The one-dimensional Potts model with long-range interactions: a renormalization group approach

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Abstract. The one-dimensional q-state Potts model with ferromagnetic pair interactions which decay with the distance r as $1/r^{\alpha}$ is considered. We calculate, through a real-space renormalization group technique using Kadanoff blocks of length b, the critical temperature $T_{\mathrm{c}}(b, q, \alpha)$ and the correlation length critical exponent $\nu(b, q, \alpha)$ as a function of α for different values of q. Some of the very few known rigorous results for general q are reproduced by our approach. Several asymptotic behaviours are derived analytically for $q = 2$, 3 in the $b \to \infty$ limit. We also obtain extrapolated critical temperatures ($b = \infty$) for arbitrary values of $\alpha > 1$ and for $q = 2$, 3, 4, which we believe approximate the exact ones well, except in the region near $\alpha = 2$. Furthermore, the use of another extrapolation procedure suitable only in the vicinity of $\alpha = 2$ led us to conjecture that the *exact* critical temperature $T_{\mathrm{c}}(q, \alpha = 2)$ is the same for *any value of* q. We also verify that $T_{\mathrm{c}}(q, \alpha \to 1) \propto (\alpha - 1)^{-1}\ \forall q$, which is consistent with a recent conjecture of Tsallis.

1. Introduction

It is well known that one-dimensional spin models can present an ordered state at low temperatures if the microscopic interactions fall off slowly enough with the distance [1–6]. For example, in the case of the spin-$\frac{1}{2}$ Ising ferromagnet with long-range interactions proportional to $r_{ij}^{-\alpha}$ (where r_{ij} is the distance between the spins at sites i and j), the existence of a phase transition at a non-zero critical temperature was proven by Dyson [1] for $1 < \alpha < 2$ and by Frölich and Spencer [5] for $\alpha = 2$. Moreover, the thermodynamic properties of the systems (the kind described above) near the critical point frequently present new behaviours, which are absent in short-range (SR) models. Hence, the study of such properties is needed in order to gain a deeper comprehension of the general theory of critical phenomena.

Besides their fundamental theoretical interest in physics, microscopic models with long-range (LR) interactions are of interest nowadays in view of their relationship with neural systems modelling [7], where far away localized neurons interact through an action potential which decays slowly along the axon. Other related problems are spin systems with RKKY-like interactions ($1/r_{ij}^{\alpha}\cos(ar_{ij})$) which are present in spin glasses [8], critical phenomena in highly ionic systems [9], Casimir forces between inert uncharged particles immersed in a

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fluid near the critical point [10], the kinetic Ising model with random spin exchanges (Lévy flights) [11], phase segregation in model alloys [12] and pattern recognition [13].

In this paper we address the q-state Potts model [14] with LR interactions, i.e. we consider the Hamiltonian:

$$H = -J \sum_{(i,j)} \frac{1}{r_{ij}^{\alpha}} \delta(\sigma_i \sigma_j) \qquad (\sigma_i = 1, 2, \ldots, q, \forall i; J > 0; \alpha > 0) \tag{1}$$

where to each site, i, we associate a Potts variable, σ_i, which can assume q integer values ($\sigma_i = 1, 2, \ldots, q$), r_{ij} is the distance (in crystal units) between sites i and j (i.e. $r_{ij} = \|i - j\| = 1, 2, 3 \cdots$), $J > 0$ is the ferromagnetic coupling constant between nearest neighbours, $\delta(\sigma_i, \sigma_j)$ is the Kronecker delta function, and the sum $\sum_{(i,j)}$ runs over all distinct pairs of sites of a one-dimensional lattice of N sites. The $\alpha \to \infty$ limit corresponds to the first-neighbour model, while the $\alpha = 0$ limit corresponds to the infinite-range ferromagnet which, after a rescaling $J \to J/N$, yields basically the mean-field approach.

This model, in its plain formulation ($\alpha \to \infty$ of equation (1)) or in a more general one with many-body interactions, is at the heart of a complex network of relations between geometrical and/or thermal statistical models, such as for example various types of percolation, vertex models, generalized resistor and diode network problems, classical spin models, etc (see [15] and references therein).

On the other hand, the one-dimensional Potts model with LR interactions has definitely not been studied so much. In particular, very few rigorous results for general q are known. Let us summarize some of the most relevant results to date: (i) this model exhibits LR order at *finite* temperatures [16] $T \leqslant T_c(q, \alpha)$ for $1 < \alpha \leqslant 2$; for $\alpha \to 1$ the critical temperature diverges and for $\alpha \leqslant 1$ the thermodynamic limit is not defined and the system becomes non-extensive; (ii) for $\alpha > 2$ (SR interactions) it has no phase transition at *finite* temperatures [16] for all $q \geqslant 1$, more precisely, $T_c = 0$; (iii) it has been proved that for $\alpha = 2$ the order parameter is discontinuous at $T = T_c \neq 0$ for any q [16]; (iv) for $q = 1$ the percolation threshold satisfies $1/p_c \leqslant 2\zeta(\alpha)$ for $1 < \alpha \leqslant 2$, where $\zeta(\alpha)$ is the Riemann Zeta function [17].

All the following additional results correspond to the $q = 2$ case, which is, up to now, the best one studied: (v) for $1 < \alpha < 1.5$ the critical exponents are classical [18]; (vi) the region $1.5 < \alpha < 2$ shows non-trivial critical exponents, which are not known exactly. Approximate results in the latter region were obtained by different methods such as (among others): series expansions [19], finite-range scaling approximations [20], coherent anomaly method [21], real-space renormalization group [22], ϵ-expansions [3, 6], around $\alpha = 2$ where the critical behaviour is of an essential singularity type [23], and $\alpha = 1.5$.

Some approximate results for the critical temperature and the correlation length critical exponent ν were obtained for a wide range of values of q using finite-range-scaling calculations [24].

The $\alpha = 2$ (i.e. the $1/r^2$ potential) case is of particular interest because for $q = 2$ it can be mapped onto the spin-$\frac{1}{2}$ Kondo problem [25] (which is related to recent developments in high temperature superconductivity [26]) and for a general value of $q > 2$ it may be related to higher spin generalizations of the Kondo problem [23].

In order to calculate the critical temperature and the critical exponent ν of the q-state LR Potts model in the extensive region $1 < \alpha \leqslant 2$ we use a real-space renormalization group (RG) method, the cumulant method of Niemeijer and van Leeuwen [27], based on a construction of Kadanoff blocks using the majority rule. Although the convergence of the cumulant method for a fixed block size can become questionable in some cases (for a discussion on the advantages and disadvantages of the method see, for example [28, 29]), a

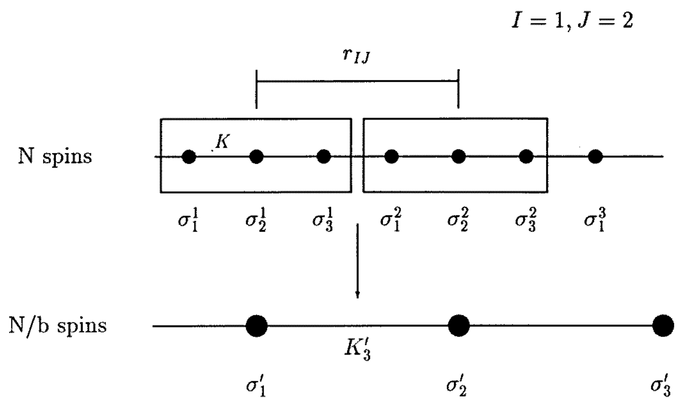

Figure 1. Renormalization group transformation using Kadanoff blocks of length $b = 3$ in the one dimensional lattice; r_{IJ} is the distance between the blocks I and J.

