

REPRESENTATIONS OF THE CATEGORY OF MODULES OVER POINTED HOPF ALGEBRAS OVER \mathbb{S}_3 AND \mathbb{S}_4

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ABSTRACT. We classify exact indecomposable module categories over the representation category of all non-trivial Hopf algebras with coradical \mathbb{S}_3 and \mathbb{S}_4 . As a byproduct, we compute all its Hopf-Galois extensions and we show that these Hopf algebras are cocycle deformations of their graded versions.

1. INTRODUCTION

Given a tensor category \mathcal{C} , an *exact module category* [EO1] over \mathcal{C} is an Abelian category \mathcal{M} equipped with a biexact functor $\otimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ subject to natural associativity and unity axioms, such that for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$ the object $P \otimes M$ is again projective.

Exact module categories, or *representations* of \mathcal{C} , are very interesting objects to consider. They are implicitly present in many areas of mathematics and mathematical physics such as subfactor theory [BEK], affine Hecke algebras [BO], extensions of vertex algebras [KO], [HuKo], Calabi-Yau algebras [Gi] and conformal field theory, see for example [BFRS], [FS], [CS1], [CS2].

Module categories have been used in the study of fusion categories [ENO1], [ENO1], and in the theory of (weak) Hopf algebras [O1], [M1], [N].

The classification of exact module categories over a fixed finite tensor category \mathcal{C} has been undertaken by several authors:

1. When \mathcal{C} is the semisimple quotient of $U_q(\mathfrak{sl}_2)$ [KO], [EO2],
2. over the tensor categories of representations of finite supergroups [EO1],
3. over $\text{Rep}(D(G))$, $D(G)$ the Drinfeld double of a finite group G [O2],
4. over the tensor category of representations of the Lusztig's small quantum group $u_q(\mathfrak{sl}_2)$ [M1],
5. and more generally over $\text{Rep}(H)$, where H is a lifting of a quantum linear space [M2].

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The main goal of this paper is the classification of exact module categories over the representation category of any non-trivial (i.e. different from the group algebra) finite-dimensional Hopf algebra with coradical $\mathbb{k}\mathbb{S}_3$ or $\mathbb{k}\mathbb{S}_4$.

Finite-dimensional Hopf algebras with coradical $\mathbb{k}\mathbb{S}_3$ or $\mathbb{k}\mathbb{S}_4$ were classified in [AHS], [GG], respectively. For all these Hopf algebras the associated graded Hopf algebras $\text{gr } H$ are of the form $\mathfrak{B}(X, q) \# \mathbb{k}\mathbb{S}_n$, $n = 3, 4$ where X is a finite set equipped with a map $\triangleright : X \times X \rightarrow X$ satisfying certain axioms that makes it into a *rack* and $q : X \times X \rightarrow \mathbb{k}^\times$ is a 2-cocycle. We have the following result:

Let $n = 3, 4$ and let \mathcal{M} be an exact indecomposable module category over $\text{Rep}(\mathfrak{B}(X, q) \# \mathbb{k}\mathbb{S}_n)$, then there exist

- a subgroup $F < \mathbb{S}_n$ and a 2-cocycle $\psi \in Z^2(F, \mathbb{k}^\times)$,
- a subset $Y \subseteq X$ invariant under the action of F ,
- a family of scalars $\{\xi_C\}$ compatible (Definition 7.1) with (F, ψ, Y) ,

such that $\mathcal{M} \simeq_{\mathcal{B}(Y, F, \psi, \xi)} \mathcal{M}$, where $\mathcal{B}(Y, F, \psi, \xi)$ is a left $\mathfrak{B}(X, q) \# \mathbb{k}\mathbb{S}_n$ -comodule algebra constructed from data (Y, F, ψ, ξ) . We also show that if H is a finite-dimensional Hopf algebra with coradical $\mathbb{k}\mathbb{S}_3$ or $\mathbb{k}\mathbb{S}_4$ then H and $\text{gr } H$ are cocycle deformations of each other. This implies that there is a bijective correspondence between module categories over $\text{Rep}(H)$ and $\text{Rep}(\text{gr } H)$.

The content of the paper is as follows. In Section 3 we recall the basic results on module categories over finite-dimensional Hopf algebras. We recall the main result of [M2] that gives an isomorphism between Loewy-graded comodule algebras and a semidirect product of a twisted group algebra and a homogeneous coideal subalgebra inside the Nichols algebra. In Section 4 we show how to distinguish Morita equivariant classes of comodule algebras over pointed Hopf algebras.

In Section 5 we recall the definition of a rack X and a ql-datum \mathcal{Q} , and how to construct (quadratic approximations to) Nichols algebras $\widehat{\mathfrak{B}}_2(X, q)$ and pointed Hopf algebras $\mathcal{H}(\mathcal{Q})$ out of them. In particular, we recall a presentation of all finite-dimensional Hopf algebras with coradical $\mathbb{k}\mathbb{S}_3, \mathbb{k}\mathbb{S}_4$. In Section 6, we give a classification of connected homogeneous left coideal subalgebras of $\widehat{\mathfrak{B}}_2(X, q)$ and also a presentation by generators and relations.

In Section 7 we introduce a family of comodule algebras large enough to classify module categories. We give an explicit Hopf-biGalois extension over $\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}\mathbb{S}_n$, $n \in \mathbb{N}$, and a lifting $\mathcal{H}(\mathcal{Q})$, proving that there is a bijective correspondence between module categories over $\text{Rep}(\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}\mathbb{S}_n)$ and $\text{Rep}(\mathcal{H}(\mathcal{Q}))$, $n = 3, 4$. In particular we obtain that any pointed Hopf algebra over \mathbb{S}_3 or \mathbb{S}_4 is a cocycle deformation of its associated graded algebra, a result analogous to a theorem of Masuoka for abelian groups [Ma]. Finally, the classification of module categories over $\text{Rep}(\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}\mathbb{S}_n)$ is

presented in this section and as a consequence all Hopf-Galois objects over $\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}\mathbb{S}_n$ are described.

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2. PRELIMINARIES AND NOTATION

We shall denote by \mathbb{k} an algebraically closed field of characteristic zero. The tensor product over the field \mathbb{k} will be denoted by \otimes . All vector spaces, algebras and categories will be considered over \mathbb{k} . For any algebra A , ${}_A\mathcal{M}$ will denote the category of finite-dimensional left A -modules.

The symmetric group on n letters is denoted by \mathbb{S}_n and by \mathcal{O}_j^n we shall denote the conjugacy class of all j -cycles in \mathbb{S}_n . For any group G , a 2-cocycle $\psi \in Z^2(G, \mathbb{k}^\times)$ and any $h \in G$ we shall denote $\psi^h(x, y) = \psi(h^{-1}xh, h^{-1}yh)$ for all $x, y \in G$.

If H is a Hopf algebra, a 2-cocycle σ in H is a convolution invertible linear map $\sigma : H \times H \rightarrow \mathbb{k}$ such that

$$(2.1) \quad \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)})$$

and $\sigma(x, 1) = \sigma(1, x) = \varepsilon(x)$, for every $x, y, z \in H$. The set of 2-cocycles in H is denoted by $Z^2(H)$.

If A is an H -comodule algebra via $\lambda : A \rightarrow H \otimes A$, we shall say that a (right) ideal J is H -costable if $\lambda(J) \subseteq H \otimes J$. We shall say that A is (right) H -simple if there is no nontrivial (right) ideal H -costable in A .

If $H = \bigoplus H(i)$ is a coradically graded Hopf algebra we shall say that a left coideal subalgebra $K \subseteq H$ is *homogeneous* if $K = \bigoplus K(i)$ is graded as an algebra and, for any n , $K(n) \subseteq H(n)$ and $\Delta(K(n)) \subseteq \bigoplus_{i=0}^n H(i) \otimes K(n-i)$. K is said to be *connected* if $K \cap H(0) = \mathbb{k}$.

If $H = \mathfrak{B}(V) \# \mathbb{k}G$, where V is a Yetter-Drinfeld module over G and $K \subseteq H$ is a coideal subalgebra, we shall denote by $\text{Stab } K$ the biggest subgroup of G such that the adjoint action of $\text{Stab } K$ leaves K invariant.

If H is a finite-dimensional Hopf algebra then $H_0 \subseteq H_1 \subseteq \dots \subseteq H_m = H$ will denote the coradical filtration. When $H_0 \subseteq H$ is a Hopf subalgebra then the associated graded algebra $\text{gr } H$ is a coradically graded Hopf algebra. If (A, λ) is a left H -comodule algebra, the coradical filtration on H induces a filtration on A , given by $A_n = \lambda^{-1}(H_n \otimes A)$. This filtration is called the *Loewy series* on A .

The associated graded algebra $\text{gr } A$ is a left $\text{gr } H$ -comodule algebra. The algebra A is right H -simple if and only if $\text{gr } A$ is right $\text{gr } H$ -simple, see [M1, Section 4].

3. REPRESENTATIONS OF TENSOR CATEGORIES

Given $\mathcal{C} = (\mathcal{C}, \otimes, a, \mathbf{1})$ a tensor category, a *module category* over \mathcal{C} or a *representation* of \mathcal{C} is an Abelian category \mathcal{M} equipped with an exact bifunctor $\bar{\otimes} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ and natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y) \otimes M \rightarrow X \otimes (Y \otimes M)$, $\ell_M : \mathbf{1} \otimes M \rightarrow M$ satisfying natural associativity and unit axioms, see [EO1], [O1]. We shall assume, as in [EO1], that all module categories have only finitely many isomorphism classes of simple objects.

A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories. A module category \mathcal{M} over a finite tensor category \mathcal{C} is *exact* [EO1] if for any projective $P \in \mathcal{C}$ and any $M \in \mathcal{M}$, the object $P \otimes M$ is again projective in \mathcal{M} .

If \mathcal{M} is an exact module category over \mathcal{C} then the dual category $\mathcal{C}_{\mathcal{M}}^*$, see [EO1], is a finite tensor category. There is a bijective correspondence between the set of equivalence classes of exact module categories over \mathcal{C} and over $\mathcal{C}_{\mathcal{M}}^*$, see [EO1, Theorem 3.33]. This implies that for any finite-dimensional Hopf algebra there is a bijective correspondence between the set of equivalence classes of exact module categories over $\text{Rep}(H)$ and $\text{Rep}(H^*)$.

3.1. Module categories over pointed Hopf algebras. We are interested in exact indecomposable module categories over the representation category of finite-dimensional Hopf algebras. If H is a Hopf algebra and $\lambda : \mathcal{A} \rightarrow H \otimes \mathcal{A}$ is a left H -comodule algebra the category ${}^H\mathcal{M}_{\mathcal{A}}$ is the category of finite-dimensional right \mathcal{A} -modules left H -comodules where the comodule structure is a \mathcal{A} -module morphism. If \mathcal{A}' is another left H -comodule algebra the category ${}^H\mathcal{M}_{\mathcal{A}'}$ is defined analogously.

The category of finite-dimensional left \mathcal{A} -modules ${}_{\mathcal{A}}\mathcal{M}$ is a representation of $\text{Rep}(H)$. The action $\bar{\otimes} : \text{Rep}(H) \times {}_{\mathcal{A}}\mathcal{M} \rightarrow {}_{\mathcal{A}}\mathcal{M}$ is given by $V \bar{\otimes} M = V \otimes M$ for all $V \in \text{Rep}(H)$, $M \in {}_{\mathcal{A}}\mathcal{M}$. The left \mathcal{A} -module structure on $V \otimes M$ is given by the coaction λ .

If \mathcal{M} is an exact indecomposable module over $\text{Rep}(H)$ then there exists a left H -comodule algebra \mathcal{A} right H -simple with trivial coinvariants such that $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ as modules over $\text{Rep}(H)$ see [AM, Theorem 3.3].

If $\mathcal{A}, \mathcal{A}'$ are two right H -simple left H -comodule algebras such that the categories ${}_{\mathcal{A}}\mathcal{M}, {}_{\mathcal{A}'}\mathcal{M}$ are equivalent as representations over $\text{Rep}(H)$. Then there exists an equivariant Morita context (P, Q, f, g) , that is $P \in {}^H\mathcal{M}_{\mathcal{A}}, Q \in {}^H\mathcal{M}_{\mathcal{A}'}$ and $f : P \otimes_{\mathcal{A}} Q \rightarrow \mathcal{A}', g : Q \otimes_{\mathcal{A}'} P \rightarrow \mathcal{A}$ such that they are bimodule isomorphisms. Moreover, it holds that $\mathcal{A}' \simeq \text{End}_{\mathcal{A}}(P)$ as comodule algebras. The comodule structure on $\text{End}_{\mathcal{A}}(P)$ is given by $\lambda(T) = T_{(-1)} \otimes T_{(0)}$, where

$$(3.1) \quad \langle \alpha, T_{(-1)} \rangle T_{(0)}(p) = \langle \alpha, T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)},$$

for any $\alpha \in H^*, T \in \text{End}_{\mathcal{A}}(P), p \in P$. See [AM] for more details.

By the previous paragraph, we can see that the categories ${}^H\mathcal{M}_{\mathcal{A}}$ play a central role in the theory. The following theorem will be of great use in the next section.

Theorem 3.1. *Let H be a Hopf algebra and \mathcal{A} a left H -comodule algebra, both finite dimensional.*

- [Sk, Theorem 3.5] *If \mathcal{A} is H -simple and $M \in {}^H\mathcal{M}_{\mathcal{A}}$, then there exists $t \in \mathbb{N}$ such that M^t , the direct sum of t copies of M , is a free \mathcal{A} -module.*
- [Sk, Theorem 4.2] *$M \in {}^H\mathcal{M}_{\mathcal{A}}$ is free as \mathcal{A} -module if and only if there exists a maximal ideal $J \subset \mathcal{A}$ such that $M/M \cdot J$ is free as \mathcal{A}/J -module. \square*

The first statement of this theorem is actually present in the proof of [Sk, Theorem 3.5]. The second one will be particularly useful when the ideal J is such that $\mathcal{A}/J = \mathbb{k}$, since in this case $M/M \cdot J$ is automatically free.

Theorem 3.2. [M2, Theorem 3.3] *Let G be a finite group and let H be a finite-dimensional pointed Hopf algebra with coradical $\mathbb{k}G$. Assume there is $V \in {}^G\mathcal{YD}$ such that $\text{gr } H = U = \mathfrak{B}(V) \# \mathbb{k}G$. Let \mathcal{A} be a left H -comodule algebra right H -simple with trivial coinvariants. There exists*

1. *a subgroup $F \subseteq G$,*
2. *a 2-cocycle $\psi \in Z^2(F, \mathbb{k}^\times)$,*
3. *a homogeneous left coideal subalgebra $\mathcal{K} = \bigoplus_{i=0}^m \mathcal{K}(i) \subseteq \mathfrak{B}(V)$ such that $\mathcal{K}(1) \subseteq V$ is a $\mathbb{k}G$ -subcomodule invariant under the action of F ,*

such that $\text{gr } \mathcal{A} \simeq \mathcal{K} \#_{\mathbb{k}\psi} F$ as left U -comodule algebras. \square

The algebra structure and the left U -comodule structure of $\mathcal{K} \#_{\mathbb{k}\psi} F$ is given as follows. If $x, y \in \mathcal{K}$, $f, g \in F$ then

$$\begin{aligned} (x \# g)(y \# f) &= x(g \cdot y) \# \psi(g, f) gf, \\ \lambda(x \# g) &= (x_{(1)} \# g) \otimes (x_{(2)} \# g), \end{aligned}$$

where the action of F on \mathcal{K} is the restriction of the action of G on $\mathfrak{B}(V)$ as an object in ${}^G\mathcal{YD}$. Observe that F is necessarily a subgroup of $\text{Stab } \mathcal{K}$.

4. EQUIVARIANT EQUIVALENCE CLASSES OF COMODULE ALGEBRAS

In this section we shall present how to distinguish equivalence classes of some comodule algebras over pointed Hopf algebras and then we will apply this result to our cases. Much of the ideas here are already contained in [M1], [M2] although with less generality.

Let Γ be a finite group and H be a finite-dimensional pointed Hopf algebra with coradical $\mathbb{k}\Gamma$ and with coradical filtration $H_0 \subseteq H_1 \subseteq \dots \subseteq H_m = H$. Assume there is $V \in {}^\Gamma\mathcal{YD}$ such that $\text{gr } H = U = \mathfrak{B}(V) \# \mathbb{k}\Gamma$.

We begin with the following lemma.

Lemma 4.1. *Let Γ, U as above. Let $\sigma \in Z^2(\Gamma, \mathbb{k}^\times)$ be a 2-cocycle. Then there exists a 2-cocycle $\varsigma \in Z^2(U)$ such that $\varsigma_{|\Gamma \times \Gamma} = \sigma$.*

Proof. Let us consider the linear map $\varsigma : U \times U \rightarrow \mathbb{k}$ defined, on homogeneous elements $x, y \in U$ by

$$\varsigma(x, y) = \begin{cases} \sigma(x, y), & \text{if } x, y \in U(0); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\varsigma(x, 1) = \varsigma(1, x) = \epsilon(x)$ by definition. We have to check that, for $x \in U(m)$, $y \in U(n)$, $z \in U(k)$, $m, n, k \in \mathbb{N}$, (2.1) holds. Now, if $k > 0$, the LHS of (2.1) is zero. Set $\Delta(z) = \sum_{i=0}^k z^i \otimes z^{k-i}$, with $z^s \in U(s)$, $s = 0, \dots, k$. Analogously, set $\Delta(y) = \sum_{j=0}^n y^j \otimes y^{n-j}$, with $y^t \in U(t)$, $t = 0, \dots, n$. Then, the RHS is

$$\sum_{i=0}^k \sum_{j=0}^n \varsigma(x, y^{n-j} z^{k-i}) \varsigma(y^j, z^i) = \varsigma(x, y^n z^k) = 0,$$

and thus (2.1) holds. Both sides of this equation are similarly seen to be zero if $m > 0$ or $n > 0$, while the case $m = n = k = 0$ holds by definition of ς . This map is convolution invertible and its inverse ς^{-1} is defined in an analogous manner, using σ^{-1} . \square

Let $\mathcal{A}, \mathcal{A}'$ be two right H -simple left H -comodule algebras. Let $F, F' \subseteq \Gamma$ be subgroups and let $\psi \in Z^2(F, \mathbb{k}^\times)$, $\psi' \in Z^2(F', \mathbb{k}^\times)$ be two cocycles such that $\mathcal{A}_0 = \mathbb{k}_\psi F$ and $\mathcal{A}'_0 = \mathbb{k}_{\psi'} F'$. Let $K, K' \in \mathfrak{B}(V)$ be two homogeneous coideal subalgebras such that $\text{gr } \mathcal{A} = K \#_{\mathbb{k}_\psi} F$ and $\text{gr } \mathcal{A}' = K' \#_{\mathbb{k}_{\psi'}} F'$.

The main result of this section is the following.

Theorem 4.2. *The categories ${}_{\mathcal{A}}\mathcal{M}$, ${}_{\mathcal{A}'}\mathcal{M}$ are equivalent as modules over $\text{Rep}(H)$ if and only if there exists an element $g \in \Gamma$ such that $\mathcal{A}' \simeq g\mathcal{A}g^{-1}$ as comodule algebras.*

Proof. Let us assume that ${}_{\mathcal{A}}\mathcal{M} \cong {}_{\mathcal{A}'}\mathcal{M}$ as $\text{Rep}(H)$ -modules. By [AM, Proposition 1.24] there exists an equivariant Morita context (P, Q, f, h) . That is $P \in {}_{\mathcal{A}'}^H \mathcal{M}_{\mathcal{A}}$, $Q \in {}_{\mathcal{A}}^H \mathcal{M}_{\mathcal{A}'}$ and $f : P \otimes_{\mathcal{A}} Q \rightarrow \mathcal{A}'$, $h : Q \otimes_{\mathcal{A}'} P \rightarrow \mathcal{A}$ are bimodule isomorphisms and $\mathcal{A}' \simeq \text{End}_{\mathcal{A}}(P)$ as comodule algebras. The comodule structure on $\text{End}_{\mathcal{A}}(P)$ is given by $\lambda : \text{End}_{\mathcal{A}}(P) \rightarrow H \otimes \text{End}_{\mathcal{A}}(P)$, $\lambda(T) = T_{(-1)} \otimes T_{(0)}$ where

$$(4.1) \quad \langle \alpha, T_{(-1)} \rangle T_{(0)}(p) = \langle \alpha, T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)},$$

for any $\alpha \in H^*$, $T \in \text{End}_{\mathcal{A}}(P)$, $p \in P$.

For any $i = 0, \dots, m$ define $P(i) = P_i/P_{i-1}$, where $P_{-1} = 0$. The graded vector space $\text{gr } P = \bigoplus_{i=0}^m P(i)$ has an obvious structure that makes it into an object in the category ${}^U \mathcal{M}_{K \#_{\mathbb{k}_\psi} F}$. We shall denote $\bar{\delta} : \text{gr } P \rightarrow U \otimes \text{gr } P$ the coaction. In particular $\text{gr } P \in {}^U \mathcal{M}_K$, thus by Theorem 3.1 (ii) we have that $\text{gr } P \simeq M \otimes K$, where $M = \text{gr } P / (\text{gr } P \cdot K^+)$, since $K/K^+ = \mathbb{k}$.

We have that $\bar{\delta}(\text{gr } P \cdot K^+) \subset (U \otimes \text{gr } P)(K^+ \otimes 1 + U \otimes K^+)$, since $K = \mathbb{k} \oplus K^+$ and thus the map $\bar{\delta}$ induces a new map $\widehat{\delta} : M \rightarrow U' \otimes M$, where $U' = U/UK^+U$. Notice that U' is a pointed Hopf algebra with coradical $\mathbb{k}\Gamma$, since U is coradically graded and the ideal UK^+U is homogeneous and does not intersect U_0 . M is also a $\mathbb{k}_\psi F$ -module with $\bar{m} \cdot f = \overline{m \cdot f}$, for $f \in F$, $\bar{m} \in M$. This action is easily seen to be well defined and, moreover, $M \in {}^{U'}\mathcal{M}_{\mathbb{k}_\psi F}$.

Let $\Psi \in Z^2(\Gamma, \mathbb{k}^*)$ be a 2-cocycle such that $\Psi|_{F \times F} = \psi$ see [Br, Proposition III (9.5)]. Let $\zeta \in Z^2(U')$ be such that $\zeta|_{\Gamma \times \Gamma} = \Psi^{-1}$, as in Lemma 4.1. By [M1, Lemma 2.1] there exists an equivalence of categories ${}^{U'\zeta}\mathcal{M}_{\mathbb{k}F} \simeq {}^{U'}\mathcal{M}_{(\mathbb{k}F)_\Psi}$. By Theorem 3.1 (ii) any object in ${}^{U'\zeta}\mathcal{M}_{\mathbb{k}F}$ is a free $\mathbb{k}F$ -module. Thus there is an object N in ${}^{U/U(\mathbb{k}F)^+}\mathcal{M}$ such that $\text{gr } P \simeq N \otimes K \otimes \mathbb{k}_\psi F$. Whence $\dim P = (\dim N)(\dim \mathcal{A})$. Similarly we can assume that there is $s \in \mathbb{N}$ such that $\dim Q = s \dim \mathcal{A}'$.

Using Theorem 3.1 (i) there exists $t \in \mathbb{N}$ such that P^t is a free right \mathcal{A} -module, that is, there is a vector space T such that $P^t \simeq T \otimes \mathcal{A}$, hence

$$(4.2) \quad t \dim N = \dim T.$$

Since $P \otimes_{\mathcal{A}} Q \simeq \mathcal{A}'$ then $P^t \otimes_{\mathcal{A}} Q \simeq T \otimes Q \simeq \mathcal{A}'^t$, then $s \dim T \dim \mathcal{A}' = t \dim \mathcal{A}'$ and using (4.2) we obtain that $s \dim N = 1$ whence $\dim N = 1$ and thus $\dim P = \dim \mathcal{A}$.

Claim 4.1. *Let $n \in P_0$, then $P = n \cdot \mathcal{A}$.*

Notice that $P_0 \neq 0$. In fact, if $P_0 = 0$ and $k \in \mathbb{N}$ is minimal with $P_k \neq 0$, then $\lambda(P_k) \subset \sum_{j=0}^k H_{k-j} \otimes P_j = H_0 \otimes P_k$, which is a contradiction. Let $g \in \Gamma$ be such that $\lambda(n) = g \otimes n$. Now, if $J = \{a \in \mathcal{A} : n \cdot a = 0\}$, then J is a right ideal of \mathcal{A} . We shall prove that $J = 0$. Let $a \in J$ and write $\lambda(a) = \sum_{i=1}^n a^i \otimes a_i$, in such a way that the set $\{a^i : i = 1, \dots, n\} \subset H$ is linearly independent. Now, $\{ga^i : i = 1, \dots, n\} \subset H$ is also linearly independent and we have $0 = \lambda(n \cdot a) = \sum_{i=1}^n ga^i \otimes n \cdot a_i$. Thus $n \cdot a_i = 0$, $\forall i = 1, \dots, n$, that is, $\lambda(a) \in H \otimes J$ and J is H -costable. As \mathcal{A} is right H -simple, $J = 0$. Therefore, the action $\cdot : N \otimes \mathcal{A} \rightarrow P$ is injective and since $\dim P = \dim N \dim \mathcal{A}$, the claim follows.

It is not difficult to prove that the linear map $\phi : g\mathcal{A}g^{-1} \rightarrow \text{End}_{\mathcal{A}}(P)$ given by $\phi(gag^{-1})(n \cdot b) = n \cdot ab$ is an isomorphism of H -comodule algebras.

Conversely, if $\mathcal{A}' \simeq g\mathcal{A}g^{-1}$ as comodule algebras and $M \in {}_{\mathcal{A}}\mathcal{M}$, then the set gMg^{-1} has a natural structure of \mathcal{A}' -module in such a way that the functor $F : {}_{\mathcal{A}}\mathcal{M} \rightarrow {}_{\mathcal{A}'}\mathcal{M}$, $M \mapsto gMg^{-1}$ is an equivalence of $\text{Rep}(H)$ -modules. \square

5. POINTED HOPF ALGEBRAS OVER \mathbb{S}_3 AND \mathbb{S}_4

In this section we describe all pointed Hopf algebras whose coradical is the group algebra of the groups \mathbb{S}_3 and \mathbb{S}_4 . These were classified in [AHS] and [GG], respectively.

Recall that a *rack* is a pair (X, \triangleright) , where X is a non-empty set and $\triangleright : X \times X \rightarrow X$ is a function, such that $\phi_i = i \triangleright (\cdot) : X \rightarrow X$ is a bijection for all $i \in X$ satisfying: $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$, for all $i, j, k \in X$. See [AG2] for detailed information on racks.

Let (X, \triangleright) be a rack. A 2-cocycle $q : X \times X \rightarrow \mathbb{k}^\times$, $(i, j) \mapsto q_{ij}$ is a function such that for all $i, j, k \in X$

$$q_{i,j \triangleright k} q_{j,k} = q_{i \triangleright j, i \triangleright k} q_{i,k}.$$

In this case it is possible to generate a braiding c^q in the vector space $\mathbb{k}X$ with basis $\{x_i\}_{i \in X}$ by $c^q(x_i \otimes x_j) = q_{ij} x_{i \triangleright j} \otimes x_i$, for all $i, j \in X$. We denote by $\mathfrak{B}(X, q)$ the Nichols algebra of this braided vector space.

5.1. Quadratic approximations to Nichols algebras. Let $\mathcal{J} = \bigoplus_{r \geq 2} \mathcal{J}^r$ be the defining ideal of the Nichols algebra $\mathfrak{B}(X, q)$. Next, we give a description of the space \mathcal{J}^2 of quadratic relations. Let \mathcal{R} be the set of equivalence classes in $X \times X$ for the relation generated by $(i, j) \sim (i \triangleright j, i)$. Let $C \in \mathcal{R}$, $(i, j) \in C$. Take $i_1 = j$, $i_2 = i$ and, recursively, $i_{h+2} = i_{h+1} \triangleright i_h$. Set $n(C) = \#C$ and

$$\mathcal{R}' = \left\{ C \in \mathcal{R} \mid \prod_{h=1}^{n(C)} q_{i_{h+1}, i_h} = (-1)^{n(C)} \right\}.$$

Let \mathcal{T} be the free associative algebra in the variables $\{T_l\}_{l \in X}$. If $C \in \mathcal{R}'$, consider the quadratic polynomial

$$(5.1) \quad \phi_C = \sum_{h=1}^{n(C)} \eta_h(C) T_{i_{h+1}} T_{i_h} \in \mathcal{T},$$

where $\eta_1(C) = 1$ and $\eta_h(C) = (-1)^{h+1} q_{i_2 i_1} q_{i_3 i_2} \cdots q_{i_h i_{h-1}}$, $h \geq 2$. Then, a basis of the space \mathcal{J}^2 is given by

$$(5.2) \quad \phi_C(\{x_i\}_{i \in X}), \quad C \in \mathcal{R}'.$$

We denote by $\widehat{\mathfrak{B}}_2(X, q)$ the quadratic approximation of $\mathfrak{B}(X, q)$, that is the algebra defined by relations $\langle \mathcal{J}^2 \rangle$. For more details see [GG, Lemma 2.2].

Let G be a finite group. A *principal YD-realization* over G of (X, q) , [AG2, Def. 3.2], is a way to realize this braided vector space $(\mathbb{k}X, c^q)$ as a Yetter-Drinfeld module over G . Explicitly, it is a collection $(\cdot, g, (\chi_i)_{i \in X})$ where

- \cdot is an action of G on X ,
- $g : X \rightarrow G$ is a function such that $g_{h \cdot i} = h g_i h^{-1}$ and $g_i \cdot j = i \triangleright j$,

- the family $(\chi_i)_{i \in X}$, where $\chi_i : G \rightarrow \mathbb{k}^*$ is a 1-cocycle, that is

$$\chi_i(ht) = \chi_i(t) \chi_{t \cdot i}(h),$$

for all $i \in X$, $h, t \in G$, satisfying $\chi_i(g_j) = q_{ji}$.

If $(\cdot, g, (\chi_i)_{i \in X})$ is a principal YD-realization of (X, q) over G then $\mathbb{k}X \in {}_G^G\mathcal{YD}$ as follows. The action and coaction of G are determined by:

$$\delta(x_i) = g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h) x_{h \cdot i} \quad i \in X, h \in G.$$

Lemma 5.1. *Assume that for any pair $i, j \in X$, $(i \triangleright j) \triangleright i = j$, then*

$$(5.3) \quad \chi_i(f) q_{f \cdot i \triangleright f \cdot j, f \cdot i} = \chi_j(f) q_{i \triangleright j, i} \quad \text{for any } f \in G, i, j \in X. \quad \square$$

5.2. Nichols algebras over \mathbb{S}_n . Let $X = \mathcal{O}_2^n$ or $X = \mathcal{O}_4^4$ considered as racks with the map \triangleright given by conjugation. Consider the applications:

$$\text{sgn} : \mathbb{S}_n \times X \rightarrow \mathbb{k}^*, \quad (\sigma, i) \mapsto \text{sgn}(\sigma),$$

$$\chi : \mathbb{S}_n \times \mathcal{O}_2^n \rightarrow \mathbb{k}^*, \quad (\sigma, i) \mapsto \chi_i(\sigma) = \begin{cases} 1, & \text{if } i = (a, b) \text{ and } \sigma(a) < \sigma(b) \\ -1, & \text{if } i = (a, b) \text{ and } \sigma(a) > \sigma(b). \end{cases}$$

We will deal with the cocycles:

$$\begin{aligned} -1 : X \times X &\rightarrow \mathbb{k}^*, & (j, i) &\mapsto \text{sgn}(j) = -1, & i, j &\in X; \\ \chi : \mathcal{O}_2^n \times \mathcal{O}_2^n &\rightarrow \mathbb{k}^*, & (j, i) &\mapsto \chi_i(j) & i, j &\in \mathcal{O}_2^n. \end{aligned}$$

The quadratic approximations of the corresponding Nichols algebras are

$$\begin{aligned} \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, -1) &= \mathbb{k}\langle x_{(lm)}, 1 \leq l < m \leq n \mid x_{(ab)}^2, x_{(ab)}x_{(ef)} + x_{(ef)}x_{(ab)}, \\ & \quad x_{(ab)}x_{(bc)} + x_{(bc)}x_{(ac)} + x_{(ac)}x_{(ab)}, \\ & \quad 1 \leq a < b < c \leq n, 1 \leq e < f \leq n, \{a, b\} \cap \{e, f\} = \emptyset \rangle, \end{aligned}$$

$$\begin{aligned} \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, \chi) &= \mathbb{k}\langle x_{(lm)}, 1 \leq l < m \leq n \mid x_{(ab)}^2, x_{(ab)}x_{(ef)} - x_{(ef)}x_{(ab)}, \\ & \quad x_{(ab)}x_{(bc)} - x_{(bc)}x_{(ac)} - x_{(ac)}x_{(ab)}, \\ & \quad x_{(bc)}x_{(ab)} - x_{(ac)}x_{(bc)} - x_{(ab)}x_{(ac)}, \\ & \quad 1 \leq a < b < c \leq n, 1 \leq e < f \leq n, \{a, b\} \cap \{e, f\} = \emptyset \rangle, \end{aligned}$$

$$\begin{aligned} \widehat{\mathfrak{B}}_2(\mathcal{O}_4^4, -1) &= \mathbb{k}\langle x_i, i \in \mathcal{O}_4^4 \mid x_i^2, x_i x_{i-1} + x_{i-1} x_i, \\ & \quad x_i x_j + x_k x_i + x_j x_k, \text{ if } ij = ki \text{ and } j \neq i \neq k \in \mathcal{O}_4^4 \rangle. \end{aligned}$$

Example 5.2. A principal YD-realization of $(\mathcal{O}_2^n, -1)$ or (\mathcal{O}_2^n, χ) , respectively $(X, q) = (\mathcal{O}_4^4, -1)$, over \mathbb{S}_n , respectively \mathbb{S}_4 , is given by the inclusion $X \hookrightarrow \mathbb{S}_n$ and the action \cdot is the conjugation. The family $\{\chi_i\}$ is determined by the cocycle. In any case g is injective. For $n = 3, 4, 5$, this is in fact the only possible realization over \mathbb{S}_n .

Remark 5.3. Notice that all $(\mathcal{O}_2^n, -1)$, (\mathcal{O}_2^n, χ) , for any n and $(\mathcal{O}_4^4, -1)$ satisfy that $\mathcal{R} = \mathcal{R}'$.

When $n = 3, 4, 5$, we have from [AG1, GG] $\widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, -1) = \mathfrak{B}(\mathcal{O}_2^n, -1)$, $\widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, \chi) = \mathfrak{B}(\mathcal{O}_2^n, \chi)$ and $\dim \mathfrak{B}(\mathcal{O}_2^n, -1), \dim \mathfrak{B}(\mathcal{O}_2^n, \chi) < \infty$.

5.3. Pointed Hopf algebras constructed from racks. A *quadratic lifting datum* $\mathcal{Q} = (X, q, G, (\cdot, g, (\chi_l)_{l \in X}), (\gamma_C)_{C \in \mathcal{R}'})$, or ql-datum, [GG, Definition 3.5], is a collection consisting of a rack X , a 2-cocycle q , a finite group G , a principal YD-realization $(\cdot, g, (\chi_l)_{l \in X})$ of (X, q) over G such that $g_i \neq g_j g_k$, for all $i, j, k \in X$, and a collection $(\gamma_C)_{C \in \mathcal{R}'} \in \mathbb{k}$ satisfying that for each $C = \{(i_2, i_1), \dots, (i_n, i_{n-1})\} \in \mathcal{R}'$, $k \in X$,

$$(5.4) \quad \gamma_C = 0, \quad \text{if } g_{i_2} g_{i_1} = 1,$$

$$(5.5) \quad \gamma_C = q_{ki_2} q_{ki_1} \gamma_{k \triangleright C}, \quad \text{if } k \triangleright C = \{k \triangleright (i_2, i_1), \dots, k \triangleright (i_n, i_{n-1})\}.$$

To each ql-datum \mathcal{Q} there is attached a pointed Hopf algebra $\mathcal{H}(\mathcal{Q})$ generated as an algebra by $\{a_l, H_t : l \in X, t \in G\}$ subject to relations:

$$(5.6) \quad H_e = 1, \quad H_t H_s = H_{ts}, \quad t, s \in G;$$

$$(5.7) \quad H_t a_l = \chi_l(t) a_{t \cdot l} H_t, \quad t \in G, l \in X;$$

$$(5.8) \quad \phi_C(\{a_l\}_{l \in X}) = \gamma_C (1 - H_{g_i g_j}), \quad C \in \mathcal{R}', (i, j) \in C.$$

Here ϕ_C is as in (5.1) above. The algebra $\mathcal{H}(\mathcal{Q})$ has a structure of pointed Hopf algebra setting

$$\Delta(H_t) = H_t \otimes H_t, \quad \Delta(a_i) = g_i \otimes a_i + a_i \otimes 1, \quad t \in G, i \in X.$$

See [GG] for further details.

5.4. Pointed Hopf algebras over \mathbb{S}_n . The following ql-data provide examples of (possibly infinite-dimensional) pointed Hopf algebras over \mathbb{S}_n .

1. $\mathcal{Q}_n^{-1}[t] = (\mathbb{S}_n, \mathcal{O}_2^n, -1, \cdot, \iota, \{0, \alpha, \beta\})$,
2. $\mathcal{Q}_n^\chi[\lambda] = (\mathbb{S}_n, \mathcal{O}_2^n, \chi, \cdot, \iota, \{0, 0, \alpha\})$ and
3. $\mathcal{D}[t] = (\mathbb{S}_4, \mathcal{O}_4^4, -1, \cdot, \iota, \{\alpha, 0, \beta\})$,

for $\alpha, \beta, \lambda \in \mathbb{k}$, $t = (\alpha, \beta)$. We will present explicitly the algebras $\mathcal{H}(\mathcal{Q})$ associated to these data. It follows that relations (5.8) for each $C \in \mathcal{R}'$ with the same cardinality are \mathbb{S}_n -conjugated. Thus it is enough to consider a single relation for each C with a given number of elements.

1. $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ is the algebra presented by generators $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ and relations:

$$H_e = 1, \quad H_r H_s = H_{rs}, \quad r, s \in \mathbb{S}_n,$$

$$H_j a_i = -a_{jij} H_j, \quad i, j \in \mathcal{O}_2^n,$$

$$a_{(12)}^2 = 0;$$

$$a_{(12)} a_{(34)} + a_{(34)} a_{(12)} = \alpha (1 - H_{(12)} H_{(34)});$$

$$a_{(12)} a_{(23)} + a_{(23)} a_{(13)} + a_{(13)} a_{(12)} = \beta (1 - H_{(12)} H_{(23)}).$$

2. $\mathcal{H}(\mathcal{Q}_n^X[\lambda])$ is the algebra presented by generators $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ and relations:

$$\begin{aligned} H_e &= 1, & H_r H_s &= H_{rs}, & r, s &\in \mathbb{S}_n, \\ H_j a_i &= \chi_i(j) a_{jij} H_j, & i, j &\in \mathcal{O}_2^n, \\ a_{(12)}^2 &= 0; \\ a_{(12)} a_{(34)} - a_{(34)} a_{(12)} &= 0; \\ a_{(12)} a_{(23)} - a_{(23)} a_{(13)} - a_{(13)} a_{(12)} &= \alpha(1 - H_{(12)} H_{(23)}). \end{aligned}$$

3. $\mathcal{H}(\mathcal{D}[t])$ is the algebra generated by generators $\{a_i, H_r : i \in \mathcal{O}_4^4, r \in \mathbb{S}_4\}$ and relations:

$$\begin{aligned} H_e &= 1, & H_r H_s &= H_{rs}, & r, s &\in \mathbb{S}_n, \\ H_j a_i &= -a_{jij} H_j, & i &\in \mathcal{O}_4^4, & j &\in \mathcal{O}_2^4, \\ a_{(1234)}^2 &= \alpha(1 - H_{(13)} H_{(24)}), \\ a_{(1234)} a_{(1432)} + a_{(1432)} a_{(1234)} &= 0, \\ a_{(1234)} a_{(1243)} + a_{(1243)} a_{(1423)} + a_{(1423)} a_{(1234)} &= \beta(1 - H_{(12)} H_{(13)}). \end{aligned}$$

These Hopf algebras have been defined in [AG1, Def. 3.7], [GG, Def. 3.9], [GG, Def. 3.10] respectively. Each of these algebras $\mathcal{H}(\mathcal{Q})$ satisfy $\text{gr } \mathcal{H}(\mathcal{Q}) = \widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}G$, for $G = \mathbb{S}_n$, n as appropriate [GG, Propositions 5.4, 5.5, 5.6].

Remark 5.4. Classification results:

- (a) [AHS] $\mathcal{H}(\mathcal{Q}_3^{-1}[t])$, $t = (0, 0)$ or $t = (0, 1)$ are all the non-trivial finite-dimensional pointed Hopf algebras over \mathbb{S}_3 .
- (b) [GG] $\mathcal{H}(\mathcal{Q}_4^{-1}[t])$, $\mathcal{H}(\mathcal{Q}_4^X[\zeta])$, $\mathcal{H}(\mathcal{D}[t])$, $t \in \mathbb{P}_{\mathbb{k}}^1 \cup \{(0, 0)\}$, $\zeta \in \{0, 1\}$ is a complete list of the non-trivial finite-dimensional pointed Hopf algebras over \mathbb{S}_4 .

We will classify module categories over the category of representations of any pointed Hopf algebra over \mathbb{S}_3 or \mathbb{S}_4 , that is, of the algebras listed in Remark 5.4.

6. COIDEAL SUBALGEBRAS OF QUADRATIC NICHOLS ALGEBRAS

A fundamental piece of information to determine simple comodule algebras is the computation of homogeneous coideal subalgebras inside the Nichols algebra. This is part of Theorem 3.2. The study of coideal subalgebras is an active field of research in the theory of Hopf algebras and quantum groups, see for example [HK], [HS], [K] and [KL].

In this section we present a description of all homogeneous left coideal subalgebras in the quadratic approximations of the Nichols algebras constructed from racks.

Fix $n \in \mathbb{N}$, $X = \{i_1, \dots, i_n\}$ a rack of n elements and $q : X \times X \rightarrow \mathbb{k}^*$ a 2-cocycle. Let \mathcal{R} be as in 5.1. Assume that, for any equivalence class C in \mathcal{R} and $i, j, k \in X$,

$$(6.1) \quad (i, j), (i, k) \in C \Rightarrow j = k \quad \text{and} \quad (i, j), (k, i) \in C \Rightarrow k = i \triangleright j.$$

Let G be a finite group and let $(\cdot, g, (\chi_i)_{i \in X})$ be a principal YD-realization of (X, q) over G . We shall further assume that

$$(6.2) \quad g \quad \text{is injective and } \mathcal{R} = \mathcal{R}'.$$

For each subset $Y \subseteq X$, $Y = \{i_{j_1}, \dots, i_{j_r}\} \subseteq X$, denote by \mathcal{K}_Y the subalgebra of $\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}1$ generated by x_{j_1}, \dots, x_{j_r} . Set $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}G$.

Proposition 6.1. *For each set $Y = \{i_{j_1}, \dots, i_{j_r}\} \subseteq X$ the algebra \mathcal{K}_Y is an homogeneous coideal subalgebra of \mathcal{H} . For each such selection, if $S = \{g_{i_{j_1}}, \dots, g_{i_{j_r}}\}$ then*

$$\text{Stab } \mathcal{K}_Y = S_Y^G = \{h \in G : hS_Y h^{-1} = S_Y\}.$$

Moreover, if \mathcal{K} is a homogeneous coideal subalgebra of \mathcal{H} generated in degree one, then there exists a unique $Y \subseteq X$ such that

$$\mathcal{K} = \mathcal{K}_Y.$$

In particular, the set of homogeneous coideal subalgebras of \mathcal{H} generated in degree one inside $\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}1$ is in bijective correspondence with the set 2^X of parts of X .

Proof. It is clear that $\mathcal{K} = \mathcal{K}_Y$ is a homogeneous coideal subalgebra. Now, to describe $\text{Stab } \mathcal{K}$ it is enough to compute the stabilizer of the vector space $\mathbb{k}\{x_{j_1}, \dots, x_{j_r}\}$. But $h \cdot x_{j_k} = \chi_{j_k}(h)x_{h \cdot j_k}$, $k = 1, \dots, r$ and $x_{h \cdot j_k} \in \{x_{j_1}, \dots, x_{j_r}\}$ if and only if $h \cdot j_k \in \{j_1, \dots, j_r\}$, if and only if $g_{h \cdot j_k} = g_{j_l}$ for some $l = 1, \dots, r$. And the first part of the proposition follows since $g_{h \cdot j_k} = hg_{j_k}h^{-1}$.

Now, let \mathcal{K} be a homogeneous coideal subalgebra of \mathcal{H} generated in degree one. If $\mathcal{K} = \mathbb{k}$ the result is trivial, so let us assume that $\mathcal{K} \neq \mathbb{k}$. Since \mathcal{K} is homogeneous then $\mathcal{K}(1) \neq 0$. Let $0 \neq y = \sum_i \lambda_i x_i \in \mathcal{K}(1)$, then

$$\Delta(y) = y \otimes 1 + \sum_i \lambda_i H_{g_i} \otimes x_i \Rightarrow \sum_i \lambda_i H_{g_i} \otimes x_i \in \mathcal{H}_0 \otimes \mathcal{K}(1).$$

Let $\sum_i \lambda_i H_{g_i} \otimes x_i = \sum_{t \in G} H_t \otimes \kappa_t$, $\kappa_t = \sum_{j \in X} \eta_{tj} x_j \in \mathcal{K}(1)$, $\eta_{tj} \in \mathbb{k}$, $\forall t, j$.

As $H_t = H_{g_j}$ if and only if $t = g_j$ and $g_i = g_j$ if and only if $i = j$, for every $i, j \in X$, $t \in G$, (6.2), then $\eta_{tk} = 0$ if $t \neq g_k$ for some $k \in X$. Let us denote $\eta_{ij} = \eta_{g_i j}$, thus,

$$\sum_i \lambda_i H_{g_i} \otimes x_i = \sum_{i, j} \eta_{ij} H_{g_i} \otimes x_j.$$

Therefore, $\lambda_i \neq 0$ implies $\eta_{ij} = \delta_{i, j} \lambda_i$ and so $\kappa_i = x_i$. Thus, $\{x_i \mid \lambda_i \neq 0\} \subseteq \mathcal{K}$, $\mathcal{K}(1) = \bigoplus_{x_i \in \mathcal{K}(1)} \mathbb{k}\{x_i\}$ and therefore if $Y = \{i \in X : x_i \in \mathcal{K}(1)\}$ then

$\mathcal{K} = \mathcal{K}_Y$. Finally, if $Y \neq Y'$ then it follows from the injectivity of g that $\mathcal{K}_Y \not\cong \mathcal{K}_{Y'}$ as coideal subalgebras. \square

The next general lemma will be useful in 6.1 to prove that certain subalgebras are generated in degree one. Given a rack X , let us recall the notion of derivations δ_i associated to every element of the canonical basis $\{e_i\}_{i \in X}$. If $\{e^i\}_{i \in X}$ denotes the dual basis of this basis, then $\delta_i = (\text{id} \otimes e^i)\Delta$. If $i \in X$ we denote by X_i the set $X \setminus \{i\}$ and thus $\mathbb{k}X_i = \mathbb{k}\{x_j \mid j \in X_i\}$. Let us assume, furthermore, that

$$(6.3) \quad q_{ii} = -1, \quad \text{for all } i \in X.$$

This condition is satisfied, for example, if $\dim \widehat{\mathfrak{B}}_2(X, q) < \infty$ or X is such that $i \triangleright i = i$ for all i , by (6.2).

Lemma 6.2. *Let $\mathcal{K} \subset \widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}1$ be a homogeneous coideal subalgebra of \mathcal{H} . Let $i \in X$ and let us assume that there is $\omega \in \mathcal{K}$ such that $\delta_i(\omega) \neq 0$. Then $x_i \in \mathcal{K}(1)$.*

Proof. Let $\mathcal{K} = \bigoplus_s \mathcal{K}(s)$, $\omega \in T(\mathbb{k}X)$ and $i \in X$. In \mathcal{H} ,

$$\omega = \alpha_i(\omega) + \beta_i(\omega)x_i, \quad \alpha_i(\omega), \beta_i(\omega) \in \mathcal{K}_{X_i}.$$

It suffices to see this for a homogeneous monomial ω . We see it by induction in $\ell = \ell(\omega) \in \mathbb{N}$ such that $\omega \in T^\ell(\mathbb{k}X)$. If $\ell = 0$, or $\ell = 1$ this is clear. Let us assume it holds for $\ell = n - 1$, for some $n \in \mathbb{N}$. If $\ell(\omega) = n$ and $\omega = x_{j_1} \dots x_{j_n}$, two possibilities hold, that is $j_1 \neq i$ or $j_1 = i$. In the first case, let $\omega' = x_{j_2} \dots x_{j_n}$. Thus, $\ell(\omega') \leq n - 1$ and therefore there exist $\alpha_i(\omega'), \beta_i(\omega') \in \mathcal{K}_{X_i}$ such that $\omega' = \alpha_i(\omega') + \beta_i(\omega')x_i$. As $x_{j_1}\alpha_i(\omega')$, $x_{j_1}\beta_i(\omega') \in \mathcal{K}_{X_i}$ the claim follows in this case.

In the second case, let $j = j_2$ and let us note that $j \neq i$, by (6.3). By (6.2), we can consider the relation

$$x_i x_j = q_{ij} x_{i \triangleright j} x_i - q_{ij} q_{i \triangleright j} x_j x_{i \triangleright j}.$$

Thus, if $\omega'' = x_{j_3} \dots x_{j_n}$, $\omega = q_{ij} x_{i \triangleright j} x_i \omega'' - q_{ij} q_{i \triangleright j} x_j x_{i \triangleright j} \omega''$ and both members of this sum belong to $\mathcal{K}_{X_i} + \mathcal{K}_{X_i} x_i$ because of the previous case and thus the claim follows.

Let $\pi : \bigoplus_{s=0}^m \mathcal{H}(s) \otimes \mathcal{K}(m-s) \rightarrow \mathcal{H}(m-1) \otimes \mathcal{K}(1)$ be the canonical linear projection. Let $\omega \in T(\mathbb{k}X)$, $i \in X$ and $\alpha_i(\omega)$, $\beta_i(\omega)$ as above. Then,

$$\pi \Delta(\omega) \in \beta_i(\omega) \otimes x_i + \bigoplus_{j \neq i} \mathcal{H} \otimes x_j.$$

Notice that $\delta_i(\omega) = \beta_i(\omega)$, and therefore if $\delta_i(\omega) \neq 0$ it follows that $x_i \in \mathcal{K}(1)$ using (6.2) as in the proof of Proposition 6.1. \square

In this part we shall assume that X is one of the racks \mathcal{O}_2^n , $n \in \mathbb{N}$, or \mathcal{O}_4^4 , q one of the cocycles in 5.2. Notice that (6.1) is satisfied in these cases. Using the previous results we shall describe explicitly all connected homogeneous

coideal subalgebras of the bosonization of the quadratic approximations to Nichols algebras described in 5.2.

We first introduce the following notation. Let $Y \subset X$ be a subset and define

$$\begin{aligned}\mathcal{R}_1^Y &= \{C \in \mathcal{R} : C \subseteq Y \times Y\}, \\ \mathcal{R}_2^Y &= \{C \in \mathcal{R} : |C \cap Y \times Y| = 1\}, \text{ and} \\ \mathcal{R}_3^Y &= \{C \in \mathcal{R} : C \cap Y \times Y = \emptyset\}.\end{aligned}$$

Remark 6.3. For the ql-data in 5.4, $\mathcal{R} = \mathcal{R}_1^Y \cup \mathcal{R}_2^Y \cup \mathcal{R}_3^Y$, for any subset Y . Moreover, if $f \in \text{Stab } \mathcal{K}_Y$ then $f \cdot \mathcal{R}_s^Y \subseteq \mathcal{R}_s^Y$ for any $s = 1, 2, 3$. Also, (6.3) holds.

Definition 6.4. Take the free associative algebra \mathcal{T} in the variables $\{T_l\}_{l \in Y}$. According to this, we set $\vartheta_{C,Y}(\{T_l\}_{l \in Y})$ in \mathcal{T} as

$$(6.4) \quad \vartheta_{C,Y}(\{T_l\}_{l \in Y}) = \begin{cases} \phi_C(\{T_l\}_{l \in X}), & \text{if } C \in \mathcal{R}_1^Y; \\ T_i T_j T_i - q_{i \triangleright j, i} T_j T_i T_j, & \text{if } C \in \mathcal{R}_2^Y, (i, j) \in C \cap Y \times Y; \\ 0, & \text{if } C \in \mathcal{R}_3^Y. \end{cases}$$

We define the algebra \mathcal{L}_Y as follows

$$(6.5) \quad \mathcal{L}_Y = \mathbb{k}\langle \{y_i\}_{i \in Y} \rangle / \langle \vartheta_{C,Y}(\{y_l\}_{l \in Y}) : C \in \mathcal{R} \rangle.$$

Notice that, if $Y = X$ then $\mathcal{L}_X \cong \mathfrak{B}(X, q)$. For simplicity we shall sometimes denote $\vartheta_C = \vartheta_{C,Y}$.

We now take \mathfrak{B} one of the quadratic (Nichols) algebras $\widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, -1)$, $\widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, \chi)$, or $\mathfrak{B}(\mathcal{O}_4^4, -1)$. Accordingly, let $X = \mathcal{O}_2^n$, $q = -1, \chi$ or $(X, q) = (\mathcal{O}_4^4, -1)$. Consider a YD-realization for (X, q) such that (6.2) is satisfied (for instance, the ones in Example 5.2). Set $\mathcal{H} = \mathfrak{B} \# \mathbb{k}G$.

Theorem 6.5. *Let $Y \subset X$. \mathcal{L}_Y is an \mathcal{H} -comodule algebra with coaction*

$$\delta(y_i) = g_i \otimes y_i + x_i \otimes 1, \quad i \in Y.$$

The application $y_i \mapsto x_i$, $i \in Y$ defines an epimorphism of \mathcal{H} -comodule algebras $\mathcal{L}_Y \twoheadrightarrow \mathcal{K}_Y$. Moreover, if $n = 3$, it is an isomorphism and $\mathcal{L}_Y \cong \mathcal{K}_Y$.

Proof. The relations that define \mathcal{L}_Y are satisfied in \mathfrak{B} . In fact, it suffices to check this in the case $C \in \mathcal{R}_2^Y$ since in the other ones $\vartheta_C = 0$ or $\vartheta_C = \phi_C$, and $\phi_C = 0$ in \mathfrak{B} , see (5.2). Now, if $C \in \mathcal{R}_2^Y$ and $(i, j) \in C \cap Y \times Y$, let $k = i \triangleright j$. By the definition of \mathcal{R}_2^Y , we have that necessarily $k \neq i, j$. Then, if we multiply the relation $x_i x_j - q_{ij} x_{i \triangleright j} x_i + q_{ij} q_{i \triangleright j} x_j x_{i \triangleright j} = 0$ by x_i on the right and apply this relations to the outcome, we get

$$\begin{aligned}0 &= x_i x_j x_i + q_{ij} q_{i \triangleright j} x_j x_{i \triangleright j} x_i = x_i x_j x_i + q_{i \triangleright j} x_j (x_i x_j + q_{ij} q_{i \triangleright j} x_j x_{i \triangleright j}) \\ &= x_i x_j x_i + q_{i \triangleright j} x_j x_i x_j.\end{aligned}$$

Thus, we have an algebra projection $\pi : \mathcal{L}_Y \rightarrow \mathcal{K}_Y$. It is straightforward to see that, for every $C \in \mathcal{R}$,

$$\delta(\vartheta_{C,Y}(\{y_l\}_{l \in Y})) = \vartheta_{C,Y}(\{x_l\}_{l \in Y}) \otimes 1 + g_{C,Y} \otimes \vartheta_{C,Y}(\{y_l\}_{l \in Y}),$$

where

$$g_{C,Y} = \begin{cases} g_i g_j & \text{if } C \in \mathcal{R}_1^Y, (i, j) \in C, \\ g_i g_j g_i & \text{if } C \in \mathcal{R}_2^Y, (i, j) \in C \cap Y \times Y, \\ 0, & \text{if } C \in \mathcal{R}_3^Y. \end{cases}$$

Therefore, δ provides \mathcal{L}_Y with a structure of \mathcal{H} -comodule in such a way that π becomes a homomorphism.

We analyze now the particular case $n = 3$. If $|Y| = 1$, the result is clear. Let us suppose then that $Y = \{i, j\} \subset \mathcal{O}_2^3$. Notice that the map π is homogeneous. Then, if $\gamma \in \ker(\pi)$, $\pi(\gamma) = 0$ in $\mathfrak{B}(\mathcal{O}_2^3, -1)$. By (5.2), we have that necessarily $\deg \gamma \geq 3$. Now, if $\deg \gamma = 3$,

$$\gamma = \alpha y_i y_j y_i + \beta y_j y_i y_j = (\alpha + \beta) y_j y_i y_j.$$

for $\alpha, \beta \in \mathbb{k}$. Then, $\pi(\gamma) = 0$ implies that $\alpha = -\beta$ and $\gamma = 0$. Finally, we can see that there are no elements $\gamma \in \mathcal{L}_Y$ with $\deg \gamma \geq 4$. In fact, an element γ with $\deg \gamma = 4$ would be of the form

$$\gamma = \alpha y_i y_j y_i y_j + \beta y_j y_i y_j y_i = \alpha y_i y_i y_j y_i + \beta y_j y_j y_i y_j = 0.$$

This also shows that there are no elements of greater degree. Therefore, $\mathcal{L}_Y = \mathcal{K}_Y$. \square

Remark 6.6. If $n \neq 3$, then in general $\mathcal{L}_Y \neq \mathcal{K}_Y$. In fact, if $n = 4$, $q = -1$ and we take $Y = \{(13), (23), (34)\} \subseteq \mathcal{O}_2^4$, then we have

$$\mathcal{L}_Y \cong \mathbb{k}\langle x, y, z : x^2, y^2, z^2, xyx - yxy, yzy - zyz, xzx - zxz \rangle.$$

Now, in the subalgebra of $\mathfrak{B}(\mathcal{O}_2^4, -1)$ generated by $x = x_{(23)}$, $y = x_{(34)}$, $z = x_{(13)}$ we have the relation

$$\begin{aligned} (xyz)^2 &= x_{(23)} x_{(34)} x_{(13)} x_{(23)} x_{(34)} x_{(13)} \\ &= -x_{(23)} x_{(34)} (x_{(23)} x_{(12)} + x_{(12)} x_{(13)}) x_{(34)} x_{(13)} \\ &= x_{(23)} x_{(34)} x_{(23)} x_{(34)} x_{(12)} x_{(13)} \\ &\quad + x_{(23)} x_{(12)} x_{(34)} x_{(13)} x_{(34)} x_{(13)} \\ &= x_{(23)} x_{(23)} x_{(34)} x_{(23)} x_{(12)} x_{(13)} \\ &\quad + x_{(23)} x_{(12)} x_{(34)} x_{(34)} x_{(13)} x_{(34)} \\ &= 0. \end{aligned}$$

But $(xyz)^2 \neq 0$ in \mathcal{L}_Y . We prove this by using the computer program [GAP] together with the package [GBNP]. See Proposition 6.9 (6), for a description of \mathcal{K}_Y in this case.

6.1. Coideal subalgebras of Hopf algebras over \mathbb{S}_n . Set $n = 3, 4$, \mathfrak{B} a finite-dimensional Nichols algebra over \mathbb{S}_n , $\mathcal{H} = \mathfrak{B} \# \mathbb{k}\mathbb{S}_n$. Recall that these Nichols algebras coincide with their quadratic approximations. We will describe all the coideal subalgebras of \mathcal{H} . We will also calculate their stabilizer subgroups.

We start out by proving that in this case these coideal subalgebras are generated in degree one.

Theorem 6.7. *Let \mathcal{K} be a homogeneous left coideal subalgebra of \mathcal{H} . Then \mathcal{K} is generated in degree one. In particular, $\mathcal{K} = \mathcal{K}_Y$ for a unique $Y \subseteq X$.*

Proof. We will see that, given $\omega \in \mathcal{K}$, $\omega \in \langle x_i : \delta_i \omega \neq 0 \rangle$. Then, by Lemma 6.2, it will follow that $\omega \in \langle \mathcal{K}(1) \rangle$. Let $I = \{i \in X : \delta_i \omega = 0\}$ and let us assume I has m elements. We will see this case by case, for $m = 0, \dots, 6$.

Cases $m = 0$ (that is, for all $i \in X$ $x_i \in \mathcal{K}(1)$) and $m = 6$, or $m = n$ in general, (since in this case $\omega = 0$, see [AG1, Section 6]) are clear. Case $m = 1$ is Lemma 6.2, which also holds for any $n \in \mathbb{N}$.

Let us see case $m = 2$, for any $n \in \mathbb{N}$. Let $I = \{i, p\}$. We know that there is an expression of ω without, v.g., x_i . Let us see that we can write ω without x_i nor x_p . Let $j \in X$ such that $p \triangleright j = i$. Using the relations as in Lemma 6.2, and using that $x_l x_r x_l = -q_{l \triangleright r} x_r x_l x_r$ and that $x_r x_l x_r x_l = 0$ for all $l, r \in X$, we can assume that ω can be written as

$$\omega = \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_j$$

with $\gamma^0, \gamma^1, \gamma^2$, such that they do not contain factors x_i, x_p in their expressions. Let us see this in detail. We can assume that $\omega \in T^\ell(\mathbb{k}X)$ is a homogeneous monomial. For each appearance of a factor $x_p x_l$, with $l \neq j$ we change it by $q_{pl} x_l x_{p \triangleright l} + q_{pl} q_{p \triangleright l p} x_{p \triangleright l} x_p$. That is, we change for an expression in which x_p is located more to the right and an expression that does not contain x_i nor x_p (in the place where we had an x_p). If we have a factor of the form $x_p x_j$ we move it to the right, until we get to $x_p x_j x_p$, but we can change this expression by $-q_{p \triangleright j p} x_j x_p x_j$.

Now, $0 = \delta_p \omega = \gamma^1 g_p + \gamma^2 g_p x_j = (\gamma^1 + q_{pj} \gamma^2 x_i) g_p$, and therefore we have

$$\omega = \gamma^0 + \gamma^1 x_p + q_{pj} \gamma^2 x_i x_p + q_{pj} q_{ip} \gamma^2 x_j x_i = \gamma^0 + q_{pj} q_{ip} \gamma^2 x_j x_i.$$

But then $0 = \delta_i \omega = q_{pj} q_{ip} \gamma^2 x_j g_i$ and therefore $\omega = \gamma^0$ can be written without x_i, x_p .

This finishes the case $n = 3$, since in this case $|X| = 3$. We now fix $n = 4$, to deal with the cases $m = 3, 4, 5$.

Let us see case $m = 3$. Fix $I = \{i_1, i_2, p\}$. There are three possibilities

$$(6.6) \quad I = \{i, j, i \triangleright j\};$$

$$(6.7) \quad I = \{i, j, k\}, \quad \text{such that } i \triangleright k = k \text{ or } j \triangleright k = k;$$

$$(6.8) \quad I = \{i, j, l\}, \quad \text{the remaining case.}$$

Set $j_1, j_2 \in X$ be such that $p \triangleright j_s = i_s$, $s = 1, 2$. We can assume that ω is written without x_{i_s} , $s = 1, 2$. Notice that not always j_1, j_2 exist. For instance, in (6.6) there are no j_1 nor j_2 and in (6.7) j_1 or j_2 does not exist. We analyze the three cases separately.

In (6.6), as there are no j_1, j_2 , we can write ω in the form $\omega = \gamma^0 + \gamma^1 x_p$ with γ^0, γ^1 without factors x_j , $j \in I$. But from $\delta_p \omega = 0$ it follows that $\omega = \gamma^0$ and therefore we can write ω without factors x_j , $j \in I$.

Case (6.7) is similar. Assume, for example, that $i_2 \triangleright p = p$. Thus, we have no j_2 . Accordingly, we can assume that ω is of the form

$$\omega = \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_{j_1} = \gamma^0 + \gamma^1 x_p + q_{pj_1} \gamma^2 x_{i_1} x_p + q_{pj_1} q_{j_1 i_1} \gamma^2 x_{j_1} x_{i_1}$$

with $\gamma^1, \gamma^2, \gamma^3$ without factors x_j , $j \in I$. Now, $0 = \delta_p \omega = (\gamma^1 p + q_{pj_1} \gamma^2 x_{i_1}) g_p$ and thus $\omega = \gamma^0 + q_{pj_1} q_{j_1 i_1} \gamma^2 x_{j_1} x_{i_1}$ but as $\delta_{i_1} \omega = 0$, it follows $\omega = \gamma^0$ and therefore ω is written without factors x_j , $j \in I$.

It remains to see (6.8). The existence of j_1, j_2 makes this case more subtle than the previous ones. Let us analyze the set $I = \{i_1, i_2, p\}$. We have that $k = i_1 \triangleright i_2 = i_2 \triangleright i_1 \notin I$, but, moreover, we have that $X = \{i_1, i_2, p, k, j_2, j_1\}$. In fact, we cannot have $i_1 \triangleright i_2 = j_1$ since this implies $i_2 = p$ and neither $i_1 \triangleright i_2 = j_2$ because this implies $i_1 = p$. Moreover, we have that $i_2 \triangleright j_1 = j_1$, and therefore $x_{i_2} x_{j_1} = q_{i_2 j_1} x_{j_1} x_{i_2}$. Set

$$\begin{aligned} a &= x_p, & b &= x_{j_1}, & c &= x_{j_2}, \\ d &= x_{i_1}, & e &= x_{i_2}, & f &= x_k. \end{aligned}$$

Now we analyze which are the longest words we can write with the ‘‘conflictive’’ factors a, b and c , starting with a . Recall that $aba = \pm bab$, and $abb = 0$. Starting with ab , we can preliminary form the words $abca$ and $abcb$. Now, $abcac = \pm babca$, and thus we discard it. Consider $abcb$. $abcabc = 0$, so we are left with $abcaba$. As $abcabab = 0$, we reach to $abcabac$. As $abcabaca = abcabacb = 0$, we keep this word. In the case of $abcb$, arguing similarly, we reach to $abcbacb$. If we start with acb , as $acbc = \pm acbc$, we consider those words starting with $acba$. The longest one is $acbabc$, but this is $\pm abcbacb$. So the longest word we can form not considered before is $acbabc$.

In consequence, we can assume there exist $\gamma^i \in \mathcal{K}$, $i = 0, \dots, 15$ without factors x_j , $j \in I$, such that ω is of the form

$$\begin{aligned} \omega &= \gamma^0 + \gamma^1 a + \gamma^2 ab + \gamma^3 abc + \gamma^4 abca + \gamma^5 abcab + \gamma^6 abcaba + \gamma^7 abcabac \\ &\quad + \gamma^8 abcb + \gamma^9 abcba + \gamma^{10} abcbac + \gamma^{11} abcbacb + \gamma^{12} ac + \gamma^{13} acb \\ &\quad + \gamma^{14} acba + \gamma^{15} acbac. \end{aligned}$$

Using the relations and the fact that $\delta_s \omega = 0$, $s = p, i_1, i_2$ we will show that we can write ω without factors x_s , $s = p, i_1, i_2$. When using the relations, by abuse of notation, we will omit the scalars q that may appear, including them in the (new) factors γ^i . We will denote by $\gamma^{i'}, \gamma^{i''}, \gamma^{i'''} \in \mathcal{K}$ to some of these scalar multiple of the factors γ^i , $i = 0, \dots, 15$, when needed.

As $\delta_p\omega = 0$, we can re-write ω as

$$\begin{aligned}\omega = & \gamma^0 + \gamma^2bd + \gamma^3bdc + \gamma^{3'}dce + \gamma^5bdcbd + \gamma^{5'}dcebd + \gamma^7abcabce \\ & + \gamma^8bdc b + \gamma^{8'}dceb + \gamma^{8''}debd + \gamma^{10}abc bce + \gamma^{11}abcbebd + \gamma^{11'}abc bceb \\ & + \gamma^{12}ce + \gamma^{13}ebd + \gamma^{13'}ceb + \gamma^{15}acbea + \gamma^{15'}ac bce.\end{aligned}$$

Now, using that $\delta_{i_1}\omega = 0$, and the relations $dc = \pm cd$, $be = \pm eb$, $bc b = \pm cbc$ and $abcabc = bc b c = 0$, we see that

$$\begin{aligned}\omega = & \gamma^0 + \gamma^2bd + \gamma^3bcd + \gamma^{3'}cde + \gamma^5bdc b d + \gamma^{5'}dcebd \\ & + \gamma^7abcdeae + \gamma^8bdc b + \gamma^{8'}dcbe + \gamma^{8''}dbed + \gamma^{11}abcbed a \\ & + \gamma^{12}ce + \gamma^{13}bed + \gamma^{13'}cbe + \gamma^{15}ac bce + \gamma^{15'}eda e a.\end{aligned}$$

Using that $\delta_{i_2}\omega = 0$ together with the relations, we get to

$$\begin{aligned}\omega = & \gamma^0 + \gamma^2bd + \gamma^3bcd + \gamma^5bc b ad + \gamma^{5'}cbafe + \gamma^{5''}cbaed \\ & + \gamma^8bcad + \gamma^{8'}bc b a + \gamma^{8''}baed + \gamma^{8'''}baf e + \gamma^{11}abcbebd \\ & + \gamma^{11'}abcbeab + \gamma^{11''}abc b f e a + \gamma^{11'''}abc b f a c + \gamma^{13}bed + \gamma^{13'}bfe.\end{aligned}$$

Using now that $\delta_{i_1}\omega = 0$,

$$\begin{aligned}\omega = & \gamma^0 + \gamma^5cbafe + \gamma^8bc b a + \gamma^{8'}baf e \\ & + \gamma^{11}abc b a c b + \gamma^{11'}abc b f c e + \gamma^{11''}abc b f a c + \gamma^{13}bfe.\end{aligned}$$

Using again that $\delta_{i_2}\omega = 0$,

$$\begin{aligned}\omega = & \gamma^0 + \gamma^8bc b a + \gamma^{11}abc b a c b + \gamma^{11'}abc b a f c \\ = & \beta^0 + \beta^1a + \beta^2abc b a c b + \beta^3abc b a b f\end{aligned}$$

for $\beta^i \in \mathcal{K}$, $i = 0, \dots, 3$ without factors x_j , $j \in I$. Using that $\delta_p\omega = 0$,

$$\omega = \beta^0 + \beta^2deda e d a + \beta^3ded a d a f = \beta^0,$$

since $eda e d a = dada = 0$. That is, we can write ω without factors x_j , $j \in I$.

In the case $m = 4$, we look at the different subsets of three elements of I . If we have a subset of three elements that corresponds to the case (6.8) it follows that ω can be written without the factors x_j with j in that subset, and then ω is in an algebra isomorphic to $\mathfrak{B}(\mathcal{O}_2^3, -1)$, for which we have already proved the result. If we have a subset as in the case (6.6) when we add to this subset a fourth element we obtain another subset as in the case (6.8). If our subset corresponds to the case (6.7), in order to get to a case different from (6.8), we necessarily have to add a fourth element such that I is

$$I = \{i, j, k, l\}, \quad \text{with } i \triangleright k = k \text{ and } j \triangleright l = l.$$

We analyze this case. If $p \in I$ is fixed and ω is written without factors x_j , $j \in I \setminus \{p\} = \{i_1, i_2, i_3\}$, notice that if $p \triangleright i_3 = i_3$ there is no other j_3 such that $p \triangleright j_3$ and, moreover, if j_1, j_2 are such that $p \triangleright j_s = i_s$, $s = 1, 2$, then

$x_{j_1}x_{j_2} = \pm x_{j_2}x_{j_1}$. Therefore, we can assume that there are γ^i , $i = 0, \dots, 4$ such that they do not contain factors x_j , $j \in I$ and such that ω can be written as

$$\begin{aligned} \omega &= \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_{j_1} + \gamma^3 x_p x_{j_1} x_{j_2} + \gamma^4 x_p x_{j_1} x_{j_2} x_p \\ \delta_p \omega &\stackrel{=0}{=} \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} + \gamma^{3'} x_{i_1} x_{j_2} x_{i_2} + \gamma^4 x_{i_1} x_{i_2} x_p \\ \delta_p \omega &\stackrel{=0}{=} \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} + \gamma^{3'} x_{i_1} x_{j_2} x_{i_2} \\ \delta_{i_2} \omega &\stackrel{=0}{=} \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} \\ \delta_{i_1} \omega &\stackrel{=0}{=} \gamma^0 + \gamma^3 x_{j_1} x_{j_2} x_{i_2} \\ \delta_{i_2} \omega &\stackrel{=0}{=} \gamma^0. \end{aligned}$$

Then, we can write ω without x_j , for $j \in I$. In the case $m = 5$, ω necessarily belongs to an algebra isomorphic to $\mathfrak{B}(\mathcal{O}_2^3, -1)$. \square

Now we apply Theorems 6.5 and 6.7 to calculate the coideal subalgebras and stabilizer subgroups of $\mathcal{H} = \mathfrak{B}(\mathcal{O}_2^3) \# \mathbb{k}\mathbb{S}_3$.

Corollary 6.8. *The following are all the proper homogeneous left coideal subalgebras of $\mathfrak{B}(\mathcal{O}_2^3, -1) \# \mathbb{k}\mathbb{S}_3$:*

- (1) $\mathcal{K}_i = \langle x_i \rangle \cong \mathbb{k}[x]/\langle x^2 \rangle$, $i \in \mathcal{O}_2^3$;
- (2) $\mathcal{K}_{i,j} = \langle x_i, x_j \rangle \cong \mathbb{k}\langle x, y \rangle / \langle x^2, y^2, xyx - yxy \rangle$, $i, j \in \mathcal{O}_2^3$.

The non trivial stabilizer subgroups of \mathbb{S}_3 are, on each case

- (1) $\text{Stab } \mathcal{K}_i = \mathbb{Z}_2 \cong \langle i \rangle \subset \mathbb{S}_3$;
- (2) $\text{Stab } \mathcal{K}_{i,j} = \mathbb{Z}_2 \cong \langle k \rangle \subset \mathbb{S}_3$, $k \neq i, j$. \square

Next, we use the computer program [GAP], together with the package [GBNP], to compute the coideal subalgebras of the finite-dimensional Nichols algebras over \mathbb{S}_4 associated to the rack of transpositions \mathcal{O}_2^4 . In the same way, the coideal subalgebras of the Nichols algebra $\mathfrak{B}(\mathcal{O}_4^4, -1)$ associated to the rack of 4-cycles can be computed. The presentation of these algebras may not be minimal, in the sense that there may be redundant relations. Moreover, in the general case, non-redundant relations in a coideal subalgebra \mathcal{K} may become redundant when computing the bosonization with a subgroup $F \leq \text{Stab } \mathcal{K}$.

First, we need to establish some notation and conventions. Let $\mathbb{k}\langle x, y, z \rangle$ be the free algebra in the variables x, y, z . We set the ideals

$$R^\pm(x, y, z) = \langle x^2, y^2, z^2, xy + yz \pm zx \rangle \subset \mathbb{k}\langle x, y, z \rangle.$$

Set $\mathfrak{B}_4^+ = \mathfrak{B}(\mathcal{O}_2^4, -1)$, $\mathfrak{B}_4^- = \mathfrak{B}(\mathcal{O}_2^4, \chi)$. Recall that Y stands for a subset of \mathcal{O}_2^4 .

Proposition 6.9. *Let $\varepsilon = \pm$. Any homogeneous proper coideal subalgebra \mathcal{K}^ε of $\mathfrak{B}_4^\varepsilon \# \mathbb{k}1$ is isomorphic to one of the algebras in the following list:*
 $\dim \mathcal{K}^\varepsilon(1) = 1$,

(1) $Y = \{i\}$, $\mathcal{K}^\varepsilon = \mathbb{k}[x]/\langle x^2 \rangle$, and $\dim \mathcal{K}^\varepsilon = 2$.

$\dim \mathcal{K}^\varepsilon(1) = 2$,

(2) $Y = \{i, j\}$, $i \triangleright j = j$,
 $\mathcal{K}^\varepsilon = \mathbb{k}\langle x, z \rangle / \langle x^2, z^2, xz + \varepsilon zx \rangle$, and $\dim \mathcal{K}^\varepsilon = 4$.

(3) $Y = \{i, j\}$, $i \triangleright j \neq j$,
 $\mathcal{K}^\varepsilon = \mathbb{k}\langle x, y \rangle / \langle x^2, y^2, xyx - \varepsilon yxy \rangle$, and $\dim \mathcal{K}^\varepsilon = 6$.

$\dim \mathcal{K}^\varepsilon(1) = 3$:

(4) $Y = \{i, j, k\}$, $i \triangleright j = k$,
 $\mathcal{K}^\varepsilon = \mathbb{k}\langle x, y, z \rangle / \langle R^\varepsilon(x, y, z) \rangle$, and $\dim \mathcal{K}^\varepsilon = 12$.

(5) $Y = \{i, j, k\}$, $i \triangleright j \neq j, k$, $i \triangleright k = k$
 $\mathcal{K}_{i,j,k}^\varepsilon := \mathbb{k}\langle x, y, z \rangle / \langle x^2, y^2, z^2, xyx - \varepsilon yxy, zyz - \varepsilon yzy, xz + \varepsilon zx \rangle$,
and $\dim \mathcal{K}^\varepsilon = 24$.

(6) $Y = \{i, j, k\}$, $i \triangleright j, j \triangleright k, i \triangleright k \notin \{i, j, k\}$,

$$\begin{aligned} \mathcal{K}_Y^\varepsilon = \mathbb{k}\langle x, y, z : x^2, y^2, z^2, \\ yxy - \varepsilon xyx, xzx - \varepsilon xzx, zyz - \varepsilon yzy, \\ zxyz + yzxy + xyzx, zyxz + yxzy + xzyx \\ zxyxzx + \varepsilon yzxyxz, zxyxzy + \varepsilon xzxyxz \rangle, \end{aligned}$$

and $\dim \mathcal{K}^\varepsilon = 48$.

$\dim \mathcal{K}^\varepsilon(1) = 4$:

(7) $Y = \{i, j, k, l\}$, $i \triangleright j = k$, $i \triangleright l = l$;

$$\begin{aligned} \mathcal{K}_Y^\varepsilon = \mathbb{k}\langle x, y, z, w : x^2, y^2, z^2, w^2, \\ zx + \varepsilon yz + \varepsilon xy, zy + yx + \varepsilon xz, wz + \varepsilon zw, \\ yxy - \varepsilon xyx, wxw - \varepsilon xwx, wyw - \varepsilon ywy, \\ wyx + \varepsilon wxz - \varepsilon zwy, wyz + wxy - zwx \\ wxyz - zwxz, wxzw + xwxz, \\ wxyw + ywxy + xywx, wxyxz - \varepsilon zwxyx, \\ wxyxwx + \varepsilon ywxyxw, wxyxwy + \varepsilon xwxyxw \rangle, \end{aligned}$$

and $\dim \mathcal{K}^\varepsilon = 96$.

(8) $Y = \{i, j, k, l\}$, $i \triangleright j \neq j, k$, $i \triangleright k = k$, $j \triangleright l = l$,

$$\begin{aligned} \mathcal{K}_Y^\varepsilon = \mathbb{k}\langle x, y, z, w : x^2, y^2, z^2, w^2, zy + \varepsilon yz, wx + \varepsilon xw, \\ yxy - \varepsilon xyx, xzx - \varepsilon xzx, wyw - \varepsilon ywy, wzw - \varepsilon zwz, \\ zxyx + yzxy, zxyz + \varepsilon xzxy, \\ wyx - \varepsilon zwy - yxz + \varepsilon xzw, wxz - \varepsilon zxy - ywz + \varepsilon xyw, \\ wyzxy - \varepsilon ywyzx - xyzwy + xyxzw, \\ wyzax + zxywz - yxzwy - xwyzx, \\ wyzaw - \varepsilon zaxw - yzaw + yxwy + \varepsilon xwyz - \varepsilon xyza \rangle, \end{aligned}$$

and $\dim \mathcal{K}^\varepsilon = 144$.

$\dim \mathcal{K}^\varepsilon(1) = 5$:

$$(9) \ Y = \{i, j, k, l, m\}, \ i \triangleright j = k, \ i \triangleright l = m, \ j \triangleright l \neq l, \ k \triangleright m \neq m, \\ j \triangleright m = m, \ k \triangleright l = l,$$

$$\begin{aligned} \mathcal{K}^\varepsilon = \mathbb{k}\langle x, y, z, w, u : x^2, y^2, z^2, w^2, u^2, wz + \varepsilon zw, uy + \varepsilon yu, \\ zx + \varepsilon yz + \varepsilon xy, zy + yx + \varepsilon xz, \\ ux + \varepsilon wu + \varepsilon xw, uw + wx + \varepsilon xu, \\ yxy - \varepsilon xyx, wxw - \varepsilon xwx, wyw - \varepsilon ywy, uzu - \varepsilon zuz, \\ wyx + \varepsilon wxz - \varepsilon zwy, wyz + wxy - zwx, \\ uzw - \varepsilon wxz - xuz, wxyz - zwxz, \\ wxyw + ywxy + xywx, \\ wxyxz - \varepsilon zwxxy, wxzw + xwxz, \\ wxyxwx + \varepsilon ywxyxw, wxyxwy + \varepsilon xwxyxw \rangle, \end{aligned}$$

and $\dim \mathcal{K}^\varepsilon = 288$.

The stabilizers subgroups of \mathbb{S}_4 are, in each case, the following:

- (1) $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle g_i, g_j \rangle \subset \mathbb{S}_4$ with $i \triangleright j = j$;
- (2) $D_4 \cong \langle g_i, \sigma \rangle \subset \mathbb{S}_4$ (if, e.g., $g_i = (12)$, $\sigma = (1324)$);
- (3) $\mathbb{Z}_2 \cong \langle g_k \rangle \subset \mathbb{S}_4$, $k = i \triangleright j$;
- (4) $\mathbb{S}_3 \cong \langle g_i, g_j, g_k \rangle \subset \mathbb{S}_4$, $i \triangleright j = k$;
- (5) $\mathbb{Z}_2 \cong \langle g_j g_l \rangle$, $j \neq l, j \triangleright l = l$;
- (6) $\mathbb{S}_3 \cong \langle g_{i \triangleright j}, g_{j \triangleright k}, g_{k \triangleright i} \rangle \subset \mathbb{S}_4$;
- (7) If \mathcal{K}^ε belongs to items (7) to (8) then $\text{Stab } \mathcal{K}^\varepsilon = 1$. □

Examples 6.10. We give, as an illustration, an example of a subset $Y \subseteq \mathcal{O}_2^4$ for each case in the previous proposition. Note that any comodule algebra $\mathcal{K}_{Y'}$ such that Y' is not in the next list, is \mathbb{S}_4 -conjugated to another, \mathcal{K}_Y , with Y a set in the list.

- (1) $Y = \{(12)\}$,
- (2) $Y = \{(12), (34)\}$,
- (3) $Y = \{(12), (13)\}$,
- (4) $Y = \{(12), (13), (23)\}$,
- (5) $Y = \{(12), (13), (34)\}$,
- (6) $Y = \{(12), (13), (14)\}$,
- (7) $Y = \{(12), (13), (23), (14)\}$,
- (8) $Y = \{(12), (13), (24), (34)\}$,
- (9) $Y = \{(12), (13), (23), (14), (24)\}$. □

Remark 6.11. Let $Y \subset \mathcal{O}_2^4$ and let $Z \subset \mathcal{O}_2^4$ be such that $\mathcal{O}_2^4 = Y \sqcup Z$, as sets. Denote by Y_j one of the subsets of item j of Proposition 6.9, and by Z_j the corresponding complement. Notice that we have the following bijections

$$Z_1 \cong Y_9, \quad Z_2 \cong Y_8, \quad Z_3 \cong Y_7, \quad Z_4 \cong Y_6, \quad Z_5 \cong Y_5.$$

Therefore, we have that $\dim \mathcal{K}_Y \dim \mathcal{K}_Z = \dim \mathfrak{B}^\varepsilon$, for every Y . An analogous statement holds for the case $X = \mathcal{O}_4^4$.

7. REPRESENTATIONS OF $\text{Rep}(\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}G)$

In this Section, we take $\mathcal{Q} = (X, q, G, (\cdot, g, (\chi_l)_{l \in X}), (\lambda_C)_{C \in \mathcal{R}'})$ as one of ql-data from Section 5.4. Note that in this case, the set $C_i = \{(i, i)\}$ belongs to $\mathcal{R} = \mathcal{R}'$ and $(i \triangleright j) \triangleright i = j$, for any $i, j \in X$. Let $\mathcal{H}(\mathcal{Q})$ be the corresponding Hopf algebra defined in Section 5.3 and set $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}G$. We will assume $\dim \widehat{\mathfrak{B}}_2(X, q) < \infty$ (and thus $\dim \mathcal{H}(\mathcal{Q}) < \infty$, [GG, Proposition 4.2]). In particular, this holds for $n = 3, 4, 5$.

7.1. $\widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}G$ -Comodule algebras. We shall construct families of comodule algebras over quadratic approximations of Nichols algebras. These families are large enough to classify module categories in all of our examples.

Definition 7.1. Let $F < G$ be a subgroup and $\psi \in Z^2(F, \mathbb{k}^\times)$. If $Y \subseteq X$ is a subset such that $F \cdot Y \subseteq Y$, that is $F < \text{Stab } \mathcal{K}_Y$, we shall say that a family of scalars $\xi = \{\xi_C\}_{C \in \mathcal{R}}$, $\xi_C \in \mathbb{k}$ is *compatible* with the triple (Y, F, ψ) if for any $f \in \text{Stab } \mathcal{K}_Y$,

$$\begin{aligned} \xi_{f \cdot C} \chi_i(f) \chi_j(f) &= \xi_C \psi(f, g_i g_j) \psi(f g_i g_j, f^{-1}), & \text{if } C \in \mathcal{R}_1^Y, (i, j) \in C; \\ \xi_{f \cdot C} \chi_i^2(f) \chi_j(f) &= \xi_C \psi(f, g_i g_j g_i) \psi(f g_i g_j g_i, f^{-1}), & \text{if } C \in \mathcal{R}_2^Y, (i, j) \in C; \\ \xi_{C_i} &= \xi_{C_j} = 0, & \text{if } C \in \mathcal{R}_2^Y, (i, j) \in C. \end{aligned}$$

We will assume that the family ξ is normalized by $\xi_C = 0$ if either $C \in \mathcal{R}_1^Y$, $(i, j) \in C$, and $g_i g_j \notin F$ or if $C \in \mathcal{R}_2^Y$, $(i, j) \in C$, and $g_i g_j g_i \notin F$.

We now introduce the comodule algebras we shall work with.

Definition 7.2. Let $F < G$ be a subgroup, $\psi \in Z^2(F, \mathbb{k}^\times)$, and let $Y \subseteq X$ be a subset such that $F \cdot Y \subseteq Y$ and let $\xi = \{\xi_C\}_{C \in \mathcal{R}'}$ be compatible with (Y, F, ψ) . Define $\mathcal{A}(Y, F, \psi, \xi)$ to be the algebra generated by $\{y_l, e_f : l \in Y, f \in F\}$ and relations

$$(7.1) \quad e_1 = 1, \quad e_r e_s = \psi(r, s) e_{rs}, \quad r, s \in F,$$

$$(7.2) \quad e_f y_l = \chi_l(f) y_{f \cdot l} e_f, \quad f \in F, l \in Y,$$

$$(7.3) \quad \vartheta_{C, Y}(\{y_l\}_{l \in X}) = \begin{cases} \xi_C e_C & \text{if } e_C \in F \\ 0 & \text{if } e_C \notin F \end{cases} \quad C \in \mathcal{R}.$$

Here $\vartheta_{C, Y}$ was defined in (6.4) and the element e_C is defined as

$$(7.4) \quad e_C = \begin{cases} e_{g_i g_j} & \text{if } C \in \mathcal{R}_1^Y, (i, j) \in C, \\ e_{g_i g_j g_i} & \text{if } C \in \mathcal{R}_2^Y, (i, j) \in C \cap Y \times Y, \\ 0, & \text{if } C \in \mathcal{R}_3^Y. \end{cases}$$

If $Z \subseteq X$ is a subset invariant under the action of F we define $\mathcal{B}(Z, F, \psi, \xi)$ as the subalgebra of $\mathcal{A}(X, F, \psi, \xi)$ generated by elements $\{y_l, e_f : l \in Z, f \in F\}$.

Remark 7.3. (a) Applying $ad(f)$, $f \in \text{Stab } \mathcal{K}_Y$ to equation (7.3) and using (5.3) one can deduce the equations in Definition 7.1.

(b) It may happen that $\mathcal{B}(Z, F, \psi, \xi) \neq \mathcal{A}(Z, F, \psi, \xi)$.

Let $\lambda : \mathcal{A}(Y, F, \psi, \xi) \rightarrow \mathcal{H} \otimes \mathcal{A}(Y, F, \psi, \xi)$ be the map defined by

$$(7.5) \quad \lambda(e_f) = f \otimes e_f, \quad \lambda(y_l) = x_l \otimes 1 + g_l \otimes y_l,$$

for all $f \in F, l \in Y$.

Lemma 7.4. $\mathcal{A}(Y, F, \psi, \xi)$ is a left \mathcal{H} -comodule algebra with coaction λ as in (7.5) and $\mathcal{B}(Z, F, \psi, \xi)$ is a subcomodule algebra of $\mathcal{A}(X, F, \psi, \xi)$.

Proof. Let us prove first that the map λ is well-defined. It is easy to see that $\lambda(e_f y_l) = \chi_l(f) \lambda(y_{f \cdot l} e_g)$ for any $f \in F, l \in X$.

Let $C \in \mathcal{R}_1^Y$ and $(i, j) \in C$. In this case $\vartheta_C = \phi_C$. We shall prove that $\lambda(\phi_C(\{y_l\}_{l \in X})) = \lambda(\xi_C e_{g_i g_j})$. Using the definition of the polynomial ϕ_C we obtain that

$$\begin{aligned} \lambda(\phi_C(\{y_l\}_{l \in X})) &= \sum_{h=1}^{n(C)} \eta_h(C) x_{i_{h+1}} x_{i_h} \otimes 1 + x_{i_{h+1}} g_{i_h} \otimes y_{i_h} + g_{i_{h+1}} x_{i_h} \otimes y_{i_{h+1}} + \\ &+ g_{i_{h+1}} g_{i_h} \otimes y_{i_{h+1}} y_{i_h} = \phi_C(\{x_l\}_{l \in X}) \otimes 1 + g_i g_j \otimes \phi_C(\{y_l\}_{l \in X}) \\ &= \xi_C g_i g_j \otimes e_{g_i g_j} = \lambda(\xi_C e_{g_i g_j}). \end{aligned}$$

The second equality follows since $i_{n(C)+1} = i_1$ and

$$g_{i_{h+1}} x_{i_h} = q_{i_{h+1} i_h} x_{i_{h+2}} g_{i_{h+1}}, \quad \eta_h(C) q_{i_{h+1} i_h} = -\eta_{h+1}(C).$$

Now, let $C \in \mathcal{R}_2^Y$, $(i, j) \in C$ and $i \triangleright j \notin Y$. In this case relation (7.3) is $y_i y_j y_i + q_{i \triangleright j} y_j y_i y_j = \xi_C e_{g_i g_j g_i}$. Note that assumption $\xi_{C_i} = \xi_{C_j} = 0$ implies that $y_i^2 = 0 = y_j^2$. The proof that $\lambda(y_i y_j y_i + q_{i \triangleright j} y_j y_i y_j) = \xi_C \lambda(e_{g_i g_j g_i})$ is a straightforward computation. \square

Theorem 7.5. Let $Y \subseteq X$ be an F -invariant subset and assume that $\mathcal{A}(X, F, \psi, \xi) \neq 0$, then the following statements hold:

1. The algebras $\mathcal{A}(X, G, \psi, \xi)$ are left \mathcal{H} -Galois extensions.
2. If ξ satisfies

$$(7.6) \quad \xi_C = \begin{cases} -\lambda_C & \text{if } \lambda_C \neq 0, \\ 0 & \text{if } \lambda_C = 0 \text{ and } g_j g_i \neq 1, \\ \text{arbitrary} & \text{if } \lambda_C = 0 \text{ and } g_j g_i = 1. \end{cases}$$

then $\mathcal{A}(X, G, 1, \xi)$ is a $(\mathcal{H}, \mathcal{H}(\mathcal{Q}))$ -biGalois object.

3. $\mathcal{B}(Y, F, \psi, \xi)_0 = \mathbb{k}_\psi F$ and thus $\mathcal{B}(Y, F, \psi, \xi)$ is a right \mathcal{H} -simple left \mathcal{H} -comodule algebra.
4. There is an isomorphism of comodule algebras $\text{gr } \mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{K}_Y \# \mathbb{k}_\psi F$.
5. There is an isomorphism $\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{B}(Y', F', \psi', \xi')$ of comodule algebras if and only if $Y = Y', F = F', \psi = \psi'$ and $\xi = \xi'$.

Proof. 1. To prove that $\mathcal{A}(X, G, \psi, \xi)$ is a Galois observe that the canonical map

$$\text{can} : \mathcal{A}(X, G, \psi, \xi) \otimes \mathcal{A}(X, G, \psi, \xi) \rightarrow \mathcal{H} \otimes \mathcal{A}(X, G, \psi, \xi),$$

$\text{can}(x \otimes y) = x_{(-1)} \otimes x_{(0)} y$, is surjective. Indeed for any $f \in G$, $l \in X$ $\text{can}(e_f \otimes e_{f^{-1}}) = f \otimes 1$, $\text{can}(y_l \otimes 1 - e_{g_l} \otimes e_{g_l^{-1}} y_l) = x_l \otimes 1$.

2. Define the map $\rho : \mathcal{A}(X, G, 1, \xi) \rightarrow \mathcal{A}(X, G, 1, \xi) \otimes \mathcal{H}(\mathcal{Q})$, by

$$\rho(e_f) = e_f \otimes H_f, \quad \rho(y_l) = y_l \otimes 1 + e_{g_l} \otimes a_l, \quad l \in X, f \in G.$$

The map ρ is well-defined. Indeed, if $C \in \mathcal{R}$ and $(i, j) \in C$ then

$$\begin{aligned} \rho(\phi_C(\{y_l\}_{l \in X})) &= \phi_C(\{y_l\}_{l \in X}) \otimes 1 + e_{g_i g_j} \otimes \phi_C(\{a_l\}_{l \in X}) \\ &= \xi_C e_{g_i g_j} \otimes 1 + \lambda_C e_{g_i g_j} \otimes (1 - H_{g_i g_j}). \end{aligned}$$

Clearly if ξ satisfies (7.6) then $\rho(\phi_C(\{y_l\}_{l \in X})) = \xi_C \rho(e_{g_i g_j})$. The proof that $\mathcal{A}(X, G, 1, \xi)$ is a $(\mathcal{H}, \mathcal{H}(\mathcal{Q}))$ -bicomodule and a right $\mathcal{H}(\mathcal{Q})$ -Galois object is done by a straightforward computation.

3. If $\mathcal{A}(X, F, \psi, \xi) \neq 0$ then there is a group \overline{F} with a projection $F \twoheadrightarrow \overline{F}$ such that $\mathcal{A}(Y, F, \psi, \xi)_0 = \mathbb{k}_{\overline{F}} \overline{F}$. The map $\mathcal{A}(Y, F, \psi, \xi)_0 \otimes \mathcal{A}(Y, F, \psi, \xi)_0 \rightarrow \mathbb{k}F \otimes \mathcal{A}(Y, F, \psi, \xi)_0$ defined by $e_f \otimes e_g \mapsto f \otimes \psi(f, g) e_{fg}$ is surjective. Hence $F = \overline{F}$. This implies that $\mathcal{B}(Z, F, \psi, \xi)_0 = \mathbb{k}_{\psi} F$ and by [M1, Prop. 4.4] follows that $\mathcal{B}(Z, F, \psi, \xi)$ is a right \mathcal{H} -simple left \mathcal{H} -comodule algebra.

4. Follows from Theorem 3.2 (3) that $\text{gr } \mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{K} \# \mathbb{k}_{\psi} F$ for some homogeneous left coideal subalgebra $\mathcal{K} \subseteq \widehat{\mathfrak{B}}_2(X, q)$. Recall that \mathcal{K} is identified with the subalgebra of $\text{gr } \mathcal{B}(Y, F, \psi, \xi)$ given by

$$\{a \in \text{gr } \mathcal{A}(Y, F, \psi, \xi) : (\text{id} \otimes \pi)\lambda(a) \in \mathcal{H} \otimes 1\}.$$

See [M1, Proposition 7.3 (3)]. In *loc. cit.* it is also proved that the composition

$$\text{gr } \mathcal{B}(Y, F, \psi, \xi) \xrightarrow{(\theta \otimes \pi)\lambda} \mathcal{K} \# \mathbb{k}_{\psi} F \xrightarrow{\mu} \text{gr } \mathcal{B}(Y, F, \psi, \xi),$$

is the identity map, where $\theta : \mathcal{H} \rightarrow \widehat{\mathfrak{B}}_2(X, q)$, $\pi : \text{gr } \mathcal{B}(Y, F, \psi, \xi) \rightarrow \mathbb{k}_{\psi} F$ are the canonical projections and μ is the multiplication map. Both maps are bijections and since for any $l \in Y$, $(\theta \otimes \pi)\lambda(y_l) = x_l$, then $\mathcal{K} = \mathcal{K}_Y$.

5. Let $\beta : \mathcal{B}(Y, F, \psi, \xi) \rightarrow \mathcal{B}(Y', F', \psi', \xi')$ be a comodule algebra isomorphism. The restriction of β to $\mathcal{B}(Y, F, \psi, \xi)_0$ induces an isomorphism between $\mathbb{k}_{\psi} F$ and $\mathbb{k}_{\psi'} F'$, thus $F = F'$ and $\psi = \psi'$. Since β is a comodule morphism it is clear that $Y = Y'$ and $\xi_C = \xi'_C$ for any $C \in \mathcal{R}$. \square

Corollary 7.6. *If $\mathcal{A}(X, G, 1, \xi) \neq 0$ for some ξ satisfying (7.6), then*

1. *The Hopf algebras $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}G$ and $\mathcal{H}(\mathcal{Q})$ are cocycle deformations of each other.*
2. *There is a bijective correspondence between equivalence classes of exact module categories over $\text{Rep}(\mathcal{H})$ and $\text{Rep}(\mathcal{H}(\mathcal{Q}))$.* \square

Remark 7.7. Under the assumptions in Corollary 7.8, we obtain, in particular, that $\text{gr } \mathcal{H}(\mathcal{Q}) = \widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}G$, since the latter is a quotient of the first.

The following corollary uses Propositions 8.1 and 8.5, where certain algebras are shown to be not null. These propositions will be proven in the Appendix and their proofs are independent of the results in the article.

Corollary 7.8. *Let H be a non-trivial pointed Hopf algebra over \mathbb{S}_3 or \mathbb{S}_4 . Then H is a cocycle deformation of $\text{gr } H$.*

Proof. Finite-dimensional Nichols algebras over \mathbb{S}_3 and \mathbb{S}_4 coincide with their quadratic approximations. That is, if H is a finite-dimensional pointed Hopf algebra over \mathbb{S}_n , $n = 3, 4$, then $\text{gr } H \cong \widehat{\mathfrak{B}}_2(X, q) \# \mathbb{k}\mathbb{S}_n$. By [GG, Main Theorem] we know that $H \cong \mathcal{H}(\mathcal{Q})$. Therefore, the theorem follows from Corollary 7.6, since in Propositions 8.1, 8.5 we show the existence of non-zero $(\text{gr } \mathcal{H}(\mathcal{Q}), \mathcal{H}(\mathcal{Q}))$ -biGalois objects in these cases.

When dealing with $\mathcal{Q}_4^{-1}[t]$ or $\mathcal{D}[t]$, notice that condition $\xi_2 = 2\xi_1$ in Proposition 8.5 does not interfere with the proof, since, by equation (7.6), ξ_1 , resp. ξ_2 , can be chosen arbitrarