modules).

Crude way to view it:

$$\begin{array}{c} \text{points of } D_G^0 \text{ as simple modules} \\ \Big\downarrow \pi \\ \text{simple } B_G^0 - mods, (\text{ points of } \mathbb{C}^2/G). \end{array}$$

How is π defined? Consider M a D_G^0-module: $M = \oplus_{\mathcal{O}\subset\mathbb{C}^2\ G-orbit} M_{\mathcal{O}}$, with $M_{\mathcal{O}}$ supported on $\mathcal{O}$ as $\mathbb{C}[x,y]$-module. In particular, if M is simple, then $M = M_{\mathcal{O}}$ for some orbit $\mathcal{O}$.

If $p \in \mathbb{C}^2$ is not zero, then $\pi^{-1}(p)$ is a single simple module: $\mathbb{C}[G\cdot p]$ (as $\mathbb{C}[G\cdot p] = \oplus_{g\in G}\mathbb{C}[g\cdot p]$ as $\mathbb{C}[x,y]$-module and G permutes the factors, it is simple).

For each $M \in \pi^{-1}(0)$, x, y act by 0 on M, so $M \cong \rho$ for some irreducible representation of G. Therefore $\pi^{-1}(G)$ has more than one point if $G \neq 1$.

We have the philosophy that for A associative algebra, $\mathrm{Rep}_n A$ are commutative aproximations of "Spec"A.

If we think which $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$ has the property that $\mathrm{Rep}_\alpha D_G^0 = \mathrm{Rep}_\alpha \Pi_Q$ contains $\pi^{-1}(p)$ for $p \neq 0$, the answer is $\delta = \underline{\dim}_{\mathbb{C}[G]}\mathbb{C}[G]$.

Consider therefore $\pi : \mathrm{Rep}_\delta \Pi_Q^0//GL_\delta \twoheadrightarrow \mathbb{C}^2/G$: it is 1-1 away from $0 \in \mathbb{C}^2/G$.

The point is: $\mathrm{Rep}_\delta \Pi_Q^\lambda//GL_\delta$ is smooth if $\lambda\cdot\alpha \neq 0$ for $\alpha < \delta$ and $\lambda \cdot \delta = 0$.

Remark 1.23. Define:

$$\begin{aligned} \mathfrak{h} &:= \{\lambda : \lambda\cdot\delta = 0\}, \\ \mathfrak{h}^{res} &:= \{\lambda : \lambda\cdot\delta = 0, \lambda\cdot\alpha \neq 0 \forall \alpha < \delta\}. \end{aligned}$$

Then we have the following:

$$\begin{tikzcd} B_G^\lambda \text{ NC defs of } \mathbb{C}^2/G \arrow[d, hook] & B_G^\lambda \cong \mathbb{C}[\mathrm{Rep}_\delta \Pi_Q^\lambda]^{GL_\delta} \arrow[d, hook] & B_G^\lambda \text{ smooth resolution of } \mathbb{C}^2/G \arrow[d, hook] \\ \Pi_Q^\lambda \simeq D_G^\lambda \arrow[d] & \Pi_Q^\lambda \simeq D_G^\lambda \arrow[l, hook'] \arrow[d] & \Pi_Q^\lambda \simeq D_G^\lambda \arrow[l, hook'] \arrow[d] \\ \mathbb{C}Q_0 & \mathfrak{h} \arrow[l, hook', "closed"'] & \mathfrak{h}^{res}. \arrow[l, hook', "open"'] \end{tikzcd}$$

B_G^λ is a semi universal deformation of $\mathbb{C}[\mathbb{C}^2/G]$.

Note 1.24. $\Pi_Q^\lambda \neq 0$ for Q extended Dynkin diagram and any λ, but

$$\left.\begin{array}{l} \lambda\cdot\delta \neq 0 \\ \lambda\cdot\alpha \neq 0 \quad \alpha < \delta \end{array}\right\} \mathrm{Rep}_\delta \Pi_Q^\lambda = \emptyset.$$

Remember that a *resolution* for singularities of a singular variety X is a smooth variety $\tilde{X}$ with a morphism $\pi : tildeX \twoheadrightarrow X$, which is an isomorphism over X_{smooth}. In this sense, $\mathrm{Rep}_\delta \Pi_Q^\lambda // PGL_\delta$ is not a resolution of $\mathbb{C}^2/G$, but it is a smooth deformation of $\mathbb{C}^2/G$.

Note 1.25. For $G < SL_2\mathbb{C}$ finite, $\mathbb{C}^2/G$ is singular, with a *Kleinian singularity.*

For $\lambda \in \mathbb{C}Q_0$, $\Pi_Q^\lambda \simeq_{Mor} D_G^\lambda$, which is smooth non commutative: that is, with finite Hochschild dimension (it coincides with usual smoothness for communtative algebras).

Each Π_Q^λ is a resolution of singularities of $B_G^\lambda := e_0 D_G^\lambda e_0 = e_0 \Pi_Q^\lambda e_0$, where $\mathbb{C}Ge_0$ is the trivial representation (corresponding to the extending vertex of Q). Remember that $D_G^0 = \mathbb{C}[x,y]\#G$, and $B_G^0 = \mathbb{C}[\mathbb{C}/G]$. We have:

$$\begin{tikzcd}
\{\text{"}\,\mathrm{Spec}\,\text{"}\Pi_Q^\lambda\} \arrow[r] \arrow[d, "\pi"] & \mathbb{C}Q_0 \\
\{\mathrm{Spec}\, B_G^\lambda\} \arrow[r] & \mathbb{C}Q_0 \supseteq \mathfrak{h} \supseteq \mathfrak{h}^r eg
\end{tikzcd}.$$

So we have non-commutative deformations and non-commutative resolutions.

Proposition 1.26. (i) $\lambda \in \mathfrak{h}$*:* $B_G^\lambda \cong \mathbb{C}[\mathrm{Rep}_\delta \Pi_Q^\lambda]^{GL_\delta}$*.*

(ii) $\lambda \in \mathfrak{h}^{reg}$*:* $B_G^\lambda \simeq_{Mor} \Pi_Q^\lambda (\simeq D_G^\lambda)$*; this says that* $\mathrm{Spec}\, B_G^\lambda \simeq \text{"}\,\mathrm{Spec}\,\text{"}\Pi_Q^\lambda$*.*

(iii) $\mathrm{Rep}_\delta \Pi_Q^\lambda // PGL_\delta$ *and* $\mathrm{Spec}\, B_G^\lambda$ *are smooth (Hochschild dimension is preserved by* $\simeq$*).*

Remark 1.27. In (i), for $\lambda = 0$ we have $\mathbb{C}^2/G$, the G-orbits in $\mathbb{C}^2$ as G-representations, $\mathbb{C}[Gx] \cong \mathbb{C}[G]$. We have $\mathbb{C}[\mathrm{Rep}_\delta \Pi_Q^0]^{GL_\delta} \cong \mathbb{C}[\mathbb{C}^2/G] = B_G^0$.

Proof. (Sketch) (ii) We have $B_G^\lambda = e_0 \Pi_Q^\lambda e_0$ for e_0 the extending vertex, and a surjective map $\mathbb{C}\bar{Q} \twoheadrightarrow \Pi_Q^\lambda$. We want to prove that $\Pi_Q^\lambda e_0 \Pi_Q^\lambda = \Pi_Q^\lambda$.

Note that $\Pi_Q^\lambda/(e_0) = \Pi_{Q'}^\lambda$, where $Q' := Q \setminus e_0$, which is a Dynkin quiver. Therefore $\Pi_{Q'}^\lambda$ is finite dimensional ($\Pi_{Q'}^0$ is the direct sum of all the Q' irreducible representations, and we have $\Pi_{Q'}^0 \twoheadrightarrow \Pi_{Q'}^\lambda$). So $\Pi_{Q'}^\lambda = 0$ if and only if it has no simple modules. But $\lambda \cdot \alpha \neq 0$ for all $\alpha \in \Delta_+(Q')$ so no Q'-representations extend to $\Pi_{Q'}^\lambda$-representations. Then $\Pi_{Q'}^\lambda = 0$, and $\Pi_Q^\lambda e_0 \Pi_Q^\lambda = \Pi_Q^\lambda$.

(iii)

Claim 1.28. $\mathrm{Rep}_\delta \Pi_Q^\lambda$ *is smooth for* $\lambda \in \mathfrak{h}^{reg}$*.*

That is, $d\mu$ has constant rank on $\mathrm{Rep}_\delta \Pi_Q^\lambda = \mu^{-1}(\lambda) \subseteq \mathrm{Rep}_\delta \mathbb{C}\bar{Q} = T^*\mathrm{Rep}_\delta \mathbb{C}Q$.

The proof is similar to the one for $\mathrm{Hilb}^n \mathbb{C}^2$, which is coming up.

Claim 1.29. PGL_δ *acts freely.*

I.e., the isotropy of GL_δ of $M \in \mathrm{Rep}_\delta \Pi_Q^\lambda$ is $\mathbb{C}^*$, or $\dim \mathrm{End}_{\bar{Q}} M = 1$.

If there exists a non zero endomorphism of M which is not invertible, it has empty image in $\mathrm{Rep}_\alpha \Pi_Q^\lambda$ for $\alpha < \delta$, by the previous remark about extensions of Q'-modules. Therefore, $\mathrm{End}_{\bar{Q}} M$ is a division algebra over $\mathbb{C}$, so $\mathrm{End}_{\bar{Q}} M = \mathbb{C}$. ∎

Remark 1.30. For $\lambda \in \mathfrak{h} \setminus \mathfrak{h}^{reg}$, B_G^λ is smooth, Π_Q^λ is not smooth and $B_G^\lambda \not\simeq \Pi_Q^\lambda$; e.g. Π_Q^λ has a simple module V_α for $\alpha \cdot \lambda = 0$, $\alpha < \delta$ and B_G^λ only has a simple for $\alpha = \epsilon_{e_0}$ (the trivial, of dimension 1).

If $\Pi_Q^\lambda \simeq B_G^\lambda$, then $\alpha \cdot \lambda = 0$, $\alpha < \delta$ only for $\alpha = \epsilon_{e_0}$, so $(\delta - \epsilon_{e_0}) \cdot \lambda = 0$, which is a contradiction.

More generally, $\Pi_Q^\lambda \simeq B_G^\lambda$ if and only if $\alpha \cdot \lambda \neq 0$ for all $\alpha \in \Delta_+(Q')$.

Remark 1.31. There exists a commutative resolution of $\mathbb{C}^2/G$, in fact for the whole family:

$$\begin{tikzcd}
\{\tilde{X}^\lambda\} \arrow[r, two heads] \arrow[d] & \{\operatorname{Spec} B_G^\lambda\} \arrow[d] \\
\mathfrak{h} \arrow[r, equal] & \mathfrak{h}
\end{tikzcd}$$

π is an isomorphism over $\mathfrak{h}^{reg}$, $\tilde{X}^\lambda$ is not affine for $\lambda \in \mathfrak{h}^{reg}$. We can't have

$$\underbrace{\tilde{X}^\lambda}_{\text{(comm. resolution)}} \qquad \underbrace{\text{" Spec "}\Pi_Q^\lambda}_{\text{(non comm. resolution)}} \qquad (Coh(\tilde{X}^\lambda) \not\cong \Pi_Q^\lambda - mod,$$

BUT $D^b(Coh\tilde{X}^\lambda) \simeq D^b(\Pi^\lambda - mod)$ (McKay correspondance)

Philosophy: Any two 'minimal' resolutions of a singular variety are derived-equivalent. Precisely, minimal corresponds to *crepant*: i.e. if $\tilde{X} \twoheadrightarrow X = \mathbb{C}^2/G$ for $G < SL_n(\mathbb{C})$ finite, $K_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}$, that is, $\tilde{X}$ is Calabi-Yau. This is proved for $n = 3$ by Bridgeland, King and Reid in their celebrated JAMS paper (2001), and for $G < Sp_n(\mathbb{C})$, n even, by Kaladin (and it might not be true for general $G < SL_n(\mathbb{C})$).

2. Hilbert schemes

Our goal now is to construct $\pi : \tilde{X}^0 \twoheadrightarrow \mathbb{C}^2/G$ using Hilbert schemes. This also gets a resolution of $S^n(X)$ for X a smooth surface (we use it in case $X = \mathbb{C}^2$, $n =\mid G \mid$ to define $\mathbb{C}^2/G$).

Given X a variety, we have $S^nX := \{$ unordered n-tuples of points in $X\}$.

Proposition 2.1. *Given X a smooth curve, S^nX is smooth.*

Proof. As this a local property, set $X = \mathbb{C}$:

$$S^n\mathbb{C} \xleftrightarrow{\sim} \mathbb{C},$$

$$(\lambda_1, \dots, \lambda_n) \longmapsto \text{coefficients of } (x - \lambda_1)\cdots(x-\lambda_n),$$

$$\text{set of roots} \longleftarrow\!\shortmid \text{monic polynomials in } x.$$

Algebraically, this says that the ring of symmetric polynomials is itself a polynomial ring in elementary symmetric polynomials. ☐

Remark 2.2. If $\dim X = 2$, S^nX is not smooth:

$$S^2\mathbb{C}^2 = \{x, y\} = \{x + y(\in \mathbb{C}^2), x - y(\in \mathbb{C}^2/\mathbb{Z}_2)\} = \mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_2,$$

and $\mathbb{C}^2/\mathbb{Z}_2 = \{x^2, y^2, xy\}$ is singular.

We need a better notion than the unordered n-tuples of points.

One solution is to consider subschemes of length n: given $X = \operatorname{Spec} A$, consider $Y \subseteq X$, $\mathbb{C}[Y] = A/I_Y$, satisfying $\dim\mathbb{C}[Y] = n$; i.e. $\operatorname{codim} I_Y = n$. We have a natural map:

$$\begin{tikzcd}
\operatorname{Hilb}^n X := \{\text{length n-subschemes}\} \arrow[d] & Y \arrow[d, maps to] & n_{x,Y} = \dim\mathbb{C}[Y]_{\mathfrak{m}_x} \\
S^nY & \sum_{x \in X} n_{x,Y} x. &
\end{tikzcd}$$

$n_{x,Y}$ is the multiplicity of x in Y. Note that if $X = \mathbb{C}^n$, $A = \mathbb{C}[x_1, ..., x_n]$. Any finite dimensional A-module is $M = \oplus M_{\tilde{x}}$, where $M_{\tilde{x}} = M_{\mathfrak{M}_x}$ is the part of M supported by $\tilde{x}$. Therefore

$$\mathbb{C}[Y] = \oplus \mathbb{C}[Y]_{\mathfrak{M}_{\tilde{x}}}.$$

Claim 2.3. *π is surjective.*

Proof. For this, is enough to cook up a subscheme of length n concentrated at $\tilde{x}$, for any $\tilde{x}$ and n.

But as $N \to \infty$, $\dim A/\mathfrak{M}_{\tilde{x}}^N \to \infty$ ($A = \mathbb{C}[X]$), so we can take a quotient module of dimension n (the composition series has all the subquotients equal to $\mathbb{C}[\tilde{x}] \cong \mathbb{C}$. □

Claim 2.4. *π is an isomorphism over $(S^n X)^{reg} = \{(x_1, ..., x_n) :$ all distinct$\}$.*

Proof. Let Y be a subscheme of length n, with support on $\tilde{x}_1, ..., \tilde{x}_n$. Then $\mathbb{C}[Y] = \oplus \mathbb{C}[Y]_{\mathfrak{M}_{\tilde{x}}}$, all 1-dimensional isomorphic to $\mathbb{C}[\tilde{x}_i]$. Therefore, $\pi^{-1}(x_1, ..., x_n)$ is a single point when the x_i's are all distinct. □

A brief explanation about $\mathrm{Hilb}^n X$ for $X = \operatorname{Spec} A$. Consider $\pi_1 : \mathrm{Hilb}^n X \times X \to \mathrm{Hilb}^n X$ and $\pi_2 : \mathrm{Hilb}^n X \times X \to X$ the canonical projections.

$\mathrm{Hilb}^n X$ is a fine moduli space: there exists a universal family $Z \subseteq \mathrm{Hilb}^n X \times X$ such that if $\pi_1([Y]) = Y$, $Y \subseteq X$ a length-n subscheme and $\mathcal{O}_Z$ is the structural sheaf of Z, $(\pi_1)_* \mathcal{O}_Z =: \mathcal{V}$ the universal sheaf on $\mathrm{Hilb}^n X$, then $\mathcal{V}|_{[Y]} = \Gamma(\mathcal{O}_Y)$.

Example 2.5. (i) $\mathrm{Hilb}^n \mathbb{C} = \mathbb{C}^n$.

(ii) For $\mathrm{Hilb}^2 \mathbb{C}^2$, the problem is over $\Delta = \{(x, x)\}$. For $(x_0, y_0) \in \mathbb{C}^2$ we want $I \subseteq \mathbb{C}[x, y]$ such that $\mathbb{C}[x, y]/I$ is 2-dimensional. Therefore $x - x_0, y - y_0$ are nilpotent and up to coordinate choice (swapping x by y), we have $x \leftrightarrow \begin{pmatrix} x_0 & 1 \\ 0 & x_0 \end{pmatrix}$ and $y \leftrightarrow \begin{pmatrix} y_0 & \lambda \\ 0 & y_0 \end{pmatrix}$ (x, y cannot act both by a scalar), so if $\pi : \mathrm{Hilb}^2 \mathbb{C}^2 \to S^2\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2/(\mathbb{Z}/2)$, then $\pi^{-1}(z, z) \cong \mathbb{P}_1$: we blow up.

Our next goal is to prove that $\mathrm{Hilb}^n X$ is smooth for X smooth of dimension 2. As it is a local question, we can consider $X = \mathbb{C}^2$.

Theorem 2.6 (following Nakajima's monograph)**.** (1) $\mathrm{Hilb}^n \mathbb{C}^2$ *is isomorphic to the set of representations of*

$$\begin{tikzcd} \circ \arrow[loop left, "x"] \arrow[loop right, "y"] \\ \circ \arrow[u, "i"] \end{tikzcd}$$

of dimension $\begin{smallmatrix} n \\ 1 \end{smallmatrix}$ *such that*

(i) $[x, y] = 0$,

(ii) *there exist no proper submodules containing* $\begin{smallmatrix} 0 \\ \mathbb{C} \end{smallmatrix}$.

(2) $\mathrm{Hilb}^n \mathbb{C}^2$ *is smooth.*

Proof. Generally, if I is an ideal of A of codimension n, A/I is a cyclic module of length n. This gets

$$\{(M, V) | M \in \mathrm{Rep}_n A, V \in M \text{ cyclic vector}\}/\tilde{GL}_n \longrightarrow \{\text{n-codimensional ideals}\}$$

$$(M, V) \longmapsto Ann(V),$$

so $\mathrm{Hilb}^n X \cong \{(M, V) \in \mathrm{Rep}_n \mathbb{C}[X] \times \mathbb{C}^n | V \text{ cyclic}\}/GL_n$.

(1) It follows from the above observation, note that $[x,y]=0$ gives a structure of $\mathbb{C}[x,y]$-module.

(2) We prove this in two steps.

Claim 2.7. *$\tilde{H} := \{(M,V) \in \mathrm{Rep}_n\mathbb{C}[x,y] \times \mathbb{C}^n | V \, cyclic\}$ is smooth.*

For this, it suffices to show that all the tangent spaces $T_x\tilde{H}$ have the same dimension for all $x \in \tilde{H} \subseteq \mathrm{Rep}_n\bar{Q} \times \mathbb{C}^n$, where $Q = \circlearrowleft \circ$. $\tilde{H}$ is the zero fiber of

$$\begin{aligned} \phi : (Rep_n\bar{Q} \times \mathbb{C}^n)' &\to \mathfrak{gl}_n \\ (A,B,v) &\mapsto [A,B]. \end{aligned}$$

$(Rep_n\bar{Q} \times \mathbb{C}^n)'$ is the subset of $(Rep_n\bar{Q} \times \mathbb{C}^n)$ where holds the condition that the last component is a cyclic vector.

We have to show that $d\phi$ has constant rank on $\tilde{H}$, because $T_x\tilde{H} = \ker(d\phi|_x)$. Now,

$$\begin{aligned} \mathrm{coker}(d\phi|_{(A,B,v)} &\cong \{\xi \in \mathfrak{gl}_n | tr\,(\xi([A,T_1]+[T_2,B]))=0, \, \forall T_1,T_2 \in \mathfrak{gl}_n\} \\ &\cong \{\xi \in \mathfrak{gl}_n | [A,\xi]=[B,\xi]=0\} \cong \mathrm{End}_{\bar{Q}}M, \end{aligned}$$

where we use that the derivative of $A,B \to [A,B]$ is $\Omega : \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathfrak{gl}_n$, $\Omega(T_1,T_2) = [A,T_1]+[T_2,B]$, and $(\mathrm{coker}\,\phi)^* = \mathrm{Ann}\,(\Omega)$.

To end this, we show that for any $(M,v) \in \tilde{H}$,
$$\begin{aligned} \mathrm{End}_{\bar{Q}}M &\xrightarrow{\sim} \mathbb{C}^n \\ \xi &\longmapsto \xi(v). \end{aligned}$$

In general, if v is cyclic, ξ is determined by $\xi(v)$, so that map is injective. On the other hand, for all $w \in \mathbb{C}^n$, there exists a polynomial $P(A,B)$ such that $P(A,B)\cdot v = w$, where $[A,B]=0$, so $P(A,B) \in \mathrm{End}_{\bar{Q}}M$.

Claim 2.8. *GL_n acts freely on $\tilde{H}$.*

If $g \in GL_n$ preserves (M,v) then $g \in \mathrm{Aut}\,M$ and satisfy $g(v)=v$. But as v is a cyclic vector, $g = \mathrm{id}$. □

If we take the quotient in the complex setting, $\mathrm{Hilb}^n\,\mathbb{C}$ is smooth.

Note 2.9. $\tilde{H}$ is not closed and $\mathrm{Hilb}^n\,\mathbb{C}^2 \neq \overline{\tilde{H}}//GL_n = \mathrm{Spec}\,\mathbb{C}[\overline{\tilde{H}}]^{GL_n}$, because in $\overline{\tilde{H}}$ some orbits have the same closure.

There is an algebraic construction of $\overline{\tilde{H}}$: the GIT quotient.

In general, let X be affine and G an affine algebraic group acting on X. Let $\chi : G \to \mathbb{C}^*$ be a appropriate character. $x \in X$ is *stable* if $G \cdot (x,z)$ is closed in $X \times \mathbb{C}$, where G acts on $X \times \mathbb{C}$ by $g \cdot (x,z) := (g \cdot x, \chi(g)z)$. We denote by X^s the set of stable point (*stable locus*).

Exercise 2.10. For $G = GL_n$, $X = \mathbb{C}^n$ and $\chi = \det^{-1}$, this is the same as v is cyclic.

The *GIT quotient* X^s/G can be defined as $\mathrm{Proj}(\sum \mathbb{C}[X]^{G,\chi^l})$, where

$$\mathbb{C}[X]^{G,\chi^l} := \{f \in \mathbb{C}[X] : g \cdot f = \chi^l(g)f, g \in G\}.$$

Theorem 2.11 (Mumford)**.** *$X^s/G(complex) = \mathrm{Proj}(\sum \mathbb{C}[X]^{G,\chi^l})$.*

For our case, $G = \mathbb{C}^*$, and $\mathbb{C}^2/\mathbb{C}^* = \mathbb{P}_1 \neq \mathbb{C}[x,y]^{\mathbb{C}^*} = \mathbb{C}$.

Consider $\chi : \mathbb{C}^* \to \mathbb{C}^*$ the inversion map:

$$\mathbb{C}^2/\mathbb{C}^* = \mathrm{Proj}_\chi\,\mathbb{C}[x,y] = \mathrm{Proj}(\sum_{l \geq 0} \mathbb{C}[X]^{\mathbb{C}^*,\chi^l}).$$

Now we have

$$\mathbb{C}[X]^{\mathbb{C}^*,\chi^l} = \{f|\lambda * f = \chi(\lambda)^d f = \lambda^{-d} f\} = \{f|\deg f = d\}.$$

Call $N = |G|$. We use the previous construction to define $\widetilde{\mathbb{C}^2/G} \subseteq \operatorname{Hilb}^N \mathbb{C}^2$. Note that G acts on $S^N\mathbb{C}^2$ by $g(x_1, ..., x_N) = (gx_1, ..., gx_N)$. If $g(x_1, ..., x_N) = (x_1, ..., x_N)$, then $g(x_1, ..., x_N) = Gx_1$. If any $x_i \neq 0$, then Gx_i is a single object, otherwise is $(0, ..., 0)$. Therefore $(S^N\mathbb{C}^2)^G = \mathbb{C}^2/G$. In that case, define $\widetilde{\mathbb{C}^2/G} := \overline{\pi^{-1}(\mathbb{C}^2/G)}$: $\widetilde{\mathbb{C}^2/G} \subseteq (\operatorname{Hilb}^N \mathbb{C}^2)^G$ is a connected component (not the whole thing). We have:

$$\begin{tikzcd}
\widetilde{\mathbb{C}^2/G} \arrow[r, hook] \arrow[d, "\pi_G"] & \operatorname{Hilb}^N \mathbb{C}^2 \arrow[d, "\pi"] \\
\mathbb{C}^2/G \arrow[r, equal] & (S^N\mathbb{C}^2)^G \arrow[r, hook] & S^N\mathbb{C}^2.
\end{tikzcd}$$

Proposition 2.12. *$\widetilde{\mathbb{C}^2/G}$ is smooth.*

Proof. As $\widetilde{\mathbb{C}^2/G}$ is irreducible (it is the closure of $\pi^{-1}(\mathbb{C}^2/G \setminus 0) \cong \mathbb{C}^2/G \setminus 0$), it is enough to show:

Lemma 2.13. *Let X be smooth, and G a finite group acting on X. Then each component of X^G is smooth.*

Proof. As G is finite, it is enough to prove that if $g \in \operatorname{Aut}(X)$ has finite order, then each component of X^g is smooth.

For this, $X^g \subseteq X$ is the fixed locus of g, so it is enough to prove that $\dim\ker(dg - \mathrm{Id})$ ($= \dim T_x X^g$ for $x \in X^g$) is constant on each connected component of X^g.

Note that dg must have finite order as an endomorphism of T_xX for $x \in X^g$, so its eigenvalues are roots of unity. These eigenvalues and the multiplicity remain constant on each connected component. □

□

Corollary 2.14. $\pi_G : \widetilde{\mathbb{C}^2/G} \twoheadrightarrow \mathbb{C}^2/G$ *is a resolution of singularities.*

Proof. π is an isomorphism restricted to $\pi_G^{-1}(\mathbb{C}^2/g \setminus 0) \to \mathbb{C}^2/g \setminus 0$ because pi is an isomorphism over the locus of N distinct points. □

Remark 2.15. If X is a symplectic variety of dimension $2n$ and ω is the symplectic form, $\omega^{\wedge n}$ is a volume form so $\mathcal{O}_X \cong K_X$.

Now $S^N\mathbb{C}^2 - \{\text{n distinct points}\}$ is holomorphic algebraic symplectic, using the symplectic structure on $\mathbb{C}^2$. This extends to $\operatorname{Hilb}^N \mathbb{C}^2$ and G acts symplectically, so each connected component of $(\operatorname{Hilb}^N \mathbb{C}^2)^G$ is symplectic. Therefore $\widetilde{\mathbb{C}^2/G}$ is symplectic and $\mathcal{O}_{\widetilde{\mathbb{C}^2/G}} \cong K_{\widetilde{\mathbb{C}^2/G}}$; i.e. $\pi_G : \widetilde{\mathbb{C}^2/G} \twoheadrightarrow \mathbb{C}^2/G$ is a crepant resolution (in fact minimal).

Let X be an affine variety over $\mathbb{C}$ and G a reductive group acting on X: $X//G = \operatorname{Spec}\mathbb{C}[X]^G$. Consider the quotient map $X \twoheadrightarrow X//G$. If G acts freely, $X//G$ is smooth; moreover, $X \twoheadrightarrow X//G$ is a principal G-bundle. This is an special case of:

Luna's slice Theorem: (Roughly:) If $x \in X$ has isotropy group $H < G$ and H is reductive, there exists an slice S to the G-action, $H \cdot S = S$, satisfying some nice conditions.

Example 2.16. For $\mathbb{C}^2 \setminus \{0\}$ and $G = \mathbb{C}^*$, we want to obtain $\mathbb{P}_1$. But $\mathbb{C}^2//\mathbb{C}^* = \{*\}$, not $\mathbb{P}_1$.

However we can localize $\mathbb{C}^2 \setminus \{0\}$, e.g. to consider $\mathbb{C}^2 \setminus \{x = 0\}$ and $\mathbb{C}^2 \setminus \{y = 0\}$, which have nice quotient, that glue to obtain $\mathbb{P}_1$. $\mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{P}_1$ is a good geometric quotient (i.e. covered by open sets with this property).

Mumford's GIT approach generalizes this example. Recall that

$$X^s = \{x \in X | G(x,z) \subseteq X \times \mathbb{C} \text{ is closed, } \forall z \in \mathbb{C}\}$$

is the stable locus (X affine). Now X^s is covered by affine open sets with a *good geometric quotient*: there exists affine open sets $U_i \subseteq X^s$ such that $U_i \twoheadrightarrow U_i//G = \operatorname{Spec} \mathbb{C}[U_i]^G$. Therefore all the orbits in U_i are sent to distinct points; i.e. all the orbits of U_i are closed.

In case that G acts freely on X^s, $X^s \twoheadrightarrow X^s//G$ (defined by gluing the U_i's or as $\operatorname{Proj}_\chi \mathbb{C}[X]$) is a principal G-bundle, and $X^s//G$ is smooth.

Example 2.17. For $\operatorname{Hilb}^n \mathbb{C}^2 = \tilde{H}/GL_n$, if $Q =$
$$\begin{tikzcd} \circ \arrow[loop left, "x"] \arrow[loop right, "y"] \\ \circ \arrow[u, "i"] \end{tikzcd}$$
, we have

$$\begin{aligned} \tilde{H} &= \Big\{M \in \operatorname{Rep}_{(n,1)} Q : [x,y] = 0, im(i) \text{ cyclic at the top vertex}\Big\} \\ &= \Big\{M \in \operatorname{Rep}_{(n,1)} Q : [x,y] = 0\Big\}^\chi, \qquad \chi := \overset{-1}{\det} : GL_n \to \mathbb{C}^*. \end{aligned}$$

Therefore we conclude from the fact that GL_n acts freely on $\tilde{H}$ that $\operatorname{Hilb}^n \mathbb{C}^2$ is smooth.

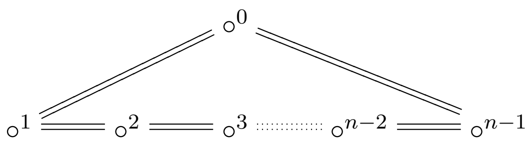
Example 2.18. Returning to $\mathbb{C}^2$, the orbits are not closed (except 0). We look at $\mathbb{C}^2 \times \mathbb{C}$, with the action of $\mathbb{C}^*$ given by $\lambda \cdot ((x,y),z) = ((\lambda x, \lambda y), \lambda^{-1} z)$. Now the orbits $\mathbb{C}^*((x,y),z)$ are closed for $z \neq 0, (x,y) \neq 0$. We can consider locally the quotient $\mathcal{U} \times \mathbb{C} \twoheadrightarrow \mathcal{U} \times \mathbb{C}//\mathbb{C}^*$. This will be geometric, moreover locally the line bundle defined by χ, $\mathcal{U} \times \mathbb{C}//\mathbb{C}^* \cong \mathcal{U}//\mathbb{C}^* \times \mathbb{C}$.

3. McKay correspondence

Recall that $\widetilde{\mathbb{C}^2/G} = \overline{\pi{-}1(\mathbb{C}/G \setminus \{0\})} \subseteq \operatorname{Hilb}^n \mathbb{C}^2$ and we have the crepant resolution of singularities: $\pi : \widetilde{\mathbb{C}^2/G} \twoheadrightarrow \mathbb{C}^2/G$.

We will prove that $\pi^{-1}(0)$ is a union of copies of $\mathbb{P}_1$, one for each vertex of the extended Dynkin diagram, where the intersection matrix between them coincides with the adjacency matrix of the diagram.

Example 3.1. Consider $G = \mathbb{Z}/m$. We have the extended Dynkin diagram



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Recall that for all $G < SL_2\mathbb{C}$ finite,

$$\begin{aligned} \mathbb{C}^2/G &= \{G - \text{orbits in } \mathbb{C}^2\} = \operatorname{Rep}_{\mathbb{C}[G]} \mathbb{C}[\mathbb{C}^2]\#G// \operatorname{Aut}_G(\mathbb{C}[G]) \\ &= \operatorname{Rep}_\delta \Pi_Q^0 // GL_\delta, \\ \widetilde{\mathbb{C}^2/G} &= \big\{\text{pairs } (M,v) \in \operatorname{Rep}_{\mathbb{C}[G]}(\mathbb{C}[\mathbb{C}^2]\#G) \times \mathbb{C}[G] \mid v \text{ is cyclic and } G\text{-invariant}\big\}// \operatorname{Aut}_G(\mathbb{C}[G]), \end{aligned}$$

and the last term can be restated in terms of the McKay quiver as follows. Define a new quiver $\widetilde{Q}$, obtained by adding to $\bar{Q}$ a new vertex denoted by ∞, and an simple arrow j from him pointing at the extending vertex. Then, we have

$$\widetilde{\mathbb{C}^2/G} = \big(\mathrm{Rep}_{\delta,1}\left(\widetilde{Q} \mid \sum_{a\in Q_1}[a,a^*]=0\right)^s\big)//GL_\delta \tag{3.2}$$
$$= \{M \in \mathrm{Rep}_{\delta,1}\widetilde{Q} \mid \sum_{a\in Q_1}[a,a^*]=0, \quad \mathrm{im}(j) \text{ is cyclic for } M\}//GL_\delta.$$

The last condition says that there are no proper submodules containing $0\oplus\mathbb{C}$ (that copy of $\mathbb{C}$ over the ∞-vertex) as subspace.

Remark 3.3. This is associated to $\chi = \prod_{i\in Q_0}\det_i^{-1}$, where $\det_i : GL_{\delta_i} \to \mathbb{C}$.

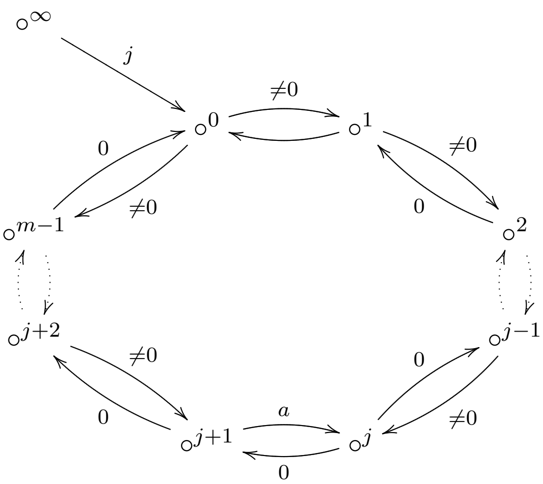
Equivalently, we can take $\widetilde{\mathbb{C}^2/G} = (\mathrm{Rep}_\delta\Pi_Q^0)^s//GL_\delta$, where $(\mathrm{Rep}_\delta\Pi_Q^0)^s$ is the space of representations for which the elementary vector at the extending vertex is cyclic.

Now $\pi^{-1}(0) \subseteq \widetilde{\mathbb{C}^2/G}$ is the set of subschemes Y of $\mathbb{C}^2$ supported at 0, satisfying $\mathbb{C}[Y]\cong\mathbb{C}[G]$. That Y is supported at 0 means that some power of (x,y) acts by zero, i.e. x,y are nilpotent.

Under Morita equivalence, $\mathrm{Rep}_\delta\mathbb{C}[x,y]\#G \cong \mathrm{Rep}_\delta\Pi_Q^0$, and this carries $\pi^{-1}(0)$ as above to the representations for which there exists $N>>0$ such that all the paths of length N act by 0.

For our example $G=\mathbb{Z}_m$, this means that any circuit in $\bar{Q}$ acts by 0 ($\delta_i=1$ for all i). In such a case, for each pair a,a^*, one of them is zero.

By the stability condition, at most one pair if opposite edges a,a^* are both 0. Otherwise, this disconnects some part of $\bar{Q}$ from extending vertex 0. Therefore, the general picture is



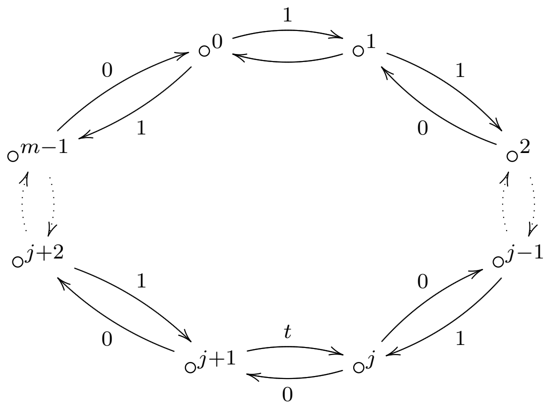
where a can be zero or non-zero.

GL_δ acts by rescaling bases at each vertex. Now

- for $a\neq 0$, consider λ_j the unique composition of non-zero arrows and inverses;
- for $a=0$, set $\lambda_j=\infty$.

This gives a bijection from the set $\pi^{-1}(0)'_j$ of isomorphism classes of representations of the above form to $\mathbb{P}^1\setminus\{0\}$. Moreover, this is an isomorphism of algebraic varieties.

However, the subvariety $\pi^{-1}(0)'_j$ is *not* closed. There is one point on the boundary, which is obtained by the limit $t \to 0$ of the following representations, call them $M_j(t)$:



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The limit is simply given by setting $t = 0$ above. Now, let $\pi^{-1}(0)_j := \overline{\pi^{-1}(0)'_j}$, which as a set is just $\pi^{-1}(0)_j = \pi^{-1}(0)'_j \sqcup \{M_j(0)\}$. The map $M_j(t) \mapsto t$ extends to an isomorphism $\pi^{-1}(0)_j \cong \mathbb{P}^1$.

Now, for each $\lambda \in \mathbb{P}^1$, let $\lambda_j \in \pi^{-1}(0)_j \subseteq \pi^{-1}(0)$ denote the corresponding point. We see from the above that $0_j = \infty_{j-1}$ and $\infty_j = 0_{j+1}$; moreover, there are no other intersections. This proves the McKay correspondence theorem in this case.

Generalization: For any $G < SL_2\mathbb{C}$, the main difference is that $\delta_i > 1$ for some i. Above for $a \neq 0$, the distinguishing feature of $j \in Q_0$ is the fact that the simple S_j is a submodule of V; i.e. there exists $v \in V_j$ such that v is in the kernel of all the outgoing arrows from j.

We define $\pi^{-1}(0)_i := \{V \in \mathrm{Rep}_\delta : V \subseteq S_j \text{ as subrepresentation of } \bar{Q}\}$.

Theorem 3.4 (Crawley-Boewey, '99)**.** *$\pi^{-1}(0)_i \cong \mathbb{P}^1$, and $\pi^{-1}(0)_i \cap \pi^{-1}(0)_j = \begin{cases} \{*\}, & i,j \text{ adjacent;} \\ 0, & \text{otherwise.} \end{cases}$*

This proves the McKay correspondence Theorem.

Geometric version: There exist locally free sheaves $\mathcal{R}_i$, $i \in Q_0 \setminus \{0\}$, a basis for $K(\widetilde{\mathbb{C}^2/G})$, such that the intersection matrix is the Cartan matrix corresponding to the Dynkin quiver.

If $\mathcal{R}$ is the universal sheaf of rank $|G|$ on $\mathrm{Hilb}^{|G|}\,\mathbb{C}^2 \supseteq \widetilde{\mathbb{C}^2/G}$, then

$$\mathcal{R} = \oplus_i \rho_i \otimes \mathcal{R}_i,\ \mathcal{R}_i \text{ a sheaf of rank } \dim\rho_i.$$

3.1. **Derived categories and derived equivalences.** We will give a somewhat informal treatment of derived categories and equivalences. We recommend Weibel's book, or Căldăraru's notes, arXiv:math/0501094, for more details.

Given an abelian category $\mathcal{A}$ (e.g. $\mathrm{Coh}(\mathbb{P}^1)$), one can define $\mathrm{Ch}(\mathcal{A})$ as the category of cochain complexes in $\mathcal{A}$. Similarly, $\mathrm{Ch}^b(\mathcal{A}), \mathrm{Ch}^+(\mathcal{A})$, and $\mathrm{Ch}^-(A)$ are the subcategories of cochain complexes which are bounded (finitely many nonzero terms), bounded below (i.e., $C^\cdot \in \mathrm{Ch}^+(\mathcal{A})$ means that there exists N such that $C^i = 0$ for $i < N$), and bounded above; note that $\mathrm{Ch}^b(A) = \mathrm{Ch}^+(A) \cap \mathrm{Ch}^-(A)$.

What we want is, given $0 \to M \hookrightarrow N \twoheadrightarrow X \to 0$, to identify $(M \hookrightarrow N) \simeq X$. This allows to say:

- when $\mathcal{A} = A - mod$, $A/(x) \simeq (A \to^{\cdot x} A)$;
- when $\mathcal{A} = \mathrm{Coh}(P^1)$, $\mathcal{O}_{[0:1]} \simeq (\mathcal{O}(-1) \to^{\cdot x})\mathcal{O})$, and similarly $\mathcal{O}_{[\lambda:\mu]} \simeq (\mathcal{O}(-1) \to^{\cdot(\lambda y - \mu x)} \mathcal{O})$.

To get $\mathcal{O}(1)$, $\mathcal{O}(-1) \xhookrightarrow{\cdot(x,y)} \mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}(1)$. Therefore all the elements of $\mathrm{Coh}(\mathbb{P}^1)$ can be rewritten as complexes using only $\mathcal{O}, \mathcal{O}(-1)$ (at least, all terms can be produced from $\mathcal{O}, \mathcal{O}(-1)$ by iterated kernels and cokernels).

Definition 3.5. $\mathcal{K}(\mathcal{A}) :=$ *homotopy category of* $\mathcal{A}$: the objects are objects of $\mathrm{Ch}(\mathcal{A})$, and the morphisms are cochain maps modulo nullhomotopic ones.

Recall that a nullhomotopic map is a cochain map $f : C^{\cdot} \to D^{\cdot}$ such that $f = dh + hd$ for some linear map $h : C^{\cdot} \to D^{\cdot -1}$.

Definition 3.6. $\mathcal{D}(\mathcal{A}) :=$ *derived category of* $\mathcal{A}$: this is the category obtained from $\mathcal{K}(\mathcal{A})$ by formally inverting quasi-isomorphisms. Thus, the objects are the same, and the morphisms are obtained by compositions of formal inverses of quasi-isomorphisms of complexes and actual cochain maps, modulo nullhomotopic maps.

The categories $\mathcal{K}(\mathcal{A}), \mathcal{D}(\mathcal{A})$ are no longer abelian categories (we no longer have a notion of short exact sequence), but is rather *triangulated categories*. Roughly, a triangulated category $\mathcal{C}$ replaces exact sequences by *exact* or *distinguished triangles*: these are sequences of maps of the form $X \to Y \to Z \to X[-1]$, where the shift $[-1]$ is an autoequivalence of the category. In the case $0 \to X \to Y \to Z \to 0$ was short exact in $\mathcal{A}$, there will always be an exact triangle of the form $X \to Y \to Z \to X[-1]$.

This is all the information we need to obtain a long exact sequence on homology: applying any suitable functor H^* to the sequence, we obtain a sequence of cohomology groups $H^0(X) \to H^0(Y) \to H^0(Z) \to H^1(X) \to \cdots$. For instance, the functor could be $H^0(X) = \mathrm{Hom}(T, X)$ for some test object T, so then $H^i(X) = \mathrm{Hom}(T, X[-i])$. In the case of the derived category, this would be nothing but $\mathrm{Ext}^i(T, X)$.

While triangulated categories do not have kernels or cokernels, they instead have the axiom that any map $f : X \to Y$ completes to a distinguished triangle $X \to Y \to Z \to X[-1]$. We may think of Z as Y/X. In the case of the derived category $\mathcal{D}(\mathcal{A})$, we actually will have $Z = \mathrm{cone}(f)$, the mapping cone of f, which in the case $X, Y \in \mathcal{A}$ has homology equal to the usual quotient $\mathrm{coker}(f) = Y/f(X)$.

Precisely, we define triangulated categories as below:

Definition 3.7. A triangulated category $\mathcal{C}$ is a $\mathbb{C}$-linear category (or $\mathbf{k}$-linear, or additive) equipped with an autoequivalence $[1] : X \mapsto X[1], X \in \mathcal{C}$, as well as a collection of *distinguished triangles*,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1], \tag{3.8}$$

satisfying the following axioms:

(1) Given any map $f : X \to Y$, there exists a distinguished triangle including f:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1]. \tag{3.9}$$

We typically will have a particular Z, g, h in mind, and will call Z the *cone* of f, denoted by $\mathrm{cone}(f)$.

Moreover, when $X = Y$ and $f = \mathrm{Id}_X$, then one may take $Z = 0$. (For us, we will always set $\mathrm{cone}(\mathrm{Id}_X) := 0$.)

T2 (The rotation axiom): Given a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1]$, then the following triangle is also distinguished:

$$Y \xrightarrow{g} Z \xrightarrow{h} X[-1] \xrightarrow{f[-1]} Y[-1], \tag{3.10}$$

and conversely.

T3 (The fill-in axiom):[1] Given maps a, b forming a commutative diagram whose rows form distinguished triangles, there exists a third map c which makes the following diagram commute:

(3.11)
$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, dashed, "c"] & X[-1] \arrow[d, "a[-1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[-1] \end{tikzcd}$$

T4 (Verdier's octahedral axiom): See `http://www.math.uchicago.edu/~may/MISC/Triangulate.pdf`. Roughly, if we interpret cone$(f : X \to Y)$ as Y/X, then this axiom says that $Z/Y \cong (Z/X)/(Y/X)$. More precisely, if $X \to Y \to Y/X$, $X \to Z \to Z/X$, and $Y \to Z \to Z/Y$ are part of three distinguished triangles, then there exist maps $Y/X \to Z/X \to Z/Y$ forming a distinguished triangle together with the composition $Z/Y \to Y[-1] \to Y/X[-1]$, making the whole obtained diagram commute.

In the case of the derived category $\mathcal{D}(\mathcal{A})$ and its bounded variants, the shift is the shift of degrees in complexes, and the distinguished triangles are those which are isomorphic to a triangle of the form

(3.12)
$$X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[-1],$$

where, as we recall, isomorphic in the derived category means connected by a sequence of quasi-isomorphisms of chain complexes and formal inverses of such.

Example 3.13. For $\mathbb{P}^1$, and $p \in \mathbb{P}^1$ we have the short exact sequence in $\mathrm{Coh}(\mathbb{P}^1)$:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_p \to 0.$$

In $\mathcal{D}(\mathbb{P}\ 1)$ it turns to

$$\ldots \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_p \to \mathcal{O}(-1)[-1] \to \ldots\ .$$

We can take global sections:

$$H^0(\mathcal{O}(-1)) \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_p) \to H^1(\mathcal{O}(-1)) \to \ldots\ .$$

Or more generally, take $\mathrm{Hom}(\mathcal{F}, -)$ (we recover the above line for $\mathcal{F} = \mathcal{O}$):

$$\mathrm{Hom}(\mathcal{F}, \mathcal{O}(-1)) \to \mathrm{Hom}(\mathcal{F}, \mathcal{O}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{O}_p) \to \mathrm{Hom}(\mathcal{F}, \mathcal{O}(-1))[-1] \to \ldots\ .$$

In general, for every triangulated category $\mathcal{C}$ and every $Z \in \mathcal{C}$,

Proposition 3.14. *Applying* $\mathrm{Hom}(Z, -)$ *turns a distinguished triangle into a long exact sequence.*

In the case $\mathcal{C} = \mathcal{D}^b(\mathcal{A})$, the derived category of an abelian category $\mathcal{A}$, then more is true: if there are enough injectives, we may define $R\,\mathrm{Hom}(Z, -) : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathrm{Vect})$ by $R\,\mathrm{Hom}(Z, Y) = \mathrm{Hom}(Z, C.)$ for $C.$ an injective resolution of Y (or, if there are enough projectives, we can define $R\,\mathrm{Hom}(-, Z)$ similarly). Then, we have that $R\,\mathrm{Hom}(Z, -)$ turns a distinguished triangle in $\mathcal{D}^b(\mathcal{A})$ into a distinguished triangle in $\mathcal{D}^b(\mathrm{Vect})$, and then applying the homology functor, one obtains the long exact sequence above. For more on this, see §3.9 below.

In general, what is the meaning of $\mathrm{Hom}(X, Y)$ in $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$? In the case that $X, Y \in \mathcal{A}$, this is just homomorphisms in $\mathcal{A}$ up to homotopy. Let us more generally consider the case that X, Y are shifts of objects in $\mathcal{A}$, but not necessarily in $\mathcal{A}$ itself:

Claim 3.15. *For* $X, Y \in \mathcal{A} \subset \mathcal{D}^b(\mathcal{A})$, $\mathrm{Hom}(X, Y[-n]) = \mathrm{Ext}^n(X, Y)$.

[1] Note: this is implied by the other three axioms: see `http://www.math.uchicago.edu/~may/MISC/Triangulate.pdf`.

Proof. (sketch) Write elements of $\mathrm{Ext}^n(X, Y)$ as an exact sequence

$$Y \to M_n \to M_{n-1} \to \cdots \to\to M_1 \to X,$$

so $(Y \to M_n \to M_{n-1} \to \cdots \to\to M_1) \cong X$. We have

$$\begin{tikzcd}
\cdots & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & X \arrow[d] & \cdots \\
\cdots & Y \arrow[r] \arrow[d] & M_n \arrow[r] & \cdots \arrow[r] & M_2 \arrow[r] & M_1 & \cdots \\
\cdots & Y \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 & \cdots
\end{tikzcd}$$

and the composition of these is our element of $\mathrm{Hom}(X, Y[-n])$. $\square$

Basic fact: If $\mathcal{A}$ has enough projectives, then $\mathcal{D}^-(\mathcal{A}) \sim \mathcal{K}^-(\text{proj in } \mathcal{A})$: in particular, every chain complex is quasi-isomorphic to a complex of projectives, and if we work with complexes of projectives $C_., C'_.$, every quasi-isomorphism $f : C_. \xrightarrow{\sim} C'_.$ actually has a quasi-inverse g, so that $fg - \mathrm{Id}_{C'_.}$ and $gf - \mathrm{Id}_{C_.}$ are both nullhomotopic.

Similarly, if $\mathcal{A}$ has enough injectives, $\mathcal{D}^+(\mathcal{A}) = \mathcal{K}^+(\text{inj in } \mathcal{A})$.

In the case of $\mathcal{D}^-(\mathbb{P}^1) := \mathcal{D}^-(\mathrm{Coh}(\mathbb{P}^1))$, or in general of $\mathcal{D}^-(X) := \mathcal{D}^-(\mathrm{Coh}(X))$, there are *not* enough projectives, and so one typically uses instead complexes of locally free sheaves (in particular locally flat; i.e. if P is locally free, we have that $M \mapsto P \otimes M$ is exact). We will explain this more below.

Given a right exact functor $F : \mathcal{A} \to \mathcal{B}$, then there exists a triangulated functor $LF : \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{B})$, obtained by applying F directly to complexes of projectives. More generally, one may apply F to complexes of F-acyclic objects: an F-acyclic object X is one such that $H^*(LF(X)) = F(X)$, concentrated in degree zero. An example of the latter is, for $\mathcal{A} = A - mod$ where A is commutative, and $M \in \mathcal{A}$ arbitrary, the case $F(-) = - \otimes_A M$, where the F-acyclic objects X include all flat A-modules: i.e., objects X such that, if $Y_.$ is an exact complex of A-modules, then $X \otimes_A Y_.$ is also an exact complex. If A is noncommutative, we can say the same words about the category of A-bimodules.

Another example is $\mathcal{A} = \mathrm{Coh}(X)$, $M \in \mathcal{A}$ arbitrary, $F(-) = - \otimes_A M$. In this case, there are not enough projectives, but one can give an alternative, universal definition of the derived functor LF, and all locally free sheaves are F-acyclic. We can therefore compute LF by $LF(X) = F(C_.)$ where $C_.$ is a bounded-above complex of locally free sheaves quasi-isomorphic to X.

More precisely, we have the following:

Theorem 3.16. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact functor. Given any $X \in \mathcal{D}^-(\mathcal{A})$, and any complex of projectives $M_.$ quasi-isomorphic to X, then $LF(X) := F(M_.) \in \mathcal{D}^-(\mathcal{B})$ does not depend on the choice of $M_.$, up to quasi-isomorphism. If there are enough projectives, this yields a triangulated functor LF called the* left derived functor *of F. Similarly, this is true for F left-exact, replacing $\mathcal{D}^-(\mathcal{A})$ with $\mathcal{D}^+(\mathcal{A})$ and projectives with injectives. If there are enough injectives, this yields a well-defined triangulated functor RF, the* right derived functor *of F.*

Definition 3.17. If $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor, then $X \in \mathcal{A}$ is called F-acyclic if $LF(X)$ is quasi-isomorphic to $F(X) \in \mathcal{B}$. Similarly, if F is right-exact, X is called F-acyclic if $RF(X)$ is quasi-isomorphic to $F(X)$.

Proposition 3.18. *Let F be left-exact. If $X \in \mathcal{D}^-(\mathcal{A})$ and $M_.$ is a complex of F-acyclic objects quasi-isomorphic to X, then $LF(X) \simeq F(M_.)$. Similarly, if F is right-exact, $X \in \mathcal{D}^+(\mathcal{A})$, and $M_.$ is a complex of F-acyclic objects quasi-isomorphic to X, then $RF(X) \simeq F(M_.)$.*

More generally, we have the

Proposition 3.19. *Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a functor of abelian categories. If $\mathcal{S} \subset \mathcal{A}$ is any collection of objects such that, for any exact cochain complex $C.$ with $C_i \in \mathcal{S}$ for all i, $F(C.)$ is also exact, then F preserves quasi-isomorphisms of chain complexes of objects in $\mathcal{S}$: $C. \simeq C'_.$ for $C_i, C'_i \in \mathcal{S}$ implies that $F(C.) \simeq F(C'_.)$.*

If $\mathcal{S}$ is an additive subcategory of $\mathcal{A}$, we obtain well-defined functors $\mathcal{K}(\mathcal{S}) \to \mathcal{K}(\mathcal{B})$ and $\mathcal{D}(\mathcal{S}) \to \mathcal{D}(\mathcal{B})$, and in nice cases, $\mathcal{D}(\mathcal{S})$ is equivalent to $\mathcal{D}(\mathcal{A})$ and therefore one obtains a triangulated functor. This is certainly true whenever F is left-exact, $\mathcal{A}$ has enough projectives, and $\mathcal{S}$ includes the projectives (e.g., $\mathcal{A} = A - mod$ for A commutative, $F(-) = - \otimes_A M$, and $\mathcal{S}$ consists of flat A-modules). However, one can also obtain a well-defined functor in other cases, for example, when $\mathcal{A} = \mathrm{Coh}(X)$ and $\mathcal{S}$ is the collection of locally free sheaves.

In general, however, even if the above procedure yields a well-defined triangulated functor $\widetilde{F}$, and the derived functor LF also exists, the two functors *need not* be the same: see the following example.

Example 3.20. *Reflection functors:* Let Q be a quiver and i a sink vertex; denote by Q' the quiver obtained by reversing the arrows at i: we have the reflection functor $F_i^+ : \mathrm{Rep}Q \to \mathrm{Rep}Q'$.

$$\begin{tikzcd}
 & V_j \arrow[dr] & & V_k \arrow[dl] & & V_j & & V_k . \\
F_i^+ : & & V_i \arrow[rr, squiggly] & & & & \ker(\oplus V_j \to V_i) \arrow[ull] \arrow[ur] \arrow[d] & \\
 & & \cdots \arrow[u] & & & & \cdots &
\end{tikzcd}$$

Reflection functors were bad when we consider $S_i \in \mathrm{Rep}Q$, $F_i^+(S_i) = 0$.

To fix this, we replace the term $\ker(\bigoplus_{j\to i} V_j \to V_i)$ above by the *complex* $(\bigoplus_{j\to i} V_j \to V_i)$ itself, which is quasi-isomorphic to the above kernel when $\bigoplus_{j\to i} V_j \to V_i$ is surjective (which was the nice case when we got an equivalence before).

This yields a triangulated equivalence: $M \mapsto F_i^+ M$ if M does not have S_i as summand, and $S_i \mapsto S_i[1]$; the inverse is the corrected F_i^- $(S_i \mapsto S_i[-1])$.

Call $\widetilde{F}_i^+$ the new functor: it is not the right or left derived functor of F_i^+, only guaranteed to exist if F_i^+ is right or left exact.

Claim 3.21. *F_i^+ is right exact.*

Proof. We know that F_i^+, F_i^- are inverse one of other on modules without S_i as summand. But $M \to S_i$ is always split because i is a sink. The only indecomposable M such that $M \to S_i$ can be non zero is $M = S_i$. □

Check that:

- projectives are acyclic,
- modules without S_i as summand are acyclic,

but not the union of these. The first family defines LF_i^+, which satisfies $LF_i^+(S_i) = 0$; the second defines $\widetilde{F}_i^+$.

Recall before a fact about the Morita equivalence: given an A, B-bimodule P and a B, A-bimodule Q such that $P \otimes_B Q = A$, $Q \otimes_A P = B$, the equivalence is given by:

$$\begin{tikzcd} A - mod \arrow[r, "\sim"] & B - mod, \\ M \arrow[r, mapsto] & M \otimes_A P, \\ N \otimes_B Q & N. \arrow[l, mapsto] \end{tikzcd}$$

For the *derived setting*, we want P, Q to be elements of $\mathcal{D}^b(X \times Y)$ ('X, Y-bimodules'):

$$\begin{tikzcd} \mathcal{D}^b(X) \arrow[r, shift left, "\otimes P"] & \mathcal{D}^b(Y) \arrow[l, shift left, "\otimes Q"] \end{tikzcd} .$$

As a particular case of the above, recall before that, given an algebra A, $A \cong \sum_i P_i^{\oplus r_i}$, where P_i are the indecomposables A-modules, $r_i \geq 1$, and $Ae_i = P_i$ for e_i idempotent. Then $A \simeq \mathrm{End}_A(\oplus P_i)$.

Replacing P_i's by generators of $\mathcal{D}^b(X)$:

$$\mathcal{D}^b(X) \simeq \mathcal{D}^b(R\mathrm{End}_{\mathcal{D}^b(X)}(\oplus_i P_i))^{op} - mod.$$

In general, $\mathrm{End}_{\mathcal{D}^b(X)}(\oplus_i P_i)^{op}$ is a *DG-algebra*: that is, an associative commutative /algebra B provided of a derivation d (i.e. $d(xy) = d(x)y + xd(y)$ for all $x, y \in B$) such that $d^2 = 0$.

Example 3.22. $\mathcal{D}^b(\mathbb{P}^1)$ is generated by $\mathcal{O}, \mathcal{O}(-1)$, so

$$\mathcal{D}^b(\mathrm{Coh}\,\mathbb{P}^1) \simeq REnd\,(\mathcal{O} \oplus \mathcal{O}(1))^{\mathrm{op}} - mod.$$

If we had $\mathrm{Ext}^i(\mathcal{O} \oplus \mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-1)) \neq 0$ for some $i \neq 0$, this would be a DG algebra. This is not the case here, because

$$\mathrm{Ext}^i(\mathcal{O}, \mathcal{O}) = \mathrm{Ext}^i(\mathcal{O}(-1), \mathcal{O}) = 0, \qquad \forall i \neq 0.$$

We get $\mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1)) \simeq \mathcal{D}^b(\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1)) - mod$; note that

$$\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \cong \langle x_0, x_1 \rangle, \quad \mathrm{End}(\mathcal{O}(-1)) = \mathrm{End}(\mathcal{O}) = \mathbb{C},$$

so this algebra corresponds to the path algebra of Kronecker quiver $\begin{tikzcd} \circ \arrow[r, bend left] \arrow[r, bend right] & \circ \end{tikzcd}$.

Lectures 20–22 follow:

3.2. **More words on reflection functors.** Some final comments on reflection functors: Let Q be a quiver with $i \in Q_0$ a sink, and Q' the associated quiver with a source at i, obtained by reversing all arrows at i. Recall again that F_i^+, F_i^- induce equivalences

$$\begin{tikzcd} \{\text{Reps of } Q \text{ without } S_i \text{ as a summand}\} \arrow[r, bend left, "F_i^+"] & \{\text{Reps of } Q' \text{ without } S_i \text{ as a summand}\}, \arrow[l, bend left, "F_i^-"] \end{tikzcd} \tag{3.23}$$

but that $F_i^{\pm}(S_i) = 0$.

Notation 3.24. For any quiver Q, let $\mathcal{R}ep(Q)$ denote the category of representations of Q.

Last time, we explained a natural replacement $\widetilde{F_i^+} : \mathcal{D}^b(\mathcal{R}ep(Q)) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{R}ep(Q'))$ and similarly $\widetilde{F_i^-}$ which replaces placing $\ker(\bigoplus_{j\to i} V_j \to V_i)$ at i with placing the map $(\bigoplus_{j\to i} V_j \to V_i)$ itself. It follows that $\widetilde{F_i^\pm}(S_i) = S_i[\pm 1]$ and hence that $\widetilde{F_i^\pm}$ are in fact quasi-inverse triangulated equivalences.

Observe that, under the natural map to K-theory, $\mathcal{D}^b(\mathcal{R}ep(Q)) \to K(\mathcal{R}ep(Q)) \cong \mathbb{Z}^{Q_0}$, sending a complex $M.$ to the alternating sum $\sum_{i\in\mathbb{Z}}(-1)^i[M_i]$, it follows that the reflection functors $\widetilde{F_i^\pm}$ act on K-theory by the simple reflections (since $[S_i] \mapsto (-1)^{\pm 1}[S_i] = -[S_i]$. One summarizes this by saying that "the functors $F_i^\pm$ categorify the simple reflection s_i."

Note that this is *not* true for the derived functors of $F_i^\pm$, although, as mentioned last time, they do exist: F_i^+ is right exact and F_i^- is left exact (this is true because, in $\mathcal{R}ep(Q)$, S_i is projective, and since it is one-dimensional, any nonzero map $X \to S_i$ must split: we have $X \cong X' \oplus S_i$ and the map factors as the projection $X \cong (X' \oplus S_i) \twoheadrightarrow S_i$; similarly for $\mathcal{R}ep(Q')$ using that there S_i is injective). Since $\mathcal{R}ep(Q)$ has enough projectives, the derived functor LF_i^+ exists and is given by applying F_i^+ to complexes of projectives. However, since S_i is projective, this means that $LF_i^+(S_i) = F_i^+(S_i) = 0$, which is not the case for $\widetilde{F_i^+}$. This is similarly true for $\widetilde{F_i^-}$.

Instead of being the derived functor, the functor $\widetilde{F_i^+}$ can be viewed as given from F_i^+ by Proposition 3.19, using the collection $\mathcal{S} \subset \mathcal{D}^b(\mathcal{R}ep(Q))$ of modules without S_i as a summand, which does not include the projective module S_i; similarly for $\widetilde{F_i^-}$.

3.3. **Integral transforms / Fourier-Mukai functors.** Let X, Y be (not necessarily affine) smooth complex algebraic varieties. There is a general philosophy that all (covariant) functors $\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ "which arise in nature" are given by the following construction: Consider the diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y. \end{tikzcd} \tag{3.25}$$

Then, for any $\mathcal{F} \in \mathcal{D}^b(X \times Y)$, consider the functor $\Phi^{\mathcal{F}}_{X\to Y}$ given by

$$\Phi^{\mathcal{F}}_{X\to Y}(\mathcal{G}) = R(\pi_Y)_*(\pi_X^*(\mathcal{G}) \otimes^L \mathcal{F}). \tag{3.26}$$

Here, we used that π_X^* already is exact, and hence its derived functor is just given by applying the functor itself to any complex of objects (i.e., all objects are π_X^*-acyclic). We note also that, in the case X and Y are not smooth and projective, the above functor may not be well-defined (at least on the level of $\mathcal{D}^b(\mathrm{Coh}(X)) \to \mathcal{D}^b(\mathrm{Coh}(Y))$), but let us ignore this issue.

These are called *integral transforms with kernel* $\mathcal{F}$, or sometimes, *Fourier-Mukai functors*, where the second name originates from the following main example, which is the most famous functor as above.

Mukai considered the case when $X = Y =$ an elliptic curve, and $\mathcal{F}$ is the *Poincaré sheaf*, which we will define in a moment. In this case, the above functor is an analogue of the Fourier transform, and was first considered by Mukai. Hence, it is called the Fourier-Mukai transform. The Poincaré sheaf is constructed as follows. We view Y as the (connected component of the trivial bundle of the) moduli space line bundles on X, which is noncanonically isomorphic to X itself: picking a basepoint $x_0 \in X$, then the line bundle associated with $x \in X$ is the bundle $\mathcal{O}(x - x_0)$ associated to the divisor $x - x_0$. Then, $\mathcal{F}$ is essentially given by the condition that (where $Y = X$ but Y still denotes the second copy of the product $X \times Y$ above).

$$\mathcal{F}|_{\pi_Y^{-1}(y)\cong X} = \mathcal{O}(y - x_0), \quad \mathcal{F}|_{\pi_X^{-1}(x)\cong Y} = \mathcal{O}(x - x_0). \tag{3.27}$$

The reason this is analogous to the Fourier transform is because the Fourier transform similarly takes functions on X to functions on $X^\vee$ via the dual pairing $\chi : X \times X^\vee \to \mathbb{C}^\times$. A main example is $X = \mathbb{R}^n = X^\vee$, and $\chi(x, y) = e^{2\pi i(x\cdot y)}$. Then, we have

$$F(f)(y) = \int_X f(x)\chi(x, y)dx = \text{“}(\pi_Y)_*(\pi_X^* f(x, y) \otimes \chi(x, y))\text{”}, \tag{3.28}$$

where $\pi_X^* f(x, y) := f(x)$, $\otimes$ is just multiplication, and the pushforward $(\pi_Y)_*$ is integration along the fibers $\pi_Y^{-1}(y) \cong X$.

3.4. **Pushforwards, the identity functor, and the diagonal.** Given any map of algebraic varieties $f : X \to Y$, we have a functor $\mathcal{D}(X) \to \mathcal{D}(Y)$; if it is a smooth map, it exists on the level of $\mathcal{D}^b$. By the above philosophy, this should be given by an integral transform, $\Phi^{\mathcal{F}_f}_{X\to Y}$.

Question: What is $\mathcal{F}_f$?

The answer is quite simple: $\mathcal{F}_f = \mathcal{O}_{\Gamma_f}$, where $\Gamma_f \subset X \times Y$ is the graph of the map $f : X \to Y$. This is not difficult to verify explicitly.

In particular, the identity functor $\mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is given by an integral transform $\Phi^{\mathcal{F}}_{X\to X}$. What is the sheaf $\mathcal{F} \in \mathcal{D}^b(X \times X)$? It is nothing but the structure sheaf of the diagonal $\mathcal{O}_\Delta$: this is because the diagonal $\Delta \subset X \times X$ is the graph of the identity map.

Remark 3.29. All of the covariant triangulated functors we have considered or will consider can be obtained using integral transforms. In particular, all the derived functors of pullbacks, tensor products, and Homs from a fixed test object, can be constructed as follows. For $f : X \to Y$, we have $Lf^* = \Phi^{\Gamma_f}_{Y\to X}$, and for $\mathcal{F} \in \mathcal{D}^b(X)$, we have $- \otimes^L \mathcal{F} = \Phi^{\Delta_*\mathcal{F}}_{X\to X}$. Next, the cohomology functor $R\Gamma : \mathcal{D}^b(X) \to \mathcal{D}^b(\mathrm{Vect})$ is given by $\Phi^{\mathcal{O}_X}_{X\to\{*\}}$. For $R\,\mathrm{Hom}$, we may use the fact that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{G}))$ and the general fact that $R(f \circ g) = Rf \circ Rg$, so it suffices to describe $R\mathcal{H}om(\mathcal{F}, -) : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$. For this we may use $\mathcal{F}^{R\vee} := R\,\mathrm{Hom}(\mathcal{F}, \mathcal{O})$ and

$$R\mathcal{H}om(\mathcal{F}, -) = \mathcal{F}^{R\vee} \otimes^L -. \tag{3.30}$$

3.5. **Resolutions of the diagonal.** As a result, if we can understand the diagonal better, we gain a better understanding of the identity map. This has nontrivial consequences: given a resolution of the diagonal by nice sheaves $\mathcal{F}_i \twoheadrightarrow \mathcal{O}_\Delta$, by applying the identity functor to an arbitrary $\mathcal{G}$, we will obtain a resolution $R\pi_*(\mathcal{F}_i \otimes^L \mathcal{G})$ of $\mathcal{G}$.

We have seen this before in the affine case $X = \operatorname{Spec} A$. Here, we may take global sections of everything, and the above is saying that, given a resolution of $\Gamma(\mathcal{O}_\Delta) = A$ as a $\Gamma(\mathcal{O}_{X\times X}) = A \otimes A$-module, i.e., a bimodule resolution

$$P_\cdot \twoheadrightarrow A, \tag{3.31}$$

we obtain a resolution of any A-module M, namely

$$P_\cdot \otimes_A M \twoheadrightarrow M. \tag{3.32}$$

The reason for this is that $\pi_X^*(M) = M \boxtimes A$, and $P_\cdot \otimes^L_{A\otimes A} \pi_X^*(M) = P_\cdot \otimes^L_{A\otimes A} (M \boxtimes A) = P_\cdot \otimes_A M$ for $P_\cdot$ a complex of projective A-bimodules. Then, since X is affine, $(\pi_X)_*$ is exact, and is the forgetful functor $A-\mathrm{bimod} \to A-\mathrm{mod}$. Thus, $(\pi_X)_*(P_\cdot \otimes_A M) = P_\cdot \otimes_A M$ is indeed the resolution we considered before.

This also has a *noncommutative geometric* generalization, that we already discussed: everything goes through for $A =$ an arbitrary associative algebra. In particular, if $P_\cdot \twoheadrightarrow A$ is a projective A-bimodule resolution of A, then $P_\cdot \otimes_A M \twoheadrightarrow M$ is a left A-module resolution of M, for every $M \in A - \mathrm{mod}$.

3.6. **Koszul complexes.** One of the most important tools for constructing resolutions, including resolutions of the structure sheaf of the diagonal (for certain smooth X, and more generally of $\mathcal{O}_Y$ for certain locally complete intersections $Y \subseteq X$), is the *Koszul complex.* This goes as follows: Suppose that $f_1, \ldots, f_n$ are functions on X whose zero loci cut out the subvariety Y (i.e., the ideal of Y is (globally) generated by $f_1, \ldots, f_n$). Then, we have the map

$$f : \mathcal{O}_X \to \mathcal{O}_X^{\oplus n}, \quad 1 \mapsto (f_1, \ldots, f_n) \in \Gamma(\mathcal{O}_X^{\oplus n}). \tag{3.33}$$

Dualizing this, we obtain the exact sequence

$$(\mathcal{O}_X^{\oplus n})^\vee \xrightarrow{f^\vee} \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Y. \tag{3.34}$$

The first arrow is not, in general, injective (unless $n = 1$ and f_1 is a nonzerodivisor), so it is not a short exact sequence. However, we can complete the sequence to a complex as follows:

$$\Lambda^\bullet(\mathcal{O}_X^{\oplus n}) \twoheadrightarrow \mathcal{O}_Y, \tag{3.35}$$

where the complex $\Lambda^\bullet(\mathcal{O}_X^{\oplus n})$ is the n-th exterior power $(\mathcal{O}_X^{\oplus n})^\vee \to \mathcal{O}_X$, i.e., $(g_1 \wedge \cdots \wedge g_j) \mapsto \sum_{i=1}^{j}(-1)^i f^\vee(g_i)(g_1 \wedge \cdots \hat{g}_i \cdots \wedge g_j)$. The fibers of the above sequence at least have the right dimension generically: on $X \setminus Y$, the fibers of the above sequence are $\Lambda^\bullet\mathbb{C}^n$, and moreover it is easy to see that the sequence is exact on $X \setminus Y$ since the complex is the n-th exterior power of the sequence $\mathbb{C}^n \twoheadrightarrow \mathbb{C}$.

Proposition 3.36. *If $f_1, \ldots, f_n$ intersect transversely, or more generally, if f_{i+1} is a nonzerodivisor in $\mathbb{C}[X]/(f_1, \ldots, f_i)$ for all $1 \leq i \leq n-1$, then* (3.35) *is exact, and hence a locally free resolution of $\mathcal{O}_Y$.*

Definition 3.37. In this case, Y is called a *complete intersection* in X.

The problem with the above is that, in general, varieties $Y \subset X$ are not cut out by global functions. For example, for $X = \mathbb{P}^n$, all global functions are constant. The solution is to replace $\mathcal{O}^{\oplus n}$ by a more general locally free sheaf that has more sections, e.g., $\mathcal{O}(1)$ for $X = \mathbb{P}^n$.

Proposition 3.38. *If $\mathcal{F}$ is a locally free sheaf, $s : \mathcal{O} \to \mathcal{F}$ a global section, and $Y \subset X$ is the subscheme with ideal sheaf locally given by the vanishing of s, then, if s is locally given on local trivializations of $\mathcal{F}$ by regular sequences, the complex*

$$\Lambda^\bullet(\mathcal{F}^\vee) \twoheadrightarrow \mathcal{O}_Y \tag{3.39}$$

is exact, and hence a locally free resolution of $\mathcal{O}_Y$.

If the Koszul complex above is exact, in particular we deduce that Y is a *locally complete intersection*, i.e., X is covered by open affine neighborhoods U in which $U \cap Y$ is a complete intersection.

3.7. **Beilinson's equivalence $\mathcal{D}^b(\mathbb{P}^n) \simeq \mathcal{D}^b(\mathcal{R}ep(Q^{(n)}|R^{(n)}))$.**

Notation 3.40. For any quiver Q and any relations R, let $\mathcal{R}ep(Q|R)$ denote the category of representations of Q subject to the relations R.

Let $Q^{(n)}$ be the quiver with vertices $0, 1, \ldots, n$ and with $n+1$ arrows from i to $i+1$ for all $0 \leq i \leq n-1$, labeled $x_0, x_1, \ldots, x_n$. Let $R^{(n)}$ be the collection of relations of the form $x_i x_j = x_j x_i$, for all $i, j \in \{0, 1, \ldots, n\}$ and beginning at any fixed vertex (so, both $x_i x_j$ and $x_j x_i$ are length-two paths from k to $k+2$ for some $k \in \{0, 1, \ldots, n-2\}$).

Our next goal is to prove the

Theorem 3.41. *$\mathcal{D}^b(\mathbb{P}^n)$ is triangulated-equivalent to $\mathcal{D}^b(\mathcal{R}ep(Q^{(n)}|R^{(n)}))$.*

The main ingredient is Beilinson's resolution of the diagonal $\Delta \subset X \times X$ in the case $X = \mathbb{P}^n$. This allows us to describe explicitly the identity functor $\mathrm{Id} = \Phi^{\mathcal{O}_\Delta}_{\mathbb{P}^n \to \mathbb{P}^n}$ and hence to prove that $\mathcal{D}^b(\mathbb{P}^n)$ is generated by $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$, and then for general reasons we will deduce that $\mathcal{D}^b(\mathbb{P}^n) \simeq R\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n)) \simeq \mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n)) = \mathbb{C}Q^{(n)}$.

3.8. **Beilinson's resolution of the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$.** Recall that the bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ has an $n+1$-dimensional space of global sections, generated by $x_0, x_1, \ldots, x_n$. More generally, local sections of $\mathcal{O}(n)$ are degree-n rational functions in the x_i.

Next, recall that local sections of the cotangent bundle $\mathcal{T}^\vee$ are sums of elements gdf for f, g sections of $\mathcal{O}$. Then, there is the Euler exact sequence of vector bundles:

$$0 \longrightarrow \mathcal{T}^\vee \longrightarrow \mathcal{O}^{\oplus(n+1)}(-1) \longrightarrow \mathcal{O} \longrightarrow 0, \tag{3.42}$$

$$df \longmapsto \left(\tfrac{df}{dx_0}, \ldots, \tfrac{df}{dx_n}\right)$$

$$(f_0, \ldots, f_n) \longmapsto \textstyle\sum x_i f_i.$$

Dualizing and tensoring by $\mathcal{O}(-1)$, we obtain an exact sequence

$$0 \to \mathcal{O}^\vee(-1) \to (\mathcal{O}^\vee)^{\oplus(n+1)} \to \mathcal{T}(-1) \to 0, \tag{3.43}$$

and taking the long exact sequence on cohomology, we obtain that $\Gamma((\mathcal{O}^\vee)^{\oplus(n+1)} \xrightarrow{\sim} \Gamma(\mathcal{T}(-1))$. Denote the image of $x_i^\vee$ by $\frac{\partial}{\partial x_i} \in \Gamma(\mathcal{T}(-1))$.

If we consider $\Gamma(\mathcal{T}(-1)) = \mathrm{Hom}(\mathcal{O}(1), \mathcal{T})$, then a calculation shows that $\frac{\partial}{\partial x_i}$ has the following form:

$$\frac{\partial}{\partial x_i} : x_j \mapsto \frac{\partial}{\partial(\frac{x_i}{x_j})}, \quad \text{if } i \neq j, \tag{3.44}$$

which completely determines the element (indeed, we need only a single $j \neq i$ and everything else is determined by $\mathcal{O}$-linearity).

Now, consider the bundle $\mathcal{O}(1) \boxtimes \mathcal{T}(-1)$ on $\mathbb{P}^n \times \mathbb{P}^n$, which has a basis of global sections $x_i \frac{\partial}{\partial y_j}$, for $0 \leq i, j \leq n$, where x_i is the i-th homogeneous coordinate on the first factor and y_j is the j-th homogeneous coordinate on the second factor.

Consider the section

$$s := \sum_{i=0}^{n} x_i \frac{\partial}{\partial y_i} \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1) \boxtimes \mathcal{T}(-1)). \tag{3.45}$$

Theorem 3.46 (Beilinson)**.** *The vanishing of s is exactly the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$, yielding an exact Koszul complex*

$$\Lambda^{\geq 1}(\mathcal{O}(1)^\vee \boxtimes (\mathcal{T}(-1))^\vee) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta. \tag{3.47}$$

Proof. Checking that the vanishing of s is exactly on the diagonal is an easy local computation using (3.44) and is omitted (see Caldararu's notes, pp. 11–12). Being a bit more careful, one may see that the vanishing locus is locally given by a complete intersection, which proves that the Koszul complex is exact. ☐

Corollary 3.48. *The locally free sheaves $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ generate the derived category $\mathcal{D}^b(\mathbb{P}^n)$, up to taking cones and shifts.*

Proof. From the resolution (3.47), which we will call $K_\cdot = \Lambda^\cdot(\mathcal{O}(1)^\vee \boxtimes (\mathcal{T}(-1))^\vee)$, of the diagonal, we obtain a locally free resolution of any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ as explained above. That is, we have

$$\mathcal{F} = \mathrm{Id}(\mathcal{F}) \simeq \Phi^{\mathcal{O}_\Delta}_{\mathbb{P}^n \to \mathbb{P}^n}(\mathcal{F}) \simeq \Phi^{K_\cdot}_{\mathbb{P}^n \to \mathbb{P}^n}(\mathcal{F}) = \Lambda^\cdot(\mathcal{O}(1)^\vee) \otimes_{\mathbb{C}} \Lambda^\cdot(\Gamma((\mathcal{T}(-1))^\vee \otimes \mathcal{F})). \tag{3.49}$$

We can also see directly that the last term is quasi-isomorphic to $\mathcal{F}$ over an open affine $U \subseteq \mathbb{P}^n$ by viewing (3.47) over U as a bimodule resolution of $\mathcal{O}(U) \otimes \mathcal{O}(U)$, and applying $\otimes_{\mathcal{O}(U)} \mathcal{F}(U)$ as we have done before in the noncommutative context.

We obtain the result that, for every actual sheaf $\mathcal{F}$ (as opposed to a complex of sheaves), we have a resolution of $\mathcal{F}$ by sums of copies of $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$.

To state this in the language of the corollary, note that, given any resolution

$$0 \to \mathcal{P}_{-n} \to \mathcal{P}_{-(n-1)} \to \cdots \to \mathcal{P}_0 \twoheadrightarrow \mathcal{F} \tag{3.50}$$

of $\mathcal{F}$ by coherent sheaves $\mathcal{P}_i$, we may iteratively construct $\mathcal{F}$ in the derived category from the sheaves $\mathcal{P}_i$ by taking cones and shifts. To do this (we will ignore shifting since this just allows us to put sheaves in any degree of a complex we like), first we can construct the cokernel, call it $\mathcal{Q}_{-(n-1)}$, of $\mathcal{P}_{-n} \to \mathcal{P}_{-(n-1)}$ by taking the cone of that map; then we can construct the cokernel of $\mathcal{Q}_{-(n-1)} \to \mathcal{P}_{-(n-2)}$ by taking the cone of that map, etc., until we get $\mathcal{F}$ after taking n cones.

Finally, every object of $\mathcal{D}^b(\mathbb{P}^n)$ is evidently obtained from actual sheaves by cones and shifts, so since every actual sheaf is obtained from $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ by cones and shifts, this yields the statement. ☐

3.9. **Equivalences $\mathcal{D}^b(\mathcal{A}) \simeq R\mathrm{End}(T_1 \oplus \cdots \oplus T_n)$.** We have stated several times the following theorem, which we will now explain.

Definition 3.51. For any abelian category $\mathcal{A}$ with either enough injectives or enough projectives (or both), let $R\,\mathrm{Hom}(-,-) : \mathcal{D}^b(\mathcal{A}) \times \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathrm{Vect})$ denote the bifunctor given by $R\,\mathrm{Hom}(X,-) := RF_X$, where $F_X(-) := \mathrm{Hom}(X,-)$ (if there are enough injectives), or $R\,\mathrm{Hom}(-,Y) = RG_Y$, where $G_Y(-) := \mathrm{Hom}(-,Y)$ (if there are enough projectives).

Note that, if there are enough projectives and injectives, it is a standard fact that the two definitions of $R\,\mathrm{Hom}(-,-)$ above coincide (up to isomorphism).

We have $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(X,Y) = H^0(R\,\mathrm{Hom}(X,Y))$ for all $X, Y \in \mathcal{D}^b(\mathcal{A})$. What this says is that the triangulated category $\mathcal{D}^b(\mathcal{A})$ has *additional structure*: rather than merely knowing the vector spaces $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(X,Y)$, or more generally the sequence of vector spaces $\mathrm{Ext}^i(X,Y) := \mathrm{Hom}_{D^b(\mathcal{A})}(X, Y[-i]), i \in \mathbb{Z}$, in fact we have *complexes* $R\,\mathrm{Hom}(X,Y)$ of which these vector spaces are the cohomology groups.

At first, this may not seem like additional structure, since any complex of vector spaces is quasi-isomorphic to its homology groups[2] (more generally, this is true for complexes of modules over any semisimple ring). However, when we consider also composition, this becomes nontrivial: for example, $R\mathrm{End}(X) := R\,\mathrm{Hom}(X,X)$ is now a *dg*-algebra: this means, an associative algebra which is also a complex (i.e., a differential-graded vector space, or *dg*-vector space), such that $d(ab) = (da)b + a(db)$. Now, for $A_\cdot$ a *dg*-algebra, while there is a quasi-isomorphism of complexes $H^\cdot(A_\cdot) \xrightarrow{\sim} A_\cdot$, where the first complex is equipped with the trivial differential, this is *not* in general an associative algebra morphism (equipping $H^\cdot(A_\cdot)$ with the induced associative algebra structure).

With this in mind, $\mathcal{D}^b(\mathcal{A})$ is more than merely a triangulated category: it also has additional homomorphism *complexes* $R\,\mathrm{Hom}(X,Y)$ for all X, Y, equipped with natural isomorphisms $H^i(R\,\mathrm{Hom}(X,Y)) \xrightarrow{\sim} \mathrm{Hom}(X, Y[-i])$ for all $i \in \mathbb{Z}$, which respect composition. Somewhat imprecisely this is sometimes called a *triangulated category enriched over complexes*, or by Kontsevich

[2]This property is called *formality*: so any complex of vector spaces is formal.

(arXiv:0801.4760) as a "$\mathbb{C}$-linear space" (provided a couple other technical conditions are satisfied).

We need this extra structure to obtain the following equivalence:

Theorem 3.52. *Suppose that $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ is generated by objects $T_i \in \mathcal{D}, i \in I$, by taking cones and shifts. Then, we have quasi-inverse triangulated equivalences*

$$\mathcal{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}^b(R\mathrm{End}(\textstyle\bigoplus_{i\in I} T_i)^{\mathrm{op}} - mod), \tag{3.53}$$

$$Y \longmapsto R\,\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(\textstyle\bigoplus_i T_i, Y),$$

$$Z \otimes^L_{R\mathrm{End}(\bigoplus_{i\in I} T_i)} (\textstyle\bigoplus_i T_i) \longleftarrow\!\shortmid\, Z,$$

provided that the functors are well-defined (i.e., up to finiteness and boundedness issues).

Above, the category $\mathcal{D}^b(R\mathrm{End}(\bigoplus_{i\in I} T_i)^{\mathrm{op}} - mod)$ must be properly interpreted: roughly, it means that one considers complexes which are dg-modules over the dg-algebra $R\mathrm{End}(\bigoplus_{i\in I} T_i)^{\mathrm{op}}$, modulo nullhomotopic maps and inverting quasi-isomorphisms. We will only actually use this in the case that $R\mathrm{End}(\bigoplus_{i\in I} T_i) = \mathrm{End}(\bigoplus_{i\in I} T_i)$, i.e. all the groups $\mathrm{Ext}^i(T_j, T_k)$ vanish for $i \neq 0$.

Remark 3.54. We may interpret the above equivalence as the integral transform Φ^T associated to $T := \bigoplus_i T_i$, which is naturally an object of both $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^b(R\mathrm{End}(T)^{\mathrm{op}} - mod)$. In general, the above theorem shows that any "$\mathbb{C}$-linear space $\mathcal{D}$," by which we mean an enriched triangulated category as above, is equivalent to $\mathcal{D}^b(A - mod)$ for some dg algebra A; then, all (covariant) functors $\mathcal{D}^b(A - mod) \to \mathcal{D}^b(B - mod)$ which "arise in nature" should be given by $\Phi^{\mathcal{F}}(-) = F \otimes^L_A -$, where F is a dg (B, A)-bimodule.

Proof. We show that the functors are quasi-inverse. First, we claim that the following natural map is a quasi-isomorphism:

$$R\,\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(\bigoplus_i T_i, Y) \otimes^L_{R\mathrm{End}(\bigoplus_{i\in I} T_i)} (\bigoplus_i T_i) \to Y. \tag{3.55}$$

To prove this, note first that it is tautologically true for $Y = \bigoplus_i T_i$; also, the left-hand-side commutes with direct sums and is a triangulated functor (sends distinguished triangles to distinguished triangles), and hence we deduce that the map must be a quasi-isomorphism for all the T_i individually and for the triangulated subcategory generated by the T_i, which is everything by assumption.

The other direction is similar: we can similarly show that the following natural map is a quasi-isomorphism:

$$Z \to R\,\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(\bigoplus_i T_i, Z \otimes^L_{R\mathrm{End}(\bigoplus_{i\in I} T_i)} (\bigoplus_i T_i)) \tag{3.56}$$

$\square$

The reason why we needed $R\,\mathrm{Hom}$ above was to ensure that the functors involved were triangulated functors. If all we had was Hom, then we would still be able to apply Hom once to a distinguished triangle and obtain a long exact sequence of vector spaces, but we would be unable to further apply another functor, $\otimes^L_{R\mathrm{End}(\bigoplus T_i)}$, which is particularly needed in (3.55).

Remark 3.57. Having $R\,\mathrm{Hom}$ above allowed us to make use of the general principle that the composition of two (or more) derived functors is still triangulated, and is in fact the derived functor of the composition: $Rf \circ Rg = R(f \circ g)$. This is *not possible* with only homology groups: it isn't true, for instance, that $\bigoplus_{i+j=m} R^i f \circ R^j g$ is the same as $R^m(f \circ g)$. (There is, however, a map from the first to the left, and in fact a spectral sequence $\bigoplus R^{\cdot} f \circ R^{\cdot} g \Rightarrow R^{\cdot}(f \circ g)$, coming from the quasi-isomorphism of the double complex $Rf \circ Rg(X)$ with the complex $R(f \circ g)(X)$ for all X. This is useful in computations.)

3.10. **The equivalence $\mathcal{D}^b(\mathbb{P}^n) \simeq \mathcal{D}^b(\mathcal{R}ep(Q^{(n)}|R^{(n)}))$, revisited.** Now we are ready to prove Theorem 3.41.

Proof of Theorem 3.41. By Theorem 3.46, we have that $\mathcal{D}^b(\mathbb{P}^n)$ is generated as a triangulated category by $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$. By Theorem 3.52, we deduce that $\mathcal{D}^b(\mathbb{P}^n) \simeq R\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n))$, provided the resulting functors are well-defined. By the claim below, $R\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n)) \simeq \mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n)) \cong \mathbb{C}Q^{(n)}/(R^{(n)})$. Once we know this, then it is easy to see that the functors $\mathcal{F} \mapsto \mathbb{R}\mathcal{H}om(\mathcal{O} \oplus \cdots \oplus \mathcal{O}(-n), \mathcal{F})$ and $M \mapsto M \otimes_{\mathbb{C}Q^{(n)}/(R^{(n)})} \mathcal{O} \oplus \cdots \oplus \mathcal{O}(-n)$ are well-defined functors between $\mathcal{D}^b(\mathbb{P}^n)$ and $\mathcal{D}^b(\mathbb{C}Q^{(n)}/(R^{(n)}) - mod)$. This proves the theorem. $\square$

Claim 3.58. (i) $\mathrm{Ext}^i(\mathcal{O}(j), \mathcal{O}(k)) = 0$ *for* $i \neq 0$ *and for* $j, k \in \{0, -1, \ldots, -n\}$. *Hence,* $\mathbb{R}\mathcal{E}nd(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n)) \simeq \mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n))$.
(ii) $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) = 0$ *if* $i > j$;
(iii) *We have that* $\mathrm{End}(\mathcal{O}(i)) = \mathbb{C}$ *and* $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i+1)) \cong \langle x_0, x_1, \ldots, x_n \rangle = \Gamma(\mathcal{O}(1)) \cong \mathbb{C}^{n+1}$;
(iv) *The algebra* $\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n))$ *identifies with the quotient of* $\mathbb{C}Q^{(n)}$ *by the relations* $R^{(n)}$.

Sketch of proof. (i) For this vanishing result we refer to Hartshorne.
(ii) This is the same as the statement that $\Gamma(\mathcal{O}(m)) = 0$ for $m < 0$, which is clear.
(iii) These statements are clear.
(iv) For this, note that $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i+m)) = \Gamma(\mathcal{O}(m)) =$ polynomials of degree m in $x_0, \ldots, x_n$. $\square$

4. The McKay correspondence $\mathcal{D}^b(\widetilde{\mathbb{C}^2/G}) \simeq \mathcal{D}^b((\mathbb{C}[x,y]\#G) - mod)$

For a reference on this topic, see Kapranov and Vasserot's paper, `arXiv:math/9812016`, *Kleinian singularities, derived categories, and Hall algebras.*

The simplest way to obtain an equivalence $\mathcal{D}^b(\widetilde{\mathbb{C}^2/G}) \simeq \mathcal{D}^b((\mathbb{C}[x,y]\#G) - mod)$ is to view the latter category as the category $\mathcal{D}^b_G(\mathbb{C}^2)$ of G-equivariant sheaves on $\mathbb{C}^2$, i.e., as objects of $\mathcal{D}^b(\mathbb{C}^2)$ equipped with an additional action by G by automorphisms in $\mathcal{D}^b(\mathbb{C}^2)$. Since G is finite, complexes of $\mathbb{C}[x,y]$-modules with a G-equivariant action are the same as complexes of G-equivariant $\mathbb{C}[x,y]$-modules, i.e., complexes of $\mathbb{C}[x,y]\#G$-modules, so this is the same.

The advantage of this formulation is that we can seek an integral transform $\Phi^{\mathcal{F}}_{\widetilde{\mathbb{C}^2/G} \to \mathbb{C}^2}$ with $\mathcal{F} \in \mathcal{D}^b_G(\widetilde{\mathbb{C}^2/G} \times \mathbb{C}^2)$, with here G acting on the second factor, which yields the desired equivalence.

Recall the universal sheaf on $\mathrm{Hilb}^{|G|}(\mathbb{C}^2) \times \mathbb{C}^2$, whose fiber over $[Y] \times \mathbb{C}^2$ is the sheaf $\mathbb{C}[Y]$, for all subschemes $Y \subset \mathbb{C}^2$ of length $|G|$. This restricts to a G-equivariant sheaf on $\mathrm{Hilb}^{|G|}(\mathbb{C}^2) \times \mathbb{C}^2$ and hence on $\widetilde{\mathbb{C}^2/G} \times \mathbb{C}^2$. Call this last sheaf $\mathcal{U}$.

Theorem 4.1 (Kapranov-Vasserot)**.** *The integral transform* $\Phi^{\mathcal{U}}_{\widetilde{\mathbb{C}^2/G} \to \mathbb{C}^2}$ *yields a triangulated equivalence* $\mathcal{D}^b(\widetilde{\mathbb{C}^2/G}) \xrightarrow{\sim} \mathcal{D}^b_G(\mathbb{C}^2)$. *Its inverse is given by*

$$\Psi^{\mathcal{U}}_{[\mathbb{C}^2/G]_{stack} \to \widetilde{\mathbb{C}^2/G}}(-) := R\,\mathrm{Hom}_{\mathbb{C}[x,y]\#G}(\mathcal{U}, -), \tag{4.2}$$

viewing here $\mathcal{D}^b_G(\mathbb{C}^2) = \mathcal{D}^b((\mathbb{C}[x,y]\#G) - mod)$.

Proof. The main tool here is a Koszul-style resolution of $\mathcal{U}$. Let $V = \mathbb{C}^2$ be the tautological two-dimensional representation of $G < \mathsf{SL}_2(\mathbb{C})$. There is a natural G-equivariant map $V^* \to \Gamma(\mathcal{U})$ sending the coordinate functions x, y to their images in $\mathbb{C}[Y]$ for all $Y \subset \mathbb{C}^2$ with $\mathbb{C}[Y] \cong \mathbb{C}[G]$.

Let $\mathcal{R} = (\pi_{\widetilde{\mathbb{C}^2/G}})_* \mathcal{U}$ be the sheaf on $\widetilde{\mathbb{C}^2/G}$ whose fiber at $[Y]$ is $\mathbb{C}[Y]$, for $Y \subset C^2$ such that $\mathbb{C}[Y] \cong \mathbb{C}[G]$. There is still a map $V^* \to \Gamma(\mathcal{R})$ as above.

On $\pi^*_{\widetilde{\mathbb{C}^2/G}} \mathcal{R}$, we may consider the coordinate functions acting on both components, so let x_i, y_i denote their actions on the i-th components, for $i \in \{1, 2\}$ (the first component is $\widetilde{\mathbb{C}^2/G}$ and the second is $\mathbb{C}^2$). We have an obvious surjection $\mathcal{R} \twoheadrightarrow \mathcal{U}$ with kernel generated by $x_1 - x_2, y_1 - y_2$. This completes to a Koszul-style complex

$$(\Lambda^{\cdot} V^*) \otimes_{\mathbb{C}} \pi^*_{\widetilde{\mathbb{C}^2/G}} \mathcal{R} \twoheadrightarrow \mathcal{U}, \tag{4.3}$$

which is exact by a local computation.

For ease of notation, let us denote the two functors simply by Φ and Ψ. Also, for simplicity, we will use the notation, for (B, A)-bimodules M, $M \otimes_A (-) : A - \text{mod} \to B - \text{mod}$, and extend this notation to $\mathcal{F} \otimes_{\mathcal{O}_X} (-) : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$, when $\mathcal{F} \in \mathrm{Coh}(X \times Y)$, and to the derived versions. Technically, the latter is given by $R(\pi_Y)_*(\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} L\pi_X^*(-))$.

We then have

$$\Psi \circ \Phi(-) = R\,\mathrm{Hom}_{\mathbb{C}[x,y]\#G}(\mathcal{U}, (\mathcal{U} \otimes^L_{\mathcal{O}_{\widetilde{\mathbb{C}^2/G}}} -)) = R\mathrm{End}_{\mathbb{C}[x,y]\#G}(\mathcal{U}) \otimes^L_{\mathcal{O}_{\widetilde{\mathbb{C}^2/G}}} (-). \tag{4.4}$$

So, $\Psi \circ \Phi \simeq \mathrm{Id}$ is equivalent to the statement that $\Phi^{R\mathrm{End}_{\mathbb{C}[x,y]\#G}(\mathcal{U})}_{\widetilde{\mathbb{C}^2/G} \times \widetilde{\mathbb{C}^2/G}} \simeq \mathrm{Id}$, i.e., that

$$R\mathrm{End}_{\mathbb{C}[x,y]\#G}(\mathcal{U}) := R\,\mathrm{Hom}_{\mathbb{C}[x,y]\#G}(\pi_1^* \mathcal{U}, \pi_2^* \mathcal{U}) \simeq \mathcal{O}_\Delta, \tag{4.5}$$

for $\Delta \subset (\widetilde{\mathbb{C}^2/G})^2$, where $\pi_1, \pi_2 : (\widetilde{\mathbb{C}^2/G} \times \mathbb{C}^2 \times \widetilde{\mathbb{C}^2/G}) \to (\widetilde{\mathbb{C}^2/G} \times \mathbb{C}^2)$ are the two projections. The first equality is a notational matter: it says that our End (as well as Hom and the derived versions) will be the external "boxed" versions here, like the difference between $\otimes$ and $\boxtimes$ (and in keeping with our simplified notation above).

Using (4.3), we have

$$R\mathrm{End}_{\mathbb{C}[x,y]\#G}(\mathcal{U}) = R\mathrm{End}_{\mathbb{C}[x,y]\#G}(\mathcal{U}) = \mathrm{Hom}_{\mathbb{C}[x,y]\#G}((\Lambda^{\cdot} V^*) \otimes_{\mathbb{C}} \pi^*_{\widetilde{\mathbb{C}^2/G}} \mathcal{R}, \mathcal{U}). \tag{4.6}$$

Since $\pi^*_{\widetilde{\mathbb{C}^2/G}} \mathcal{R}$ is free over $\mathbb{C}[x, y]$, we can rewrite the above as

$$\mathrm{Hom}_G((\Lambda^{\cdot} V^*) \otimes_{\mathbb{C}} \mathcal{R}, \mathcal{R}), \quad d\phi(v \otimes \eta) = v\phi(\eta) - \phi(v \cdot \eta), \tag{4.7}$$

where the $\mathcal{U}$ becomes $\mathcal{R}$ when we forget the $\mathbb{C}[x, y]$-structure. The differential uses the forgotten V^*-action. We make use of the convention that $\Lambda^{\cdot} V^* \subset TV^*$ is the subspace of skew-symmetric tensors above.

We have to show that (4.7) is quasi-isomorphic to $\mathcal{O}_\Delta$. This is proved in Nakajima's monograph, Lemma 4.10. We reproduce the part that says that the cohomology is supported on Δ. The kernel of the first nonzero arrow above is just $\mathrm{Hom}_{\mathbb{C}[x,y]\#G}(\mathcal{U}, \mathcal{U})$, i.e., at each point $([Y], [Z]) \in (\widetilde{\mathbb{C}^2/G})^2$, we are considering the homomorphisms of $\mathbb{C}[x, y]\#G \simeq \Pi_Q$-representations $\mathbb{C}[Y] \to \mathbb{C}[Z]$. By the stability condition, this is completely determined by the linear map $\mathbb{C} \cong \mathbb{C}[Y]^G \to \mathbb{C}[Z]^G \cong \mathbb{C}$. In particular, the kernel must be either an isomorphism or zero. Thus, off the diagonal, the first kernel is zero. Dually, the last cokernel is zero off the diagonal, so the first and last cohomology groups are zero. Since the alternating sum of the ranks of the locally free sheaves above are zero, the alternating sum of the cohomology dimensions must also be zero, so the middle cohomology must also be zero off the diagonal.

The rest of the proof is based on some fancier algebraic geometry which relies on references (e.g. the Buchsbaum-Eisenbud criterion), so we refer to Nakajima, Lemma 4.10.

In the other direction, again using the notation that $\otimes^L_{\widetilde{\mathbb{C}^2/G}}$ means derived-tensoring sheaves and applying $R(\pi_{\widetilde{\mathbb{C}^2/G}})_*$,

$$\Phi \circ \Psi(M) = \mathcal{U} \otimes^L_{\mathcal{O}_{\widetilde{\mathbb{C}^2/G}}} R\operatorname{Hom}_{\mathbb{C}[x,y]\#G}(\mathcal{U}, M)) = \mathcal{U} \otimes^L_{\mathcal{O}_{\widetilde{\mathbb{C}^2/G}}} (\mathcal{U}^{R\vee_{\mathbb{C}[x,y]}} \otimes^L_{\mathbb{C}[x,y]\#G} M). \tag{4.8}$$

So we have to show that (using that $\mathbb{C}^2$ is affine so we can just take global sections in these components),

$$R\Gamma(\mathcal{U} \otimes^L_{\mathcal{O}_{\widetilde{\mathbb{C}^2/G}}} \mathcal{U}^{R\vee_{\mathbb{C}[x,y]}}) \simeq (\mathbb{C}[x,y]\#G), \tag{4.9}$$

as complexes of $\mathbb{C}[x,y]\#G$-bimodules. Using (4.3), the LHS translates to

$$(\Lambda^{\cdot} V) \otimes_{\mathbb{C}} R\Gamma(\mathcal{U} \otimes_{\mathcal{O}_{\widetilde{\mathbb{C}^2/G}}} \pi^*_{\widetilde{\mathbb{C}^2/G}} \mathcal{R}^\vee) = (\Lambda^{\cdot} V) \otimes_{\mathbb{C}} R\Gamma(\pi^*_{\widetilde{\mathbb{C}^2/G}}(\mathcal{R} \boxtimes_{\widetilde{\mathbb{C}^2/G}} \mathcal{R}^\vee)) = (\Lambda^{\cdot} V) \otimes_{\mathbb{C}} \mathbb{C}[x,y] \otimes R\operatorname{End}_{\mathcal{D}^b(\widetilde{\mathbb{C}^2/G})}(\mathcal{R}), \tag{4.10}$$

viewed as a $\mathbb{C}[x,y]\#G$-bimodule by acting on the left on the copy of $\mathbb{C}[x,y]$, and acting on the right on the output copy of $\mathcal{R}$.

A local calculation which we omit (although will discuss more in the next section) shows that $\mathcal{R}$ restricts on each $\mathbb{P}^1$ component of $\pi^{-1}(0) \subset \widetilde{\mathbb{C}^2/G}$ to copies of $\mathcal{O}(-1)$ and $\mathcal{O}$, and hence all higher $\operatorname{Ext}^i_{\widetilde{\mathbb{C}^2/G}}(\mathcal{R}, \mathcal{R})$ vanish there; they also vanish away from the zero fiber since there we have the same result as self-Exts of a locally free (and in fact free) sheaf on $\mathbb{C}^2/G$, which is affine. Finally, $\operatorname{End}_{\widetilde{\mathbb{C}^2/G}}(\mathcal{R}) = C[x,y]\#G$ itself, given on fibers $[Y], Y \subset \mathbb{C}^2$ by specifying the image of the cyclic vector $1 \in \mathbb{C}[Y]$. Thus, we get that

Summing up, it remains to show that the following is quasi-isomorphic to $\mathbb{C}[x,y]\#G$ as a $\mathbb{C}[x,y]\#G$-bimodule:

$$\mathbb{C}[x,y] \otimes \Lambda^{\cdot} V \otimes \mathbb{C}[x,y]\#G, \tag{4.11}$$

which is easily checked to be the result of applying the (exact) smash product functor $Z \mapsto Z\#G$ to the Koszul resolution $\mathbb{C}[x,y] \otimes \Lambda^{\cdot} V \otimes \mathbb{C}[x,y] \twoheadrightarrow \mathbb{C}[x,y]$ of $\mathbb{C}[x,y]$ (in general, the smash product functor $Z \mapsto Z\#G$ is an exact functor from G-modules to G-bimodules; here, the Koszul resolution of $\mathbb{C}[x,y]$ has a natural action of $\mathsf{SL}_2(\mathbb{C})$ and hence of G).

□

4.1. **Generalization from $\mathbb{C}^2$ to arbitrary surfaces.** Given any smooth surface X equipped with a faithful action by automorphisms of a finite group G, we may again consider the smooth variety $\operatorname{Hilb}^{|G|}(X)$ and the subvariety $(\operatorname{Hilb}^{|G|}(X))^G$ whose connected components are smooth. Once again, if $X^{\text{reg}} :=$ the locus of $x \in X$ such that the stabilizer $G_x = \{1\}$ is trivial, then we have that X^{reg}/G is smooth, and that $(\operatorname{Hilb}^{|G|}(X^{\text{reg}}))^G \cong X^{\text{reg}}/G$.

4.2. **Generators of $\mathcal{D}^b(\widetilde{\mathbb{C}^2/G})$ and the zero fiber.** Using the equivalence, we immediately find generators of $\mathcal{D}^b(\widetilde{\mathbb{C}^2/G})$. Up to direct summands as well as shifts and cones, the object A is always a single generator of $\mathcal{D}^b(A - mod)$ for any algebra A. For $A = \mathbb{C}[x,y]\#G$, we see that $\mathbb{C}[x,y]\#G$ is a generator of $\mathcal{D}^b((\mathbb{C}[x,y]\#G) - mod)$. Its image under the inverse equivalence Ψ above is

$$R\operatorname{Hom}_{\mathbb{C}[x,y]\#G}(\mathcal{U}, \mathbb{C}[x,y]\#G), \tag{4.12}$$

and evaluating at the fiber over $Y \subset \mathbb{C}^2$ with $\mathbb{C}[Y] \cong \mathbb{C}[G]$ as a G-module, we have to compute

$$R\operatorname{Hom}_{\mathbb{C}[x,y]\#G}(\mathbb{C}[Y], \mathbb{C}[x,y]\#G) = \operatorname{Hom}_{\mathbb{C}[x,y]\#G}(\mathbb{C}[x,y] \otimes \Lambda^{\cdot} V^* \otimes \mathbb{C}[Y], \mathbb{C}[x,y]\#G) \tag{4.13}$$
$$= \operatorname{Hom}_G(\Lambda^{\cdot} V^* \otimes \mathbb{C}[Y], \mathbb{C}[x,y]\#G) \cong \Lambda^{\cdot} V \otimes \mathbb{C}[x,y]\#G \cong \Lambda^{\cdot} V^* \otimes \mathbb{C}[x,y]\#G,$$

since $\mathbb{C}[Y] \cong \mathbb{C}[G]$ as a representation, and $\Lambda^{\cdot} V \cong \Lambda^{\cdot} V^*$ using the volume form. The induced differential is such that the above complex resolves exactly $\mathbb{C}[Y]$ again: it is essentially saying that $\mathbb{C}[Y]$ is derived-selfdual as an object of $\mathbb{C}[x,y]-mod$, or that the Koszul complex is quasi-isomorphic to its dual.

That is, we just get out $\mathcal{R}$: so the equivalence says in particular that $\mathcal{R}$ is a generator.

In terms of (where here the G-irrep ρ_i is viewed as a $\mathbb{C}[x,y]\#G$-module with x, y acting by zero)

$$\mathbb{C}[x,y]\#G \simeq \Pi_Q, \tag{4.14}$$

$$\rho_i \leftrightarrow S_i, \tag{4.15}$$

$$(\mathbb{C}[x,y]\#G)e_i) \leftrightarrow P_i, \tag{4.16}$$

we can find the image under Ψ of the projective modules $P_i = \Pi_Q i$ and the simple modules S_i. We see that $\mathcal{R} = \bigoplus_i \mathcal{R}_i \otimes_{\mathbb{C}} \rho_i$, and $\mathcal{R}_i \simeq \Psi(P_i)$, while for S_i we claim that: if i is not the extending vertex, we get $\mathcal{O}(-1)_i$, the sheaf $\mathcal{O}(-1)$ supported on the $\mathbb{P}^1$-component of the zero fiber corresponding to i; if i is the extending vertex, we get the structure sheaf of the zero fiber.

This should be computable by writing $\Psi(S_i)_{[Y]}$ again as

$$\Lambda^{\cdot} V^* \otimes \rho_i, \tag{4.17}$$

with the differential acting by applying the action of V^*, i.e., the maps $V^* \otimes \rho_i \cong \bigoplus_{j \to i} \rho_j \to \rho_i$ are the arrows $j \to i$. We claim that, if Y is not supported at zero, this complex has first and last cohomology vanishing. This is because, in these cases, the arrows going into i are surjective, and similarly the arrows coming out are also injective. As a result, all the cohomology must vanish since the Euler characteristic is zero. So $\Psi(S_i)$ is supported on the zero fiber (as it should be). Moreover, the first cohomology group will be identically zero, since it follows that on open sets the first nontrivial arrow has no kernel (as our original complex, before passing to the fiber over Y, is a complex of locally free sheaves, obtained by replacing $\mathcal{U}$ with a resolution by locally free sheaves and applying $\operatorname{Hom}_{\mathbb{C}[x,y]\#G}(-, \mathbb{C}[x,y]\#G)$. Similarly, the second cohomology group must be zero, and so $\Psi(S_i)$ is quasi-isomorphic to a single quasi-coherent sheaf....

QUIVERS IN REPRESENTATION THEORY (18.735, SPRING 2009) LECTURES 23 TO 26

TRAVIS SCHEDLER, TYPED BY IVÁN ANGIONO

1. Deligne-Simpson Problem

The basic object of study are topological vector bundles on a Riemann surface with flat connection.

A flat connection (connection with zero curvature) is a way of differentiating sections so that we know what a locally constant section is. Flat means that the monodromy of a contractible loop is trivial; i.e. get a local constant structure in all directions simultaneously.

We get $p \in \operatorname{Hom}(\pi_1(X), GL_n)$, where V is a bundle of rank n. Pick a base point $p \in X$, a framing of $V|_p$ is the fiber at p, the monodromy of loops based at $p \in GL_n$. Up to change of framing, we get $[p] \in \operatorname{Hom}(\pi_1(X), GL_n)/conj$.

1.1. **Deligne-Simpson Problem:**

which monodromies are realizable by a vector bundle and connections. In particular, analyze the case $X = S^2 \setminus \{x_1, ..., x_p\}$. I.e. which conjugacy classes $\mathcal{C}_1, ..., \mathcal{C}_n \subseteq GL_n$ have representatives $X_1, ..., X_p$ such that $X_1 \cdots X_n = \mathrm{Id}$:

$$\pi_1(S^2 \setminus \{x_1, ..., x_p\}) = \mathbb{F}\langle x_1, ..., x_p\rangle/(x_1 \cdots x_p - 1).$$

Example 1.1. For $p = 1$ we have the trivial monodromy.

For $p = 2$, simply $\mathcal{C}_1 = \mathcal{C}_2^{-1}$.

For $p = 3$, $x_1x_2x_3 = \mathrm{Id}$. One possibility is x_1, x_2x_3 scalar, with product the identity, but there are other more complicated ones.

The problem comes so more complicated for $p = 4, 5, ...$: all of these correspond to non-Dynkin diagrams.

Additive version of this problem: For which $\mathcal{C}_1, ..., \mathcal{C}_p$ there exist representatives $X_1, ..., X_p$ such that $X_1 + ... + X_p = 0$.

1.2. **Main idea: construction of quivers and its pre-projective algebras.**

For $S^2\setminus\{x_1, ..., x_p\}$, we will associate star-shaped quivers with p branches, in such a way that monodromies correspond to representations of deformed preprojective algebras of these quivers.

For $\mathcal{C}_1, ..., \mathcal{C}_k$ conjugacy classes in GL_n, consider their minimal polynomials: for $A_i \in \mathcal{C}_i$,

$$(A_i - \xi_{i1}\mathrm{id})(A - \xi_{i2}\mathrm{id}) \cdot (A - \xi_{i,w_i}), \quad \xi_{ij} \in \mathbb{C},\ 1 \leq i \leq k, 1 \leq j \leq w_i.$$

Here we count roots with multiplicity. We consider the star-shaped quiver:

$$
Q = \begin{tikzcd}
 & \circ^{1,1} \arrow[ddl] & \circ^{1,2} \arrow[l] & \cdots \arrow[l] & \circ^{1,w_1-1} \arrow[l] \\
 & \circ^{2,1} \arrow[dl] & \circ^{2,2} \arrow[l] & \cdots \arrow[l] & \circ^{2,w_2-1} \arrow[l] \\
\circ^{0} & \circ^{3,1} \arrow[l] & \circ^{3,2} \arrow[l] & \cdots \arrow[l] & \circ^{3,w_3-1} \arrow[l] \\
 & \vdots & \vdots & & \vdots \\
 & \circ^{k,1} \arrow[uul] & \circ^{k,2} \arrow[l] & \cdots \arrow[l] & \circ^{k,w_k-1} \arrow[l]
\end{tikzcd} \tag{1.2}
$$

For this quiver, we can take its double $\overline{Q}$.

I'm sorry, my notes about the construction of the quiver weren't clear... Here you put something about the deformed preprojective algebra, and about Jordan blocks.

In we require that outward arrows are surjective and inward arrows are injective, we get Jordan blocks of maximal size.

I think all the next part of first lecture is explained in the second... but anyway please take a look if something is missed. I mixed both classes a bit

The Deligne-Simpson problem also ask for irreducible bundles with flat connections: that is, without subbundles of flat sections.

Theorem 1.3. *Let $\mathcal{C}_1, ..., \mathcal{C}_k$ be conjugacy classes in GL_n.*

(i) *Denote by $(\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})'$ the set of $\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}}$ such that the outward arrows are surjective and inward arrows are injective. There exists an isomorphism*

$$\{(X_1, ..., X_k) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_k : X_1 + \cdots + X_p = 0\}/GL_n \to (\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})'/GL_{\overrightarrow{d}}.$$

(ii) *There exists a surjective map:*

$$\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}}/GL_{\overrightarrow{d}} \twoheadrightarrow \{(X_1, ..., X_k) \in \overline{\mathcal{C}}_1 \times \cdots \times \overline{\mathcal{C}}_k : X_1 + \cdots + X_p = 0\}/GL_n.$$

(iii) *The first isomorphism restricts to an isomorphism*

$$\{(X_1, ..., X_k)\, Irrep : X_1 + \cdots + X_p = 0\}/GL_n \to Irrep_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}}/GL_{\overrightarrow{d}}.$$

Note 1.4. Map in (ii) restricts to an inverse of (i) . For (iii) , a simple representation of $\Pi_Q^{\overrightarrow{\lambda}}$ has all inward arrows injective, and all outward arrows surjective.

Corollary 1.5. (1) *There exists a solution $X_1 + \cdots X_k = 0$ with $X_i \in \mathcal{C}_i, \forall i$ iff $(\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})' \neq \emptyset$.*

(2) *There exists a solution $X_1 + \cdots X_k = 0$ with $X_i \in \overline{\mathcal{C}}_i, \forall i$ iff $\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}} \neq \emptyset$.*

(3) *There exists an irreducible solution $X_1 + \cdots X_k = 0$ with $X_i \in \mathcal{C}_i, \forall i$ iff there exists a simple $\Pi_Q^{\overrightarrow{\lambda}}$-representation of dimension $\overrightarrow{d}$.*

Then we apply combinatorial description of $\Pi_Q^{\overrightarrow{\lambda}}$.

Proof. (Theorem) It follows from the construction. Consider for $X_i \in \mathcal{C}_i$ the minimal polynomial $(x-\xi_{i,1})\cdots(x-\xi_{i,l_i}$, $l_i \geq 1$. Let

$$d_{i,j} = \mathrm{rk}\left((X_i - \xi_{i,1}\cdots(X_i - \xi_{i,l_i}\right),$$

so the d_{ij} decrease with j. Define $\overrightarrow{d} = (n, d_{i,j})_{i=1,\dots,k, j=1,\dots,l_i}$. Note that $d_{i,l_i} = 0$. Now define

$$\lambda_{i,j} := \xi_{i,j} - \xi_{i,j+1},\ 1 \leq j < l_i, \quad \lambda_{i,l_i} = 0.$$

Then call $\overrightarrow{\lambda} = (-\sum_{1\leq i\leq k}\xi_{i,1}, \lambda_{i,j})$.

Claim 1.6. *We get an injection*

$$\{(X_1,...,X_k) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_k : X_1 + \cdots + X_p = 0\}/GL_n \hookrightarrow (\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})'/GL_{\overrightarrow{d}}.$$

We construct from the left hand side the representation given in each branch i by

$$\circ^0 \underset{inc}{\overset{X_i-\xi_{i,1}\mathrm{Id}}{\rightleftarrows}} \circ^{i,1} \underset{inc}{\overset{X_i-\xi_{i,2}\mathrm{Id}}{\rightleftarrows}} \circ^{i,2} \underset{inc}{\overset{X_i-\xi_{i,3}\mathrm{Id}}{\rightleftarrows}} \circ^{i,3} \rightleftarrows \circ^{i,l_i-1} \underset{inc}{\overset{X_i-\xi_{i,l_i}\mathrm{Id}}{\rightleftarrows}} \circ^{i,l_i}.$$

Consider V_{ij} the image of $(X_i - \xi_{i_1})\cdot(X_i - \xi_{i,j})$. We have that inward arrows are injective, and outward arrows are surjective.

We have to check that we get a representation of $\Pi_Q^{\overrightarrow{\lambda}}$. This also is how we define the $\lambda_{i,j}$. At the node (vertex 0),

$$\sum X_i - \xi_{i,1} = \lambda_0\mathrm{Id} = -\sum \xi_{i,1} \iff \sum X_i = 0.$$

At the other vertices,

$$(X_i - \xi_{i,j+1}) - (X_i - \xi_{i,j}) = \xi_{i,j} - \xi_{i,j+1} = \lambda_{i,j}.$$

This gets an injection for (i) (even on level of isomorphism classes as is written), so it remains to check that it is surjective; i.e. given an element $\rho \in (\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})'$, it comes from $(X_1,...,X_k) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ satisfying $X_1 + \cdots X_k = 0$. We construct an explicit inverse.

Given $\rho \in (\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})'$, let's say by simplicity $\overrightarrow{\lambda} = 0$. In this case, for the i-branch

$$\circ \xrightarrow{a_{i,1}} \circ \xrightarrow{a_{i,2}} \circ \dashrightarrow \circ \xrightarrow{a_{i,l_i}} \circ,$$

we get in Π_Q^0: $(a_{i,1}^*a_{i,1})^j = a_{i,1}^*\cdots a_{i_j}^*a_{i,j}\cdots a_{i_1}$. This means that we can identify $\mathbb{C}^n \supsetneq V_{i,1} \supsetneq V_{i_2} \supsetneq \cdots$. Then we set $X_i - \xi_{i,1}$ from the first arrow and we get all the other arrows: $X_i - \xi_{i,j}$.

If we allow $\overrightarrow{\lambda} \neq 0$, the previous argument is just affected by adding scalars, and the same proof is correct. This says that making the above identifications, we get arrows in such way to define

$$\begin{aligned}(\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})' &\longrightarrow \{(X_1,...,X_k) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_k : X_1 + \cdots + X_p = 0\}, \\ (V,\rho) &\longmapsto \{a_{1,1}^*a_{1,1} + \chi_{1,1}, ..., a_{k,1}^*a_{k,1} + \chi_{k,1}\}\end{aligned}$$

which is an inverse of map in (i) .

For (ii) , use the inverse of map in (i) that we constructed, except for $V \in \mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}} \setminus (\mathrm{Rep}_{\overrightarrow{d}}\Pi_Q^{\overrightarrow{\lambda}})'$. For this case not covered, first pass to subquotient representation generated by V_0 (i.e. the $\mathbb{C}^n$ at the node) modulo kernels of inward arrows. This only decreases $\overrightarrow{d}$, therefore it gives $(X_1,...,X_p)$ with either a lower degree minimal polynomial (reducing the number of copies of some linear factors) or a lower rank of $(X_i - \xi_{i,1})\cdots(X_i - \xi_{i,j})$, or both. In this way, we replace $\mathcal{C}_i$ by other conjugacy classes in $\overline{\mathcal{C}}_i$ ∎

1.3. **Multiplicative version.** In this case, we have the same thing but we replace $\Pi_Q^{\vec{\lambda}}$ with

$$\Lambda_Q^{\vec{t}} := \widehat{\mathbb{C}\bar{Q}} / \prod_{a \in Q_1} (1 + aa*)(1 + a * a) = \vec{t}\,\mathrm{Id},$$

where $\widehat{\mathbb{C}\bar{Q}}$ is the algebra $\mathbb{C}\bar{Q}$ localized by inverting $1 + a * a, 1 + aa*$.

A non trivial fact is that it does not matter what order we take the product.

Now, $\mathrm{Rep}_{\vec{d}} \Lambda_Q^{\vec{t}} \subseteq \mathrm{Rep}_{\vec{d}} \bar{Q} = T^*\mathrm{Rep}_{\vec{d}} Q$ is given by "quasi-Hamiltonian" reduction using the multiplicative moment map $\prod(1 + \rho_a \rho_{a^*})(1 + \rho_{a^*} \rho_a)^{-1}$.

We get the same theorem holds for the $\Lambda_Q^{\vec{t}}$ version. Moreover, the representation theory of $\Lambda_Q^{\vec{t}}$ is analogous to the one of $\Pi_Q^{\vec{\lambda}}$: namely, replace the condition $\vec{\lambda} \cdot \alpha = 0$ for indecomposables in $\mathrm{Rep}_\alpha Q$ to lift to $\mathrm{Rep}_\alpha \Pi_Q^\lambda$, by $\vec{t}^{\,\alpha} := \prod t_i^{\alpha_i} = 1$. Again we get that simple modules of $\Lambda_Q^{\vec{t}}$ must have dimension in Δ_+.

1.4. **Examples.** The trivial example corresponds to $\mathcal{C}_i = \{\xi_i \mathrm{Id}\}$. For this:

$$\begin{tikzcd}
 & \circ^0 \\
 & \circ^0 \\
\vec{d} : \circ^n \arrow[uur] \arrow[ur] \arrow[r] \arrow[ddr] & \circ^0, \\
 & \vdots \\
 & \circ^0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \circ^0 \\
 & \circ^0 \\
\vec{\lambda} : \circ^n \arrow[uur] \arrow[ur] \arrow[r] \arrow[ddr] & \circ^0. \\
 & \vdots \\
 & \circ^0
\end{tikzcd}$$

By the Theorem, there exists a solution iff $\mathrm{Rep}_n \Pi_Q^\lambda \neq \emptyset$ iff $\sum \xi_i = 0$. this exactly says that $\mathcal{C}_1 + \cdots + \mathcal{C}_k = 0$.

Note 1.7. The type of Q determines the structure of the solution of Deligne-Simpson problem.

Example 1.8. For A_n type we have two conjugacy classes $\mathcal{C}_1$, $\mathcal{C}_2$:

$$\begin{tikzcd}
 & \circ \arrow[r] & \circ \arrow[r, dotted] & \circ \arrow[r] & \circ\,. \\
A_n : \circ^n \arrow[ur] \arrow[dr] & & & & \\
 & \circ \arrow[r] & \circ \arrow[r, dotted] & \circ \arrow[r] & \circ
\end{tikzcd}$$

By the Theorem, there exists solution iff $\mathcal{C}_1 = -\mathcal{C}_2$. Let's try with part (iii) : we know that there exists a simple representation of $\Pi_Q^{\vec{\lambda}}$ of dimension $\vec{d}$ iff $\vec{d} \in \Delta_+$. This implies that $n = 1$, which says that all irreducible representations of Deligne-Simpson problem for $k = 2$ have $n = 1$.

Example 1.9. For D_n type, we have three conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$: two branches are just of length 1,

$$
\begin{tikzcd}
 & \circ \\
D_n : \circ^n \arrow[ur] \arrow[r] \arrow[dr] & \circ \\
 & \circ \arrow[r] & \circ \arrow[r, dotted] & \circ \arrow[r] & \circ
\end{tikzcd},
$$

so $\mathcal{C}_1, \mathcal{C}_2$ are each either semisimple with two different eigenvalues, or have just one eigenvalue and Jordan blocks of dimension ≤ 2. On the other hand, $\mathcal{C}_3$ can be anything.

Using (i) we get: a description of solutions in $\overline{\mathcal{C}}_1 \times \overline{\mathcal{C}}_2 \times \overline{\mathcal{C}}_3$ allows Jordan blocks to shrink (get more Jordan blocks).

Now if we want to have a simple representation of Π_Q^λ of dimension $\overrightarrow{d}$, necessarily $n \leq 2$. We consider now irreducible solutions for

$$
\begin{tikzcd}
 & \circ^1 \\
\overrightarrow{d} : \circ^1 & \circ^2 \arrow[l] \arrow[u] \arrow[r] & \circ^1
\end{tikzcd}
$$

so $n = 2$ and $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ can now be anything (a priori).

Note 1.10. The indecomposable representation with dimension as above has subquotients of the form

$$
\begin{tikzcd}
 & \circ^0 \\
\alpha_1 = \circ^1 & \circ^1 \arrow[l] \arrow[u] \arrow[r] & \circ^1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \circ^1 \\
\alpha_2 = \circ^0 & \circ^1 \arrow[l] \arrow[u] \arrow[r] & \circ^0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \circ^1 \\
\alpha_3 = \circ^1 & \circ^1 \arrow[l] \arrow[u] \arrow[r] & \circ^1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \circ^0 \\
\alpha_4 = \circ^0 & \circ^1 \arrow[l] \arrow[u] \arrow[r] & \circ^0
\end{tikzcd}.
$$

We can guarantee that there exists a simple representation if we choose $\overrightarrow{\lambda}$ such that for these α (up to permutation) we have $\overrightarrow{\lambda} \cdot \alpha \neq 0$, and $\overrightarrow{\lambda} \cdot \overrightarrow{d} = 0$; i.e. no representations of Q with these vectors as dimensions lift to $\Pi_Q^{\overrightarrow{\lambda}}$.

Proof. First, there exists an indecomposable of dimension $\overrightarrow{d}$ since $\overrightarrow{\lambda} \cdot \overrightarrow{d} = 0$, so we can lift the one for Q to $\Pi_Q^{\overrightarrow{\lambda}}$. If it weren't simple, then there exists a decomposition $\overrightarrow{d} = \alpha^{(1)} + \cdots + \alpha^{(m)}$ such that for all i: $\alpha^{(i)} \cdot \lambda = 0$. $\square$

This will require one of the four listed above, up to permutation. In fact, there exists a simple iff these conditions are satisfied.

To prove the reciprocal statement, we know that indecomposable representations of dimension $\overrightarrow{d}$ has these four as subquotients, but there exists a unique isoclass of each indecomposable, so if any of these four lifted to $\Pi_Q^{\overrightarrow{\lambda}}$, then $\alpha \cdot \overrightarrow{\lambda} = 0$. From this $(\overrightarrow{d} - \alpha) \cdot \overrightarrow{\lambda} = 0$, so the corresponding representations are as above, and would give a subquotient/quotient of the lifted representation.

To find explicit solutions, note that

$$\begin{array}{ccccc} & & \circ^{\xi_{3,1}-\xi_{3,2}} & & \\ & & \uparrow & & \\ \overrightarrow{\lambda} = \circ^{\xi_{2,1}-\xi_{2,2}} & \longleftarrow & \circ^{\sum_i \xi_{i,1}} & \longrightarrow & \circ^{\xi_{1,1}-\xi_{1,2}}. \end{array}$$

From $\overrightarrow{\lambda} \cdot \overrightarrow{d} = 0$ we have $0 = \sum \xi_{ij} = \operatorname{tr}(X_1 + X_2 + X_3)$.

From $\overrightarrow{\lambda} \cdot \alpha_3 = 0$ and permutations of this we have $\sum_i \xi_{i,1} \neq \xi_{i,1} - \xi_{i,2}$ for $i = 1, 2, 3$. And from $\overrightarrow{\lambda} \cdot \alpha_4 \neq 0$ we obtain $\sum_i \xi_{i,1} \neq 0$.

These equations imply that at least one class is semisimple: otherwise, $\xi_{i,1} = \xi_{i,2}$ for all i so $\sum_i \xi_{i,1} = 0$, which is a contradiction.

When just one class is semisimple: $\mathcal{C}_1 = \left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right] = \mathcal{C}_2$, $\mathcal{C}_3 = \left[\begin{pmatrix} \xi & 0 \\ 0 & -\xi \end{pmatrix}\right]$ $(\xi \neq 0)$. I.e. any non semisimple classes $\mathcal{C}_1, \mathcal{C}_2$ and any non-scalar semisimple class $\mathcal{C}_3$ have a irreducible solution of Deligne-Simpson problem.

Note 1.11. The solution is unique up to conjugation since Q is Dynkin.

Consider now the semisimple case: $\mathcal{C}_i = \left[\begin{pmatrix} \xi_{i,1} & 0 \\ 0 & \xi_{i,2} \end{pmatrix}\right]$, $i = 1, 2, 3$.

Claim 1.12. *For generic $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, there exists a unique solution up to conjugation such that*

$$\sum_{i,j} \xi_{i,j} = 0, \quad \xi_{1,j_1} + \xi_{2,j_2} + \xi_{3,j_3},\ \{j_1, j_2, j_3\} \textit{ any permutation of } \{1, 2, 3\}.$$

The equality $\sum_{i,j} \xi_{i,j} = 0$ follows as above from $\overrightarrow{d} \cdot \overrightarrow{\lambda} = 0$, and also from $\alpha_3 \cdot \overrightarrow{\lambda}, \alpha_4 \cdot \overrightarrow{\lambda} \neq 0$ and their permutations, we obtain:

$$\sum_i \xi_{i,1} \neq 0, \quad \sum_i \xi_{i,1} \neq \xi_{i_0,1} - \xi_{i_0,2}, \forall i_0.$$

From these and $\sum_{i,j} \xi_{i,j} = 0$, we obtain $\xi_{1,j_1} + \xi_{2,j_2} + \xi_{3,j_3}$, for all the permutations $\{j_1, j_2, j_3\}$ of $\{1, 2, 3\}$.

Proposition 1.13. *If $\mathcal{C}_i$ are semisimple classes, for generic choice of eigenvalues (allow multiplicity), then there exists an irreducible representation of the Deligne-Simpson problem (multiplicative or additive) if and only if $\overrightarrow{d}$ is a positive root (determined by multiplicity).*

Proof. If $\overrightarrow{d}$ is indivisible, this means that $\overrightarrow{\lambda}$ is general for this $\overrightarrow{d}$: for all $\alpha < \overrightarrow{d}$, $\overrightarrow{\lambda} \cdot \alpha \neq 0$ unless α is a multiple of $\overrightarrow{d}$.

More precisely, replace generic by the following condition: $\not\exists\ m$ subsets with multiplicity $Y_i \subseteq$ eigenvalues$(\mathcal{C}_i)$ (where $m < n$) such that $\sum_i \sum_{\xi \in Y_i} \xi = 0$ $(\prod_i \prod_{\xi \in Y_i} \xi = 1)$ for the additive (multiplicative) problem.

This is not difficult to prove using the following result:

Theorem 1.14 (Crawley-Boewey '01)**.** *There exists a simple representation of $\Pi_Q^{\overrightarrow{\lambda}}$ of dimension $\overrightarrow{d}$ if and only if $\overrightarrow{d}$ satisfy:*

(i) $\overrightarrow{d} \in \Delta_+$*;*
(ii) $\overrightarrow{\lambda} \cdot \overrightarrow{d} = 0$*;*
(iii) *whenever* $\overrightarrow{d} = \alpha^{(1)} + \cdots + \alpha^{(k)}$*, where* $\alpha^{(i)} \in \Delta_+$*,* $\alpha^{(i)} \cdot \overrightarrow{\lambda} = 0$ *for all* i*, then*

$$p(\overrightarrow{d}) > p(\alpha^{(1)}) + \cdot + p(\alpha^{(k)}).$$

Here, $p(\beta) := 1 - \frac{1}{2}(\beta, \beta)$*, and when* β *is indivisible or* $(\beta, \beta) \neq 0$*,*

$$p(\beta) = \dim(\mathrm{Indec}_\beta\, Q/GL_\beta)$$

$\square$

Note 1.15. $\mathrm{Rep}_{\vec{d}}\Pi_Q^{\vec{\lambda}} \neq \emptyset$ iff $\vec{d} = \alpha^{(1)} + \cdots + \alpha^{(k)}$ for some $\alpha^{(i)} \in \Delta_+$, $\alpha^{(i)} \cdot \vec{\lambda} = 0$ for all i (possibly just one root vector).

Condition (iii) says that $\dim(\mathrm{Indec}_{\vec{d}}\, Q/GL_{\vec{d}}) > \sum_i \dim(\mathrm{Indec}_{\alpha^{(i)}}\, Q/GL_{\alpha^{(i)}})$, whenever a representation $\oplus_i V_i$ lifts to $\Pi_Q^{\vec{\lambda}}$, $V_i \in \mathrm{Indec}_{\alpha^{(i)}}\, Q$. (It seems not to be correct or I don't understand this statement.

Note 1.16. Every representation of $\Pi_Q^{\vec{\lambda}}$ of dimension $\vec{d}$ restricts as a Q representation to such $\oplus_i V_i$.

So condition (iii) is equivalent to the following: the generic $\Pi_Q^{\vec{\lambda}}$-representation of dimension $\vec{d}$ restricts to an indecomposable Q representation.

Note 1.17. There exists a simple $\Pi_Q^{\vec{\lambda}}$-representation iff the generic $\Pi_Q^{\vec{\lambda}}$-representation is simple.

Another two corollaries of previous Theorem are the following:

Corollary 1.18. *Classification of rigid solutions... Another bad statement in my notes, I'm sorry.*

Note 1.19. When $\vec{d}$ belongs to the fundamental region (i.e. $(\vec{d}, \epsilon_i) \leq 0$ for all $i \in Q_0$), the generic representation is indecomposable, and a brick:

$$0 \leq \frac{1}{2}(\vec{d}, \vec{d}) = \dim \mathrm{Hom}(V, V) - \dim \mathrm{Ext}^1(V, V), \quad \forall V \in \mathrm{Rep}_{\vec{d}} Q,$$

so $\mathrm{Ext}^1(V, V) = 0$ Why? I can't figure out this

Regarding rigid solutions, in particular any irreducible solution is rigid in the case $\mathrm{supp}(\vec{d})$ is Dynkin.

Corollary 1.20. *For the nilpotent case* $\vec{\lambda} = 0$*, if there exists an irreducible representation, then* $\vec{d} \in \Delta_+$*.*

In fact,

Proposition 1.21. *If there exists a solution, then* $\vec{d}$ *belongs to the fundamental region.*

Proof. If $(\vec{d}, \epsilon_i) > 0$ for some i, then $0 = \vec{\lambda} \cdot \epsilon_i$, so we get $\vec{d} = c\epsilon_i + (\vec{d} - c\epsilon_i)$. Let $c = (\vec{d}, \epsilon_i)$. Therefore $(c\epsilon_i) \cdot \vec{\lambda} = (\vec{d} - c\epsilon_i) \cdot \vec{\lambda} = 0$, so $d = c\epsilon_i$ or $p(\vec{d}) < p(\vec{d} - c\epsilon_i) + p(c\epsilon_i)$: note that $(\vec{d}, \epsilon_i) > 0$ implies $q(\vec{d}) > q(\vec{d} - c\epsilon_i) + q(c\epsilon_i)$. If $\vec{d} = c\epsilon_i$ then $c = 1$, so $\vec{d} = \epsilon_i$ is a real root. $\square$

In fact, being in the fundamental region is almost the same as to exist a nilpotent solution.

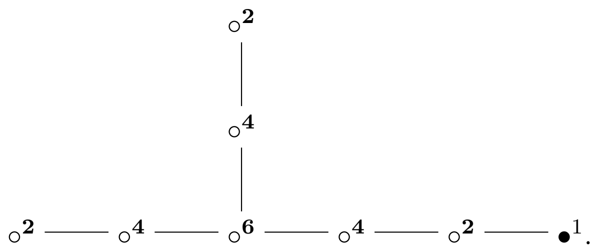
Theorem 1.22 (Crawley-Boewey)**.** *Let* $\vec{d} \in \Delta_+$*. Then* $\vec{d}$ *satisfies*

(1.23) $\quad \forall \alpha^{(i)} \in \Delta_+, \alpha^{(i)} \cdot \vec{\lambda} = 0$ *such that* $\vec{d} = \alpha^{(1)} + \cdots + \alpha^{(k)} : p(\vec{d}) > p(\alpha^{(1)}) + \cdot + p(\alpha^{(k)})$,

if and only if $\vec{d}$ *is in the fundamental region and is not one of the following:*

(1) $\vec{d} = m\delta$ *for* $m \geq 2$ *and* $\mathrm{supp}(\vec{d})$ *an extended Dynkin diagram, or*
(2) *the same as in 1. except we attach a vertex with dimension 1 to the extending vertex.*

An example of 2. for $\widetilde{E}_6$ and $m = 2$ is:



Remark 1.24. More generally for any $\overrightarrow{\lambda}$, satisfying 1.23 is almost the same as

$$\overrightarrow{d} \in Fund_{\overrightarrow{\lambda}} = \{\alpha | (\alpha, \epsilon_i) \leq 0 \text{ if } \lambda_i = 0\}.$$

Theorem 1.25. *Let Q be an arbitrary quiver. If $\overrightarrow{d} \cdot \overrightarrow{\lambda} = 0$, $\overrightarrow{d} \in \Delta_+$, then there exists a simple $\Pi_Q^{\overrightarrow{\lambda}}$-representation, and in fact the generic representation is simple.*

Remark 1.26. This is also proved for the multiplicative case. The converse is true in the additive case, and is difficult to prove; it seems not to be proved in the multiplicative case.

Proposition 1.27. *Let $\overrightarrow{d}$ be an element of the fundamental region such that $\overrightarrow{\lambda} \cdot \overrightarrow{d} = 0$. If $q(\overrightarrow{d}) \neq 0$ or $\overrightarrow{d}$ is indivisible, then there exists a simple $\Pi_Q^{\overrightarrow{\lambda}}$-representation of dimension $\overrightarrow{d}$ (and the generic representation is simple).*

Proof. If not, there exists $\alpha < \overrightarrow{d}$ such that the generic representation in $\mathrm{Rep}_{\overrightarrow{d}} \Pi_Q^{\overrightarrow{\lambda}}$ which has a subrepresentation of dimension α.

Moreover, the generic representation also has an α-dimensional quotient using duality:

$$\Pi_Q^{\overrightarrow{\lambda}} \cong (\Pi_Q^{\overrightarrow{\lambda}})^{op} : \qquad \mathrm{Rep}_{\overrightarrow{d}} \Pi_Q^{\overrightarrow{\lambda}} \xrightarrow[\sim]{v \mapsto v^*} \mathrm{Rep}_{\overrightarrow{d}} (\Pi_Q^{\overrightarrow{\lambda}})^{op} .$$

Dualization sends $V \supseteq W$ into $V \twoheadrightarrow W$.

This implies that generic representations of Q, which lift to $\Pi_Q^{\overrightarrow{\lambda}}$ of dimension $\overrightarrow{d}$, have α and $\overrightarrow{d} - \alpha$-dimensional subrepresentations.

Claim 1.28. *The generic $V \in \mathrm{Rep}_{\overrightarrow{d}} Q$ lifts to $\Pi_Q^{\overrightarrow{\lambda}}$.*

This follows from the fact $\overrightarrow{d}$ is in the fundamental region, and $q(\overrightarrow{d}) \neq 0$ or $\overrightarrow{d}$ indivisible. Therefore the generic representation of dimension $\overrightarrow{d}$ is indecomposable.

Then, generically representations of dimension $\overrightarrow{d}$ have α and $\overrightarrow{d} - \alpha$-dimensional subrepresentations. By a result of Scheefield, the generic decomposition is decomposable, which is a contradiction. So generic $\Pi_Q^{\overrightarrow{\lambda}}$-representation is simple. $\square$

Now we use reflection functors to deduce the existence of simple representations in general.

Proposition 1.29. *Let $\overrightarrow{d} \in \Delta_+$ be such that $\overrightarrow{d} \cdot \overrightarrow{\lambda} = 0$. There exists a sequence of reflections:*

$$\mathrm{Rep}_{\overrightarrow{d}} \Pi_Q^{\overrightarrow{\lambda}} \xrightarrow{\sim} \mathrm{Rep}_{s_{i_1} \overrightarrow{d}} \Pi_Q^{r_{i_1} \overrightarrow{\lambda}} \xrightarrow{\sim} \mathrm{Rep}_{s_{i_2} s_{i_1} \overrightarrow{d}} \Pi_Q^{r_{i_2} r_{i_1} \overrightarrow{\lambda}} \longrightarrow \cdots \xrightarrow{\sim} \mathrm{Rep}_{\overrightarrow{d}'} \Pi_Q^{\overrightarrow{\lambda}'}$$

such that $d' = \epsilon_j$ for some j or d' belongs to the fundamental region.

Proof. Recall that we only have a problem with applying the i-reflection functor if $\lambda_i = 0$. The strategy is: if $(\overrightarrow{d}, \epsilon_i) > 0$, then apply s_i.

We have to show that if $(\overrightarrow{d}, \epsilon_i) > 0$ and $\lambda_i = 0$, then $\overrightarrow{d} = c\epsilon_i$ for some $c \geq 1$.

Call $c = (\overrightarrow{d}, \epsilon_i) > 0$, so as above $q(\overrightarrow{d}) > q(\overrightarrow{d} - c\epsilon_i) + q(c\epsilon_i)$. Note that i is loop-free, otherwise $(\overrightarrow{d}, \epsilon_i) \leq 0$. In consequence, $q(c\epsilon_i) = 1 - c^2 \leq 0$, so $q(\overrightarrow{d}) > q(\overrightarrow{d} - c\epsilon_i)$. Therefore,

$$p(\overrightarrow{d}) < p(\overrightarrow{d} - c\epsilon_i) = p(\overrightarrow{d} - c\epsilon_i) + cp(\epsilon_i),$$

and the decomposition $\overrightarrow{d} = (\overrightarrow{d} - c\epsilon_i) + \epsilon_i + \cdot + \epsilon_i$ violates 1.23 if $\lambda_i = 0$ $\square$

As a consequence, there exists a simple representation if $q(\overrightarrow{d}) \neq 0$ or $\overrightarrow{d}$ is indivisible. In other case, More problems here with my notes!

Multiplicative case: For simple part (showing that there exists an irreducible solution of multiplicative Deligne-Simpson problem), this follows from the existence of solutions in $\overline{\mathcal{C}}_i$ not necessarily irreducible, using similar arguments for $\Lambda_Q^{\overrightarrow{t}}$.

For latter part, i.e. when $\mathrm{Rep}_{\overrightarrow{d}}\Lambda_Q^{\overrightarrow{t}} \neq \emptyset$, one uses geometry:

$$\mathrm{Rep}_{\overrightarrow{d}}\Lambda_Q^{\overrightarrow{t}} \longleftrightarrow \text{parabolic bundles} \longleftrightarrow \text{bundles with flat connection on } \mathbb{P}^1 \setminus \{x_1, ..., x_p\}.$$

A *parabolic bundle* is a holomorphic bundle on $\mathbb{P}^1$ together with flags

$$\text{whole fiber} = E_{i,0} \supsetneq E_{i,1} \supsetneq \cdots \supsetneq E_{i,l_i} = 0$$

at fiber over x_i.

It is equipped with connections with logarithmic singularities at x_i. Apply Weil: there exists a holomorphic connection bundle on a closed surface R if and only if all the indecomposable summands have degree 0.

2. Last lecture!!! Choose a title, and maybe change subsections titles...

2.1. About a result of Derksen-Weyman.

We look at some results in Derksen-Weyman's paper "Combinatorics of quiver representations". It deals with semi-invariants: $\mathrm{Rep}_{\overrightarrow{d}}Q//SL_{\overrightarrow{d}}$, where $SL_{\overrightarrow{d}} = \prod_{i \in Q_0} DL_{d_i}$.

Remember that a Young tableau $\lambda = (\lambda_1, ..., \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n = 0$ is a collection of boxes, arranged in n rows with λ_i boxes on each one: they are related with irreducible representations of GL_n. The *Littlewood-Richardson coefficients* $C^{\nu}_{\lambda,\mu}$ are the multiplicity of V_ν in $V_{\lambda,\mu}$, for any ν, λ, μ Young tableau.

Given Q a quiver and $\sigma : GL_{\overrightarrow{d}} \to \mathbb{C}^\times$ a character, define

$$SI(Q, \overrightarrow{d})_\sigma := \mathbb{C}[\mathrm{Rep}_{\overrightarrow{d}}Q]_\sigma^{SL_{\overrightarrow{d}}} = \{f \in \mathbb{C}[\mathrm{Rep}_{\overrightarrow{d}}Q]^{SL_{\overrightarrow{d}}} : g * f = \sigma(g)f, \forall g \in GL_{\overrightarrow{d}}\}.$$

Theorem 2.1 (DW)**.** *Let Q be the 3-branch star-shaped quiver:*

$$\begin{tikzcd}
\circ^1 \arrow[r, dash] & \circ^2 \arrow[r, dash] & \cdots \arrow[r, dash] & \circ^{n-1} \arrow[dr, dash] & \\
\circ^1 \arrow[r, dash] & \circ^2 \arrow[r, dash] & \cdots \arrow[r, dash] & \circ^{n-1} \arrow[r, dash] & \circ^n \\
\circ^1 \arrow[r, dash] & \circ^2 \arrow[r, dash] & \cdots \arrow[r, dash] & \circ^{n-1} \arrow[ur, dash] &
\end{tikzcd}.$$

Consider $\sigma = \prod_{i \in Q_0} \det_i^{\sigma_i}$, *where* $\det_i : GL_{d_i} \to \mathbb{C}^\times$ *is the determinant at vertex* i, *and* $\sigma_i \in \mathbb{Z}$ *are given by* a_i, b_i, c_i *as follows:*

$$\begin{tikzcd}
\circ^{a_1} \arrow[r, dash] & \circ^{a_2} \arrow[r, dash] & \cdots \arrow[r, dash] & \circ^{a_{n-1}} \arrow[dr, dash] & \\
\circ^{b_1} \arrow[r, dash] & \circ^{b_2} \arrow[r, dash] & \cdots \arrow[r, dash] & \circ^{b_{n-1}} \arrow[r, dash] & \circ^{c_n} \\
\circ^{c_1} \arrow[r, dash] & \circ^{c_2} \arrow[r, dash] & \cdots \arrow[r, dash] & \circ^{c_{n-1}} \arrow[ur, dash] &
\end{tikzcd},$$

$$\begin{aligned}
\lambda &= (a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, ..., a_{n-1}, 0),\\
\mu &= (b_1 + \cdots + b_{n-1}, b_2 + \cdots + b_{n-1}, ..., b_{n-1}, 0),\\
\nu &= (-c_n, -c_n - c_{n-1}, ..., -c_n - c_{n-1} - \cdots - c_1).
\end{aligned}$$

Then, $C^\nu_{\lambda,\mu} = \dim SI(Q, \overrightarrow{d})_\sigma$.

2.2. **About the semester, and Hall algebras.** This semester we did:

(i) **Kac's Theorem:** There exists an indecomposable representation of Q of dimension $\overrightarrow{d}$ if and only if $\overrightarrow{d} \in \Delta_+$. This is unique up to isomorphism if and only if $\overrightarrow{d} \in \Delta_+^{re}$.

In general, $\dim(Indec_{\overrightarrow{d}} Q/GL_{\overrightarrow{d}}) = 1 - q(\overrightarrow{d})$: it is proved in characteristic 0, and Kac proved it for characteristic p and algebraically closed fields.

We consider now representations over $\mathbb{F}_q$, we count it and call $F(q, Q, \overrightarrow{d})$.

Conjecture 2.2 (Kac)**.** *For fixed* $Q, \overrightarrow{d}$, *it is a polynomial in* q.

It was proved for cases $\overrightarrow{d}$ indivisible using preprojective algebras and Hall algebras by Crawley-Boewey and van der Bergh.

Δ_+ was defined using reflections (Weyl group). There is also a Lie-theoretic definition:

Definition 2.3. Let Q be a quiver, and consider its Cartan matrix $C = (c_{ij})$:

$$c_{ij} = \begin{cases} 2, & i = j; \\ -\#(i \to j) - \#(j \to i), & i \neq j \end{cases}.$$

A *realization* of this C is a $\mathbb{Q}$-vector space $\mathfrak{h}$ together with elements $\{\alpha_i\} \subseteq \mathfrak{h}^*$, $\{\alpha_i^\vee\} \subseteq \mathfrak{h}$ such that $\alpha_i(\alpha_j^\vee) = c_{ij}$ and $\dim(\mathfrak{h}) = 2|Q_0| - \mathrm{rk}C$.

$\mathfrak{h}, \{\alpha_i\}, \{\alpha_i^\vee\}$ are unique up to isomorphism.

Let $\widetilde{\mathfrak{g}}$ be the Lie algebra generated by $e_i, f_i, \mathfrak{h}$ as a Lie algebra, with relations:

$$\begin{aligned}
[\mathfrak{h}, \mathfrak{h}] &= 0, & [h, e_i] &= \alpha_i(h) e_i,\\
[e_i, f_j] &= \delta_{i,j} \alpha_i^\vee, & [h, f_i] &= -\alpha_i(h) f_i,
\end{aligned}$$

for $h \in \mathfrak{h}$. Let I be the unique maximal ideal such that $I \cap \mathfrak{h} = 0$. The Kac-Moody Lie algebra associated with C is

$$\mathfrak{g} := \widetilde{g}/I.$$

Δ_+ is the set of roots of the associated Kac-Moody Lie algebra of Q.

Note that $\mathfrak{g}$ is finite dimensional if and only if Q is Dynkin, in which case it is the usual simple algebra associated to Q.

We also talked about special cases of Kac's Theorem:

- Gabriel's Theorem: There exist finitely many isoclasses of indecomposables iff Q is Dynkin.

- Extended Dynkin case: rigid indecomposables. Indecomposables of dimension δ are bricks, and for all $m \geq 1$ we have a one-dimensional family of indecomposables of dimension $m\delta$, obtained by extensions of indecomposables of dimension δ.

(ii) **Mc Kay Correspondence:**

$$\begin{tikzcd}
\text{Dynkin diagrams} \arrow[dr, bend right=10, "\text{root system}"'] & & \text{finite subgroups of } SL_2\mathbb{C} \arrow[ll, bend right=5, "\text{zero fiber of } \widetilde{\mathbb{C}^2/G} \to \mathbb{C}^2/G"'] \arrow[ll, bend left=5, "\text{McKay quiver}"] \arrow[d, dotted, bend left=10] \\
& & \text{Simply laced f.d. simple Lie algebras.} \arrow[ull, bend right=10] \arrow[u, dotted, bend left=10]
\end{tikzcd}$$

To go from simple Lie algebras to finite dimensional subgroups of $SL_2\mathbb{C}$, we can use Slodowy slices. Consider

$$Nil := \{x \in \mathfrak{g} : \mathrm{ad}x \text{ nilpotent}\} \subseteq \mathfrak{g},$$

which is singular. Let G be the associated Lie group, with Borel subgroup B. By a result of Springer, there exists a resolution $T^*(G/B) \twoheadrightarrow Nil$.

In $G = GL_n$ case, fix $B \subseteq G$, any element $g \in G/B$ gives a unique choice of Borel gBg^{-1}, and this corresponds to a flag $\mathbb{C}^n = V_n \supseteq V_{n-1} \supseteq \cdots \supseteq V_1 \supseteq 0$. Given $x \in Nil$, if regular, $x^i V_n = V_{n-i}$ has dimension $n - i$. A choice of linear maps

$$\begin{tikzcd}
V_n \arrow[r, two heads] & V_{n-1} \arrow[r, two heads, "x"] & \cdots \arrow[r, two heads, "x"] & V_1 \arrow[r, two heads, "x"] & 0
\end{tikzcd}$$

gives a cotangent vector.

Now let $x \in Nil$ have $Ad\,G(x)$ with codimension 2 in Nil (called subprincipal) form an slice: a singular complex surface. Then we get

$$\begin{tikzcd}
\widetilde{\mathbb{C}^2/\Gamma} \arrow[r, two heads] \arrow[d, hook] & \mathbb{C}^2/\Gamma \arrow[d, hook] \\
T^*(G/B) & Nil
\end{tikzcd}$$

which is our resolution from before. This gives a map from simple Lie algebras to finite subgroups of $SL_2\mathbb{C}$.

Now we want to complete our correspondences with:

$$\begin{tikzcd}
\text{Dynkin diagrams} \arrow[d] \arrow[dr, shift left] & \text{finite subgroups of } SL_2\mathbb{C} \arrow[l] \\
\text{Rep of finite type } Q \arrow[r, "\text{Hall algebras}"] & \text{Simply laced f.d. simple Lie algebras.} \arrow[u] \arrow[ul, shift left]
\end{tikzcd}$$

Definition 2.4. Given $\mathcal{C}$ an abelian category, the *Hall algebra* of $\mathcal{C}$ is the algebra $H_{\mathcal{C}}$ generated over $\mathbb{C}$ by isomorphism classes of objects $X \in \mathcal{C}$, with product:

$$[X] \circ [Y] = \sum_{[Z]} N_{XY}^{Z} [Z],$$

$$N_{XY}^{Z} := \#\{Z' \subseteq Z : Z' \cong Y, Z/Z' \cong X\}.$$

N_{XY}^{Z} can be defined as the number of isoclasses of short exact sequences $0 \to X \to Z \to Y \to 0$, where the equivalence relation is given by the existence of

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\sim"] & Z \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & X \arrow[r] & Z \arrow[r] & Y \arrow[r] & 0.
\end{tikzcd}$$

Claim 2.5. *This forms an associative algebra. We should correct the product by*

$$[X] \cdot [Y] = \langle X, Y \rangle_m [X] \circ [Y], \quad \langle X, Y \rangle_m = \prod (\dim \operatorname{Ext}^i(X,Y))^{(-1)^i}.$$

Note that for Rep$Q$, $\langle X, Y \rangle_m = v^{\dim \operatorname{Hom}(X,Y) - \dim \operatorname{Ext}^1(X,Y)}$, where $v^2 = q$.

We define a coproduct for this algebra (really, a topological one), $\Delta : H_{\mathcal{C}} \to H_{\mathcal{C}} \widehat{\otimes} H_{\mathcal{C}}$,

$$\Delta([Z]) = \sum_{[X],[Y]} N_{Z}^{XY} [X] \widehat{\otimes} [Y],$$

where N_{Z}^{XY} is defined as the number of isoclasses of short exact sequences $0 \to X \to Z \to Y \to 0$, where the equivalence relation is given by the existence of morphisms with identities in X, Y in this case.

Theorem 2.6 (Green). *$H_{\mathcal{C}}$ is a topological Hopf algebra, which is self dual using* $([X],[Y]) = \delta_{[X],[Y]}$.

Theorem 2.7. *In case $\mathcal{C} = \operatorname{Rep}(Q|\mathbb{F}_q)$, we get a monomorphism of Hopf algebras: $U_v(\mathfrak{g})_+ \hookrightarrow \widetilde{H}_{\mathcal{C}} = H_{\mathcal{C}} \otimes \mathbf{k}$. Such morphism is an isomorphism if and only if Q is Dynkin.*

Example 2.8. For $Q : \circ^1 \longrightarrow \circ^2$, indecomposables are S_1, S_2 and $S_{1,2}$, of dimension $(1,0)$, $(0,1)$ and $(1,1)$.

$N_{S_1,S_1}^{S_1} = \#(S_1 \hookrightarrow S_1 \oplus S_1) = \#(\mathbb{P}^1_{\mathbb{F}_q}) = 1 + q$, so $[S_1] \cdot [S_1] = v(q+1)[S_1 \oplus S_1]$ (the same for S_2). Also,

$$[S_1] \cdot [S_2] = v^{-1}[S_{12}] + v^{-1}[S_1 \oplus S_2], \quad [S_2] \cdot [S_1] = [S_1 \oplus S_2].$$

This algebra has a PBW basis: $[S_1]^a [S_2]^b [S_{12}]^c$.