great improvement has been obtained in the nth cumulant result for the critical temperature of simple ferromagnets as one increases the cell size. In a recent work [22], one of us adapted the Niemeijer–van Leeuwen RG method [27] to the one-dimensional LR Ising model (in this case it can be shown that the cumulant expansion becomes a series expansion in powers of [22] $1/b^\alpha$). In the case of $q = 2$ states the tie-breaking problem in the majority rule can be easily avoided by considering only blocks with an odd number of sites; for $q \geqslant 3$ this ansatz does not work. Hence, in this paper we generalize the above technique by introducing an *equally probable tie-breaking majority rule*. We expect that this method applied to blocks of increasing lengths b, together with our proposed extrapolation for $b \to \infty$, gives good results for the Potts model, provided the phase transition is a second-order one [15]. The paper is organized as follows. In section 2 the general RG formalism is described. In section 3 we present our results which recover those of [22] for $q = 2$. Finally, the conclusions are given in section 4.

2. The RG formalism

We start by constructing Kadanoff blocks of length $b > 1$, as shown in figure 1 for the particular case $b = 3$; we will consider, for simplicity, only odd values of b herein. The parameter b characterizes the *rescaling length* of the RG transformation. The blocks will be numbered by capital letters. We will assign a block-spin variable $\sigma_I' = 1, 2, \ldots, q$ to every block I. Let us denote by $\sigma_i^I = 1, 2, \ldots, q$ ($i = 1, 2, \cdots, b$; $I = 1, 2, \cdots, N/b$) the spin state at the ith site of the block I. Then, defining the dimensionless Hamiltonian

$$\mathcal{H} \equiv -K \sum_{I=1}^{N/b} \sum_{J=1}^{N/b} \sum_{i \in I} \sum_{j \in J} \frac{1}{r_{ij}^\alpha} \delta(\sigma_i^I, \sigma_j^J) \qquad (i \neq j) \tag{2}$$

with $K \equiv \beta J$ ($\beta = 1/k_B T$; hereafter we take $k_B = 1$), a renormalized (block) Hamiltonian is determined by the following RG transformation:

$$\mathrm{e}^{-(\mathcal{H}'+\mathcal{C})} = \mathrm{Tr}_{\{\sigma_i^I\}} \{P(\{\sigma_i^I\}, \{\sigma_I'\}) \mathrm{e}^{-\mathcal{H}}\}. \tag{3}$$

The symbol $\mathrm{Tr}_{\{\sigma_i^I\}}$ denotes a sum over all the configurations of site-spins σ_i^I, $\mathcal{C}$ is a spin-independent constant and

$$P(\{\sigma_i^I\}, \{\sigma_I'\}) = \prod_{I=1}^{N/b} P_I(\{\sigma_i^I\}, \sigma_I') \tag{4}$$

is a weight function which characterizes the majority rule with *equally probable tie-breaking*, that means:

$$P_I = \begin{cases} 1/m & \text{if one of the } m \text{ major subgroups of } \{\sigma_i^I\} \text{ is in the state } \sigma_I' \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For instance, in the case $b = 5$, $q = 4$ with $\{\sigma_i^I\} = \{1, 1, 4, 4, 3\}$ and $\sigma_I' = 4$ then $P_I = \frac{1}{2}$.

The Hamiltonian, $\mathcal{H}$, can be divided into two parts: $\mathcal{H} = \mathcal{H}_0 + V$, where $\mathcal{H}_0 = \sum_I \mathcal{H}_0^I$ and $V = \sum_{(I,J)} V_{IJ}$; $\mathcal{H}_0^I$ includes only the interactions between spins *inside* the block I, whereas V_{IJ} includes the interactions between spins belonging to *different* blocks I and J. Introducing the intra-block expectation values:

$$\langle \mathcal{O} \rangle_0 \equiv \frac{1}{Z_0} \mathrm{Tr}_{\{\sigma_i^I\}} \{P(\{\sigma_i^I\}, \{\sigma_I'\}) \exp[-\mathcal{H}_0(\{\sigma_i^I\})] \mathcal{O}\} \tag{6}$$

with

$$Z_0 \equiv \prod_I Z_0^I \tag{7}$$

and

$$Z_0^I = \mathrm{Tr}_{\{\sigma_i^I\}} P_I(\{\sigma_i^I\}, \{\sigma_I'\}) \exp[-\mathcal{H}_0^I(\{\sigma_i^I\})] \tag{8}$$

we can rewrite equation (3) as:

$$\mathrm{e}^{-(\mathcal{H}' + \mathcal{C})} = Z_0 \langle \mathrm{e}^{-V} \rangle_0. \tag{9}$$

Using a cumulant expansion of $\langle \mathrm{e}^{-V} \rangle_0$, a *first-order* approximation of $\mathcal{H}'$ can be obtained through:

$$\mathcal{H}' \approx \langle V \rangle_0|_{\mathrm{sdp}} = \sum_{(I,J)} \langle V_{IJ} \rangle_0|_{\mathrm{sdp}} \tag{10}$$

where sdp refers to the spin dependent part on $\{\sigma_I'\}$ of the resulted average.

Let r_{IJ} be the distance between the centre sites of the blocks I and J (see figure 1), measured *in units of the rescaling length b*. For $r_{IJ} \gg 1$ we can approximate [22]

$$r_{ij} \approx b r_{IJ}. \tag{11}$$

Then

$$\langle V_{IJ} \rangle_0|_{\mathrm{sdp}} \approx -\frac{K}{b^\alpha r_{IJ}^\alpha} \sum_{i \in I} \sum_{j \in J} \langle \delta(\sigma_i^I, \sigma_j^J) \rangle_0|_{\mathrm{sdp}}. \tag{12}$$

Since the expectation value (6) is carried out with a block-independent probability distribution it follows that

$$\langle \delta(\sigma_i^I, \sigma_j^J) \rangle_0 = \sum_{l=1}^{q} \langle \delta(\sigma_i^I, l) \delta(\sigma_j^J, l) \rangle_0 \tag{13}$$

$$= \sum_{l=1}^{q} \langle \delta(\sigma_i^I, l) \rangle_0 \langle \delta(\sigma_j^J, l) \rangle_0. \tag{14}$$

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On the other hand, by symmetry, one has that:

$$\langle\delta(\sigma_i^I, l)\rangle_0 = a_i(K, q, \alpha)\delta(\sigma_I', l) + b_i(K, \alpha) \tag{15}$$

where a_i and b_i are block-independent functions of K, α, q and the site i. By combining equation (15) with equations (10), (12) and (14), and by using the fact that $\sum_{l=1}^{q} \delta(\sigma_I', l) = 1$ and $\sum_{l=1}^{q} \delta(\sigma_I', l)\delta(\sigma_J', l) = \delta(\sigma_I', \sigma_J')$ one obtains that:

$$\mathcal{H}' = -K_b'(K, q, \alpha) \sum_{(I,J)} \frac{1}{r_{IJ}^{\alpha}} \delta(\sigma_I', \sigma_J') \tag{16}$$

where

$$K_b'(K, q, \alpha) = \frac{K}{b^\alpha}\left[\sum_{i=1}^{b} a_i(K, q, \alpha)\right]^2 \tag{17}$$

is our RG recurrence equation. By using equation (15) we can express $a_i(K, q, \alpha)$ as:

$$a_i(K, q, \alpha) = \frac{1}{q-1}[q\langle\delta(\sigma_i^I, 1)\rangle_0|_{\sigma_I'=1} - 1]. \tag{18}$$

Since the nth cumulant of $\langle\exp(-V)\rangle$ is of the order of $1/b^{n\alpha}$, approximation (10) can be seen as the leading term in a series expansion [22] of equation (3) in powers of $1/b^\alpha$. Therefore, it is expected that the results will be systematically improved for increasingly high values of b.

3. Results

3.1. Analysis of the recurrence equation

We now analyse the recurrence equation (17) and its fixed points $K^* = K_b'(K^*, q, \alpha)$ as a function of α for different values of $q \geqslant 2$. The typical structure of equation (17) is as follows. It always shows two trivial fixed points: $K = 0$ ($T = \infty$) and $K = \infty$ ($T = 0$). From equation (18) we found that $a_i(K, q, \alpha) \sim 1\ \forall i, q, \alpha$ for $K >> 1$ ($T \to 0$); hence, from equation (17) we obtain the asymptotic behaviour $K_b'(K, q, \alpha) \sim b^{2-\alpha}K\ \forall q$. For low values of α the gradient of $K_b'(K, q, \alpha)$ at $K = 0$ is greater than one and it does not present a (non-trivial) fixed point for finite values of K. In this case the fixed point $K = 0$ is repulsive and therefore $T_c = \infty$. For intermediate values of α, $K_b'(K, q, \alpha)$ possess a non-trivial fixed point at finite $K = K_c(b, q, \alpha) \equiv J/T_c(b, q, \alpha)$. For $\alpha > 2$ the gradient of $K_b'(K, q, \alpha)$ is less than one and there is again no fixed point at finite K. In this case, however, the fixed point $K = 0$ is attractive and therefore $T_c = 0$ for all values of b, recovering the exact result. Therefore, some value $\alpha_1(b, q)$ exists such that (i) $T_c = \infty$ for $\alpha \leqslant \alpha_1(b, q)$; (ii) there is a phase transition at finite temperature $T_c(b, q, \alpha)$ for $\alpha_1(b, q) < \alpha < 2$ and (iii) $T_c = 0$ for $\alpha > 2$.

The borderline value $\alpha_1(b, q)$ is determined by the condition $\mathrm{d}K_b'/\mathrm{d}K\|_{K=0} = 1$. This equation can be solved by noting that

$$a_i(0, q, \alpha) = \gamma(b, q) \equiv \frac{1}{q-1}\left[q^{2-b}\sum_{m=1}^{m_{\max}} \frac{A_m(b, q)}{m} - 1\right] \qquad (\forall i). \tag{19}$$

The coefficient A_m gives the number of configurations of b spins (where each one can be in the states $\sigma_i^I = 1, 2, \ldots, q$) of a block where one of the m major subgroups of $\{\sigma_i^I\}$ is in a fixed state, say 1, and $\sigma_1^I = 1$.

From equation (17) we obtain that $\mathrm{d}K_b'/\mathrm{d}K\|_{K=0} = b^{2-\alpha}\gamma(b,q)^2$ and therefore

$$\alpha_1(b,q) = 2\left[1 + \frac{\ln\gamma(b,q)}{\ln b}\right].$$

For $q = 2$ we have that [22]

$$\gamma(b,2) = \frac{(b-1)!}{2^{b-1}\left(\frac{b-1}{2}!\right)^2}$$

which for $b \gg 1$ behaves as $\gamma(b,2) \sim \frac{2}{\sqrt{2\pi}} b^{-1/2}$ and we recover the exact result $\alpha_1(b,2) \to 1$ in the limit $b \to \infty$.

For higher values of q the calculation of the quantities A_m involves a lot of combinatorial analysis. For $q = 3$ we found the following expression:

$$\gamma(b,3) = \frac{1}{2}\left\{3^{2-b}\left[\sum_{l=0}^{X}\binom{b-1}{l}2^l + \sum_{j=1}^{\mathrm{Int}(X/3)}\sum_{j_1=2j}^{X-j}\binom{b-1}{X+j}\binom{X+j}{j_1}\right.\right.$$
$$+\sum_{l=0}^{\mathrm{Int}\left(\frac{X-2}{3}\right)}\binom{cb-1}{X+l+1}\binom{X+l+1}{X-l}$$
$$\left.\left.+\frac{1}{3}\binom{b-1}{2b/3}\binom{2b/3}{b/3}\delta(b,3n)\right]-1\right\} \tag{20}$$

where $n = 1, 2, \ldots$, $X \equiv (b-1)/2$ and Int$(\ldots)$ represents the integer part of its argument. This form can be easily evaluated numerically for values up to $b \sim 200$. An analysis of the log–log plot of γ versus b shows that the asymptotic regime is attained for low values of $b \sim 7$ and clearly $\gamma(b,3) \sim b^{-1/2}$ for $b \to \infty$. Therefore, $\alpha_1(b,3)$ also reproduces the exact result in such a limit. For values of $q \geqslant 4$ the combinatorial problem becomes very hard. However, we performed a numerical calculation of $\gamma(b,q)$ for $q = 4, 5$ and $b = 3, 5, 7, 9$, finding again $\gamma(b,q) \sim b^{-1/2}$. All these results suggest that $\alpha_1(b,q) \to 1$ for $b \to \infty$ *for all values of* $q \geqslant 2$.

Closed forms of the function $K_b'(K,q,\alpha)$ can be obtained analytically for low values of q and b with the aid of symbolic computer languages. With these expressions the critical temperature $T_c(b,q,\alpha)$ can be calculated numerically as a function of α for fixed values of q and b. The correlation length critical exponent can also be calculated from the expression

$$\nu(b,q,\alpha) = \frac{\ln b}{\ln\left(\frac{\mathrm{d}K_b'}{\mathrm{d}K}(K,q,\alpha)\|_{K_c}\right)}. \tag{21}$$

In figure 2 we show our results for different values of q and $b = 3$ fixed, while in figure 3 we keep $q = 3$ fixed and vary b. The corresponding curves for other values of q are qualitatively similar.

3.2. $\alpha \to 2^-$ asymptotic results

For $\alpha \to 2^-$ we see that $K_c \to \infty$ ($T_c \to 0$). The asymptotic behaviour of the recurrence equation (17) in such a limit can be obtained by adding an external field h into the Hamiltonian (2), i.e. $\mathcal{H}_0^I \to \mathcal{H}_0^I + h\sum_{i=1}^b \delta(\sigma_i^I, 1)$. Then, in the $h \to 0$ limit, it is easy to prove that

$$\sum_{i=1}^{b}\langle\delta(\sigma_i^I,1)\rangle_0 = \left.\frac{\partial \ln Z_0^I}{\partial h}\right|_{h=0}. \tag{22}$$

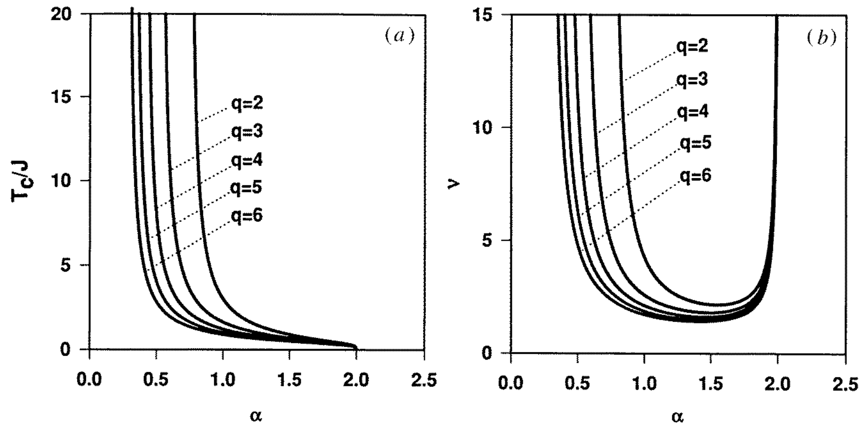

Figure 2. Numerical calculations for $b = 3$ and different values of q. (a) Critical temperature $T_c(b, q, \alpha)/J$ versus α; (b) correlation length critical exponent $\nu(b, q, \alpha)$ versus α.

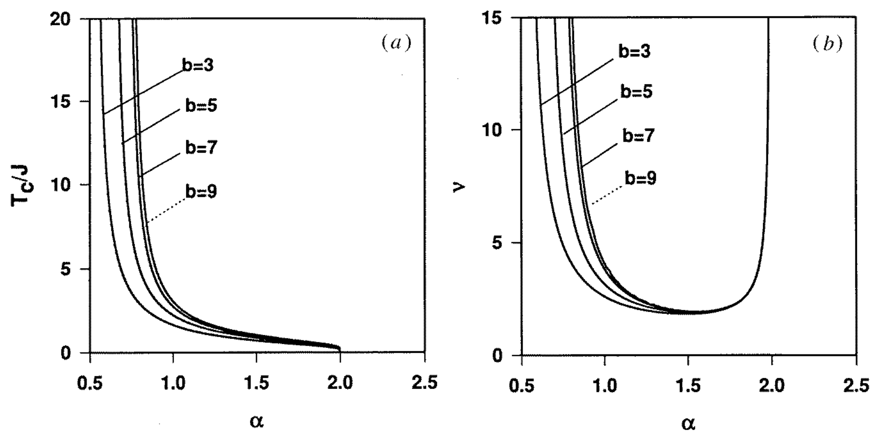

Figure 3. Numerical calculations for $q = 3$ and different values of the rescaling length b. (a) Critical temperature $T_c(b, q, \alpha)/J$ versus α; (b) correlation length critical exponent $\nu(b, q, \alpha)$ versus α.

For $K \to \infty$ we can expand

$$Z_0^I(K, h) \sim \mathrm{e}^{B_1(b,\alpha)K+bh}[1 + 2(q-1)\mathrm{e}^{-B(b,\alpha)K-h} + \cdots] \tag{23}$$

where $KB_1(b, \alpha)$ is the energy of the ground state and $KB(b, \alpha)$ is the energy difference between the ground state and the first excited state of $\mathcal{H}_0^I$. These are given by

$$B_1(b, \alpha) = \sum_{(k,j)} \frac{1}{r_{kj}^\alpha} = \sum_{n=1}^{b-1} \frac{b-n}{n^\alpha} \tag{24}$$

$$B(b, \alpha) = \sum_{n=1}^{b-1} \frac{1}{n^\alpha}. \tag{25}$$

Then, from equations (22) and (23) we obtain in the $K \to \infty$ limit, that

$$\sum_{i=1}^{b} \langle \delta(\sigma_i^I, 1) \rangle_0 \approx b - 2(q-1)\mathrm{e}^{-B(b,\alpha)K} \tag{26}$$

which combined with equations (17) and (18) leads to

$$K'_b(K, q, \alpha) \sim \frac{K}{b^\alpha} \left[b - 2q\mathrm{e}^{-B(b,\alpha)K} \right]^2 \qquad (K \to \infty). \tag{27}$$

The fixed-point equation derived from equation (27) leads to

$$T_\mathrm{c}/J \sim \frac{-B(b,\alpha)}{\ln\left(\frac{b-b^{\alpha/2}}{2q}\right)}. \tag{28}$$

For $\alpha \to 2^-$ the asymptotic behaviour of $T_\mathrm{c}(b, q, \alpha)$ is then given by the Cauchy function:

$$2 - \alpha \sim D(q, b)\mathrm{e}^{-B(b,2)J/T_\mathrm{c}} \tag{29}$$

with $D(q, b) = 4q/(b \ln b)$. In the $b \to \infty$ limit we have $B(b, 2) \to \zeta(2) = \pi^2/6$ and $D(q, b) \to 0$. $D(q, b)$ determines the region around $\alpha = 2$ in which the asymptotic regime (29) holds. Since the present RG procedure is systematically improved [22] for higher values of b, the shrinking of such a region in the $b \to \infty$ limit suggests a non-uniform convergence to a *non-zero* value $T_\mathrm{c}(\alpha = 2)$, consistently with the exact result for $q = 1, 2$. In other words, the convergence of $T_\mathrm{c}(b, q, \alpha)$ towards zero appears to be a mathematical artefact of the RG approximation, that disappears when $b \to \infty$; the leading behaviour of $T_\mathrm{c}(b, q, \alpha)$ in such a limit will therefore converge to a non-zero value, which is expected to be a good estimate of the exact result. These facts suggest that the whole region of α where the curve $T_\mathrm{c}(b, q, \alpha)$ versus α shows a negative curvature (and therefore a convergence towards zero for $\alpha \to 2$) to be spurious. Hence, the inflection point of the curve appears to be a good choice for estimating the leading behaviour of the curve in the limit $b \to \infty$. In figure 4 we show the asymptotic behaviour (28) of $T_\mathrm{c}(b, q, \alpha)$ for $q = 3$ (similar curves are obtained for other values of q) near $\alpha = 2$ as b increases. We see that the inflection points (full circles) converge to a non-zero value at $\alpha = 2$ for $b \to \infty$, while the region of negative curvature tends to disappear (notice that at these points $T_\mathrm{c}/J < 1\ \forall b$). Hence, we propose the following ansatz: a good estimate of $T_\mathrm{c}(\infty, q, 2)$ can be obtained by calculating the value of $T_\mathrm{c}(b, q, \alpha)$ at the inflection point of the Cauchy function (29) for finite b and then taking the $b \to \infty$ limit (it can be verified that for values of $b > 100$ the inflection point of equation (28) coincides with that of equation (29)). This procedure gives the value

$$T_\mathrm{c}(q, \alpha = 2)/J = B(\infty, 2)/2 = \pi^2/12 \tag{30}$$

for all values of $q \geqslant 2$. This result can be tested for $q = 2$, since for this case several results, obtained by different approximated methods, are available. Actually, the present procedure is almost the same as the one introduced in [22] for the Ising model ($q = 2$), the only difference being the criterion of extrapolation. A careful comparison of (30) with the corresponding values obtained by other methods shows that the choice of the inflection point is better than the previous one†. In particular, for the Ising model $q = 2$ (remember that $(T_\mathrm{c}/J)^{\mathrm{Ising}} = 2(T_\mathrm{c}/J)^{\mathrm{Potts}}|_{q=2}$) we have $T_\mathrm{c}(2, \alpha = 2)/J = \pi^2/6 \approx 1.64$, which compares well with other results (renormalization group [25]: 1.57; series expansions [19]: 1.63; finite range scaling [24]: 1.63; ζ function [30]: 1.69).

† The value of $T_\mathrm{c}/J = 0.79$ for Anderson and Yuval's result cited in [22] is incorrect, due to a factor of $\frac{1}{2}$ in the definition of the Hamiltonian in [25].

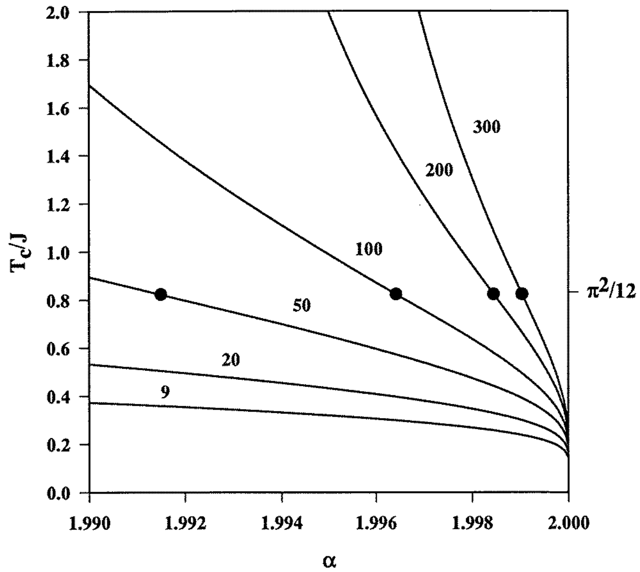

Figure 4. Asymptotic behaviour of the critical temperature $T_c(b, q, \alpha)/J$ when $\alpha \to 2$ (equation (28)) for $q = 3$. The numbers beneath the curves indicate the values of b. The full circles are the inflection points of the corresponding curves. We see a convergence of the inflection points to a constant value $\pi^2/12$ for $b \to \infty$, that results independent of q.

From equations (21), (27) and (28) one obtains the following asymptotic value for $\nu(b, q, \alpha \to 2^-)$:

$$\nu(b, q, \alpha \to 2^-) = \left\{ -(2-\alpha) \ln \left[\frac{(2-\alpha)}{D(q, b)} \right] \right\}^{-1}$$

which combined with equation (29) leads to

$$\nu(b, q, \alpha \to 2^-) = \{(2-\alpha) B(b, 2) K_c\}^{-1}.$$

Using our ansatz equation (30) one, finally, obtains in the $b \to \infty$ limit that:

$$\nu(q, \alpha \to 2^-) = [2(2-\alpha)]^{-1} \tag{31}$$

for *all values of* $q \geqslant 2$ provided that the transition is continuous when $\alpha \to 2^-$. Notice that expression (31) is in contrast with the renormalization group result of Kosterlitz [3] ($\nu_K \sim [2(2-\alpha)]^{-1/2}$) for the Ising case ($q = 2$).

3.3. High temperature asymptotic results

For $\alpha \to \alpha_1^+(b, q)$ we see that $K_c \to 0$. The fixed point equation assumes then the form

$$b^{\alpha/2} = \sum_{i=1}^{b} a_i(K_c, q, \alpha) \tag{32}$$

with

$$b^{\alpha_1/2} = \sum_{i=1}^{b} a_i(0, q, \alpha_1).$$

From equation (18) it can be verified that

$$\frac{\partial}{\partial\alpha}\left[\sum_i a_i(K,q,\alpha)\right]_{K=0}=0.$$

Then, expanding equation (32) around $K_c=0$ and $\alpha=\alpha_1$ up to first order in K_c and in $(\alpha-\alpha_1)$, and using that $b^{\alpha_1/2}=b\gamma(b,q)$ we find that:

$$\tfrac{1}{2}b\gamma(b,q)\ln b(\alpha-\alpha_1)=K_c\frac{\partial}{\partial K}\left[\sum_i a_i(K,q,\alpha_1)\right]_{K=0}.$$

Since $Z_0^I=Z^I/q$ (Z^I being the partition function of the block I) and $Z^I=q^b$ for $K=0$, we obtain from equations (18) and (24) that

$$\frac{\partial}{\partial K}\left[\sum_i a_i(K,q,\alpha_1)\right]_{K=0}=\frac{q}{(q-1)q^{b-1}}\left\{B_1(b,\alpha_1)G(b,q)-\frac{1}{q^b}\left.\frac{\partial Z^I}{\partial K}\right|_{K=0,\alpha=\alpha_1}A(b,q)\right\} \tag{33}$$

where

$$G(b,q)=\mathrm{Tr}_{\{\sigma_i^I\}}P_I(\{\sigma_i^I\},1)\delta(\sigma_k^I,\sigma_j^I)\sum_{i=1}^{b}\delta(\sigma_i^I,1) \tag{34}$$

$$\begin{aligned}A(b,q)&=\mathrm{Tr}_{\{\sigma_i^I\}}P_I(\{\sigma_i^I\},1)\sum_{i=1}^{b}\delta(\sigma_i^I,1)\\&=bq^{b-2}[1+(q-1)\gamma(b,q)]\end{aligned} \tag{35}$$

and

$$\left.\frac{\partial Z^I}{\partial K}\right|_{K=0,\alpha=\alpha_1}=B_1(b,\alpha_1)\,\mathrm{Tr}_{\{\sigma_i^I\}}\delta(\sigma_k^I,\sigma_j^I)\qquad\forall k,j\in I \tag{36}$$

$$=q^{b-1}B_1(b,\alpha_1). \tag{37}$$

Therefore, we have the asymptotic behaviour

$$\alpha-\alpha_1\sim C(b,q)K_c$$

where

$$C(b,q)=\frac{2B_1(b,\alpha_1)}{b\ln b}\left\{\frac{G(b,q)q^{2-b}-\frac{b}{q}[1+(q-1)\gamma(b,q)]}{(q-1)\gamma(b,q)}\right\}. \tag{38}$$

Since $\alpha_1\to1$ for $b\to\infty$, this result suggests that the asymptotic behaviour of T_c is *proportional to* $(\alpha-1)^{-1}$, i.e.

$$T_c(q,\alpha)/J\propto\frac{1}{\alpha-1} \tag{39}$$

for $\alpha\to1$ holds *for all values of* q, provided that the phase transition is a second-order one.

In view of the proportionality constant $C(b,q)$, equation (38) reduces, for $q=2$, to

$$C(b,2)=\frac{B_1(b,\alpha_1)}{b\ln b}\frac{\binom{b-2}{\frac{b-3}{2}}}{\gamma(b,2)2^{b-2}}.$$

Since $B_1(b, \alpha_1) \sim b \ln b$ for $b \to \infty$, we obtain that $\lim_{b\to\infty} C(b, 2) = 1$. Therefore, we find in the limit $b \to \infty$ that

$$T_c(2, \alpha)/J \sim \frac{1}{\alpha - 1} \tag{40}$$

which reproduces known results (see [22] and references therein). It is worth stressing that expression (40) recovers asymptotically the mean-field one [31].

For $q = 3$ a closed form of $G(b, q)$ (and therefore of $C(b, q)$) can also be analytically obtained. The detailed form of it can be seen in the appendix. We found numerically that $C(b, 3) \to$ constant $\approx 0.67 \approx \frac{2}{3}$ for $b \to \infty$. These results are consistent with $C(\infty, q) = 2/q \forall q \geqslant 2$, provided that the phase transition is a second-order one. Eventually it might hold also for $q = 1$, which would be consistent with Schulman's bound [17] $1/p_c \leqslant 2\zeta(\alpha)$ ($\zeta(\alpha) \sim 1/(\alpha - 1)$ for $\alpha \to 1$). It is interesting to compare this result with the mean-field one, which predicts a first-order phase transition for $q \geqslant 3$. We calculated the corresponding critical temperature following along the lines of Mittag and Stephen [32], namely

$$T_c^{\mathrm{MF}}/J = \frac{(q-2)}{(q-1)} \frac{1}{\ln(q-1)} \zeta(\alpha)$$

which for $q \gg 1$ (where mean field becomes exact [33]) and $\alpha \to 1$ behaves as

$$T_c^{\mathrm{MF}}/J \sim \frac{1}{\ln(q)} \frac{1}{\alpha - 1}.$$

Notice that this *exact* asymptotic behaviour of T_c/J for a first-order transition differs from our result $(2/q)(\alpha - 1)^{-1}$ which is expected to hold only for continuous transitions.

It is worth stressing that the asymptotic functional form (39) agrees with Tsallis' proposal [34] for unifying in a single picture both SR and LR interaction systems. This proposal has recently been verified for Lennard-Jones-like potential systems [35, 36] as well as for LR ferromagnetic Ising models [31, 37].

Finally, using the same expansion of $K_c = K_b'(K_c, q, \alpha)$ around $K_c = 0$ and $\alpha = \alpha_1$, and combining it with equation (21) we find that

$$\nu(b, q, \alpha) \sim \frac{1}{\alpha - \alpha_1} \qquad (\forall b, q \geqslant 2)$$

which, together with the result $\alpha_1(b = \infty, q) = 1$, leads, in the $b \to \infty$ limit to

$$\nu(q, \alpha) \sim \frac{1}{\alpha - 1} \qquad (\forall q \geqslant 2). \tag{41}$$

For $q = 2$ the mean-field behaviour $\nu = 1/(\alpha - 1)$ holds for $1 < \alpha < 1.5$ exactly [6, 18]. Our results suggest that such a behaviour holds, at least asymptotically for $\alpha \to 1$, *for all values of* $q \geqslant 2$. The results of Glumac and Uzelac [24] also suggest such a behaviour for all $q \leqslant 1$ and $1 < \alpha \leqslant \frac{4}{3}$. So, eventually this asymptotic behaviour for $\alpha \to 1$ might be true for all q, provided the phase transition is a continuous one.

3.4. $b \to \infty$ extrapolations for an arbitrary α

Now, we can use the asymptotic behaviours obtained in the preceding section to extrapolate the full curves $T_c(b, q, \alpha)$ versus α for $b \to \infty$ as follows [22]. First, we define the rescaled variables $x_q \equiv (2 - \alpha)/(2 - \alpha_1(b, q))$ and $y_q \equiv T_c(b, q, \alpha)(2 - \alpha_1(b, q))/JC(b, q)$, so that $y_q(x_q) \sim 1/(1 - x_q)$ for $x_q \to 1\ \forall b, q$. In figure 5 we plot, for $q = 3$, $y_q(x_q)$ versus x_q for different values of b. This figure clearly shows a data collapse for $b > 5$ (represented by

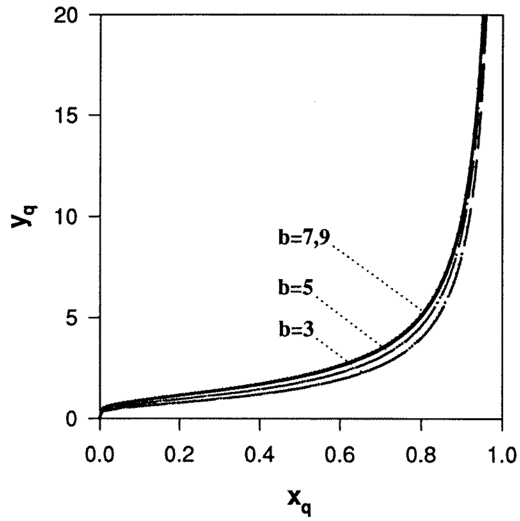

Figure 5. Rescaled critical temperature $y_q \equiv T_c(b, q, \alpha)(2 - \alpha_1(b, q))/JC(b, q)$ versus $x_q \equiv (2 - \alpha)/(2 - \alpha_1(b, q))$ for $q = 3$. All curves with $b > 5$ coincide, within the used scale, with the full curve.

a full curve in figure 5). This also appears for other values of q. Hence, such curves are expected to be good estimates of the $b = \infty$ ones. Using the results $C(\infty, q) = 2/q$ and $\alpha_1(\infty, q) = 1$ (which we know holds at least for $q = 2, 3$), we revert such curves into the (T_c, α) variables. The results, which are expected to be good estimates of the exact critical temperatures $T_c(q, \alpha)$ for values of α near one, are shown in figure 6(*a*). In table 1 we compare our results with those obtained by Glumac and Uzelac [24] through finite-range scaling (FRS) (as far as we know, these are the only published results for $q > 2$), and by Luijten and Blöte [39] (in the case $q = 2$) using Monte Carlo simulations, for some typical values of α and q. We see that our results show a good agreement with the others for values of $\alpha \sim 1$ (the percentual discrepancy is below 11% for $\alpha < 1.4$), but the difference increases for $\alpha \to 2$. However, it is not expected that our extrapolation gives reliable results near $\alpha = 2$. First, because the procedure was constructed using the asymptotic behaviours obtained for $b \to \infty$ *near* $\alpha = 1$ and therefore we expect these results to be accurate only in such regions. Second, we saw in section 3.2 that for finite b our curves present a spurious convergence to zero for $\alpha \to 2$, which can be in principle associated with a region of negative curvature of $T_c(b, q, \alpha)$ that disappears for $b \to \infty$. However, for the (relatively) small values of b used in this section (the exponential growth with b of the number of terms in the RG equations rapidly exceeds our computational capabilities) such a region extends for a wide range of values of α. Therefore, the extrapolated curves obtained in this section also presents the spurious behaviour, even far away from $\alpha = 2$ ($\alpha \gtrsim 1.7$)

The same extrapolation procedure can be applied to the critical exponent ν, using the asymptotic behaviour (41). The numerical results are depicted in figure 6(*b*) for $q = 2$, 3, 4; our asymptotic result equation (31) is also represented by a dotted curve. We also include, for comparison, the exact value [18] for $q = 2$ and $1 < \alpha \leqslant 1.5$ ($\nu_2 = 1/(\alpha - 1)$) and the asymptotic result from Kosterlitz [3] ν_K for $\alpha \to 2$ and $q = 2$ (shown by a broken curve). The discrepancy between our results in the vicinity of $\alpha = 2$ and our previous prediction (equation (31)) is not surprising since the present extrapolation method for ν, similar to that for T_c, was constructed for reproducing the expected behaviour in the other extreme region

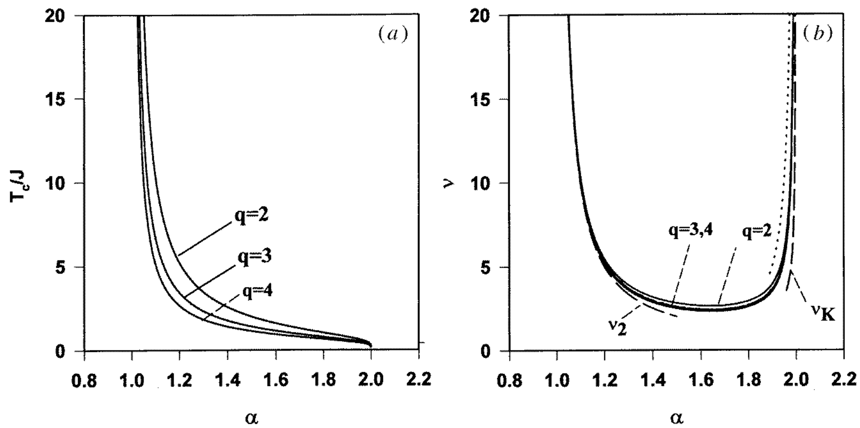

Figure 6. $b \to \infty$ extrapolations for different values of q. (a) Critical temperature T_c/J versus α; (b) critical exponent ν versus α. The dotted curve represents our asymptotic behaviour equation (31), while the broken curves correspond to the exact result ν_2 for $1 < \alpha \leqslant 1.5$ and to the Kosterlitz's asymptotic result ν_K for $\alpha \to 2$, both for $q = 2$.

Table 1. Comparison of our RG ($b \to \infty$) extrapolated values, Luijten and Blöte [39] Monte Carlo calculations and Glumac and Uzelac [24] FRS calculations of the critical temperature T_c/J for different values of q and α.

	$q = 2$			$q = 3$		$q = 4$	
α	RG	MC	FRS	RG	FRS	RG	FRS
1.1	10.40	10.500 19	10.787	6.72	7.353	5.16	4.926
1.3	3.48	3.673 61	3.680	2.33	2.589	1.89	2.045
1.5	2.00		2.179	1.41	1.663	1.14	1.402
1.7	1.28		1.463	0.95	1.194	0.78	1.048
1.9	0.77		1.003	0.61	0.874	0.51	0.797

($\alpha \to 1$). For $q > 2$ all numerical curves are quite indistinguishable within the resolution of the plot and with a little departure from the $q = 2$ case. This fact, together with the obtained q-independence for the asymptotic behaviours of ν in both the region near $\alpha = 1$ (equation (41)) and the region near $\alpha = 2$ (equation (31)), suggests that the critical exponent may be independent of q for all $1 < \alpha < 2$. In table 2 we compare our results for ν with the exact ones ($q = 2$ and $1 < \alpha \leqslant 1.5$) and the corresponding ones obtained by FRS [24] for $q = 2$ (our results for $q > 2$ show a little difference with ours for $q = 2$). Notice that although our results for $q = 2$ are worse than those of FRS when $1 < \alpha \leqslant 1.5$, we obtain a divergence for $\alpha \to 2^-$ (as expected) in contrast with their finite value for ν. The main difference between our results and those of FRS occurs for $q > 2$, where the FRS values for ν show a strong dependency on q.

4. Conclusions

The approach adopted here gives an estimate of some critical properties of the LR Potts model as a function of α for different values of $q \geqslant 2$, based on an extrapolation of a systematic series of RG calculations. This method allows us to obtain analytically several

Table 2. Comparison of our RG $b \to \infty$ extrapolation, exact results and Glumac and Uzelac FRS calculations of the critical exponent ν for $q = 2$ and some typical values of α.

α	RG	Exact	FRS
1.1	10.48	10	9.901
1.3	3.90	$3.33\dot{3}$	3.322
1.5	2.81	2	2.325
1.7	2.66	—	1.930
1.9	3.90	—	2.469
2.0	∞	∞	3.236

important quantities as functions of the rescaling parameter b. This fact, in turn, permits us to take the $b \to \infty$ limit, where the results are expected to be highly accurate and perhaps reproduce the exact ones. This last assumption is supported by the recovery of several known results for $q = 2$ and some of the very few rigorous results available for general q, giving confidence to the validity of the method. It is worth stressing that two different extrapolation methods have been used, each one corresponding to different regions of validity. The first method (section 3.2) applies only to the vicinity of $\alpha = 2$. In this case our method predicts a remarkable new result, namely that the critical temperature at $\alpha = 2$ is discontinuous with the *same value* for all $q \geqslant 2$. On the other hand, the second method (section 3.4) is based on some $b = \infty$ asymptotic results obtained for $\alpha \to 1$, and therefore we believe that the extrapolated critical curves $T_c(q, \alpha)$ approximate with high precision the exact ones for α *near one*. We also believe that the asymptotic functional form $T_c(q, \alpha)/J \propto (\alpha - 1)^{-1}$ for $\alpha \to 1$ might be *exact*. Notice that this is consistent with the recent conjectured scalings for generalized thermodynamics which allow an unification of extensive ($\alpha > 1$) and non-extensive ($0 \leqslant \alpha \leqslant 1$) regimes [34].

In view of the critical exponent $\nu(q, \alpha)$, we obtained, through the first extrapolation method, an asymptotic behaviour for $\alpha \to 2^-$ which is *the same for all* $q \geqslant 2$. Although its explicit form may not be the exact one (as it differs from the RG prediction of Kosterlitz [3] for the $q = 2$ case), it suggests that $\nu(q \geqslant 2, \alpha \to 2^-)$ might be independent of q provided that the transition is continuous when $\alpha \to 2^-$. Some other predictions for arbitrary q and continuous phase transitions, such as the asymptotic behaviour of $\nu(q, \alpha)$ for $\alpha \to 1$ and its possible q-independence for all $1 < \alpha < 2$, are also of interest. It would be worth testing our conjectures and predictions with other techniques, such as the recent Monte Carlo method for LR spin models [38, 39].

Finally, one point which requires some discussion is the possible appearance of a first-order transition for some finite $q > q_c$ (for the two-dimensional SR case it is known exactly [40] that $q_c = 4$). For the LR (as well as for the SR) case it was proved [33] that the mean-field theory becomes exact (and therefore the transition is of first order) in the limit $q \to \infty$. We have not found any evidence of a first-order transition, but it is also known that the present kind of RG approach does not detect this type of transition in the two-dimensional SR Potts model [41]. As far as we know, this question remains open since the FRS results [24] are also inconclusive in this respect. However, this problem could be solved by introducing appropriately some dilution in the RG formalism [41] and it would be interesting to apply this ansatz to the present case. Other possible extensions of the present paper concern higher-dimensional systems, where the crossover between short- and long-range regimes could be of interest for some real problems [9]. It could also be used to treat more complex interactions such as the RKKY one. Some calculations along these lines are in progress and will be published elsewhere.

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Appendix. Derivation of $G(b, 3)$

Since expression (34) is independent of the pair of sites k, j it can be written as:

$$G(b,q) = 2G^1(b,q) + (b-2)G^2(b,q) \tag{A1}$$

where

$$G^1(b,q) \equiv \mathrm{Tr}_{\{\sigma_i^I\}} P_I(\{\sigma_i^I\},1))\delta(\sigma_1^I,\sigma_2^I),\delta(\sigma_1^I,1) \tag{A2a}$$

$$G^2(b,q) \equiv \mathrm{Tr}_{\{\sigma_i^I\}} P_I(\{\sigma_i^I\},1))\delta(\sigma_2^I,\sigma_3^I)\delta(\sigma_1^I,1) \tag{A2b}$$

which can be written as:

$$G^i(b,q) = \sum_{m=1}^{m_{\max}} \frac{G_m^i(b,q)}{m} \qquad (i=1,2) \tag{A3}$$

where

• $G_m^1 \equiv$ number of configurations of b spins ($\sigma_i^I = 1, 2, \ldots, q$) of a block where one of the m major subgroups of $\{\sigma_i^I\}$ is in the state 1, and the spins $\sigma_1^I = \sigma_2^I = 1$.

• $G_m^2 \equiv$ number of configurations of b spins ($\sigma_i^I = 1, 2, \ldots, q$) of a block where one of the m major subgroups of $\{\sigma_i^I\}$ is in the state 1, and the spins $\sigma_1^I = 1$ and $\sigma_2^I = \sigma_3^I$.

We found that

$$G_1^1(b,3) = \sum_{l=0}^{X} \binom{b-2}{l} 2^l + \sum_{j=1}^{\mathrm{Int}(X/3)} \sum_{j_1=2j}^{X-j} \binom{b-2}{X+j}\binom{X+j}{j_1} \tag{A4a}$$

$$G_2^1(b,3) = 2 \sum_{l=0}^{\mathrm{Int}\left(\frac{X-2}{3}\right)} \binom{b-2}{X+l+1}\binom{X+l+1}{X-l} \tag{A4b}$$

$$G_3^1(b,3) = \binom{b-2}{2b/3}\binom{2b/3}{b/3}\delta(b,3n) \tag{A4c}$$

$$\begin{aligned} G_1^2(b,3) = {} & \sum_{l=0}^{X} \binom{b-3}{l} 2^l + \sum_{j=1}^{\mathrm{Int}(X/3)} \sum_{j_1=2j}^{X-j} \binom{b-3}{X+j}\binom{X+j}{j_1} \\ & + 2\sum_{l=2}^{X} \binom{b-3}{l-2} 2^{l-2} + 2 \sum_{j=1}^{\mathrm{Int}(X/3)} \sum_{j_1=2j}^{X-j} \binom{b-3}{X+j-2}\binom{X+j-2}{j_1} \end{aligned} \tag{A4d}$$

$$G_2^2(b,3) = 2\sum_{l=0}^{\mathrm{Int}\left(\frac{X-2}{3}\right)} \binom{b-3}{X+l+1}\binom{X+l+1}{X-l} + 2\sum_{l=0}^{\mathrm{Int}\left(\frac{X-2}{3}\right)} \binom{b-3}{X+l-1}\binom{X+l-1}{X-l-2}$$
$$+2\sum_{l=1}^{\mathrm{Int}\left(\frac{X-2}{3}\right)} \binom{b-3}{X+l-2}\binom{X+l-2}{X-l-1} \qquad \text{(A4}e\text{)}$$

$$G_3^2(b,3) = \left[\binom{b-3}{2b/3}\binom{2b/3}{b/3} + 2\binom{b-3}{2b/3-2}\binom{2b/3-2}{b/3-2}\right]\delta(b,3n) \qquad \text{(A4}f\text{)}$$

where $n = 1, 2, \ldots$ and $X \equiv (b-1)/2$. A combination of expressions (A1), (A3) and (A4) leads to $G(b,3)$.

References

[1] Dyson F J 1969 *Commun. Math. Phys.* **12** 91
Dyson F J 1969 *Commun. Math. Phys.* **12** 212
[2] Thouless D J 1969 *Phys. Rev.* **187** 732
[3] Kosterlitz J M 1976 *Phys. Rev. Lett.* **37** 1577
[4] Ellis R S 1985 *Entropy, Large Deviations and Statistical Mechanics* (Berlin: Springer) p 131–3
[5] Frölich J and Spencer T 1982 *Commun. Math. Phys.* **84** 87
[6] Fisher M E, Ma S and Nickel B G 1972 *Phys. Rev. Lett.* **29** 917
[7] Amit D J 1989 *Modeling Brain Functions* (Cambridge: Cambridge University Press)
[8] Ford P J 1982 *Comtemp. Phys.* **23** 141
[9] Pitzer K S, de Lima M C P and Schreiber D R 1985 *J. Phys. C: Solid State Phys.* **89** 1854
[10] Burkhardt T W and Eisenriegler E 1995 *Phys. Rev. Lett.* **74** 3189
[11] Bergersen B and Rácz Z 1991 *Phys. Rev. Lett.* **67** 3047
Xu H J, Bergersen B and Rácz Z 1993 *Phys. Rev.* E **47** 1520
[12] Giacomini G and Lebowitz J L 1996 *Phys. Rev. Lett.* **76** 1094
[13] Stošić B D and Fittipaldi I P 1997 Pattern recognition via ising model with long range interactions *Preprint*
[14] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235
Wu F Y 1984 *J. Appl. Phys.* **55** 2421
[15] Tsallis C and de Magalhães A C N 1996 *Phys. Rep.* **268** 305
[16] Aizenman M, Chayes J T, Chayes L and Newman C M 1988 *J. Stat. Phys.* **50** 1
[17] Schulman L S 1983 *J. Phys. A: Math. Gen.* **16** L639
[18] Aizenman M and Fernandez R 1988 *Lett. Math. Phys.* **16** 39
[19] Nagle J F and Bonner J C 1970 *J. Phys. C: Solid State Phys.* **3** 352
[20] Glumac Z and Uzelac K 1989 *J. Phys. A: Math. Gen.* **22** 4439
[21] Monroe J L, Lucente R and Hourlland J P 1990 *J. Phys. A: Math. Gen.* **23** 2555
[22] Cannas S A 1995 *Phys. Rev.* B **52** 3034
[23] Cardy J L 1981 *J. Phys. A: Math. Gen.* **14** 1407
[24] Glumac Z and Uzela K 1993 *J. Phys. A: Math. Gen.* **26** 5267
[25] Anderson P W and Yuval G 1971 *J. Phys. C: Solid State Phys.* **4** 607
[26] Chakravarty S and Anderson P W 1994 *Phys. Rev. Lett.* **72** 3859
Clarke D G, Strong S P and Anderson P W 1994 *Phys. Rev. Lett.* **72** 3218
Chakravarty S 1982 *Phys. Rev. Lett.* **49** 681
[27] Niemeijer T and van Leeuwen J M J 1974 *Physica* **71** 17
[28] Niemeijer T and van Leeuwen J M J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic)
[29] Burkhardt T W and Van Leeuwen J M J (ed) 1982 *Real-Space Renormalization (Topics in Current Physics)* vol 30 (Berlin: Springer)
[30] Mainieri R 1992 *Phys. Rev.* A **45** 3580
[31] Cannas S A and Tamarit F A 1996 *Phys. Rev.* B **54** R12 661
[32] Mittag L and Stephen M J 1974 *J. Phys. A: Math. Gen.* **7** L109
[33] Pearce P A and Griffiths R B 1980 *J. Phys. A: Math. Gen.* **13** 2143
[34] Tsallis C 1995 *Fractals* **3** 541
[35] Jund P, Kim S G and Tsallis C 1995 *Phys. Rev.* B **52** 50

[36] Grigera J R 1996 *Phys. Lett.* A **217** 47
[37] Sampaio L C, de Albuquerque M P and de Menezes F S 1997 *Phys. Rev.* B **55** to appear
[38] Luijten E and Blöte H W J 1995 *Int. J. Mod. Phys.* C **6** 359
[39] Luijten E and Blöte H W J 1996 *Phys. Rev. Lett.* **76** 1557
[40] Baxter R J 1973 *J. Phys. C: Solid State Phys.* **12** L125
Baxter R J, Temperley H N V and Ashley S E 1978 *Proc. R. Soc.* A **358** 535
[41] Nienhuis B, Berker A N, Riedel E K and Schick M 1979 *Phys. Rev. Lett.* **43** 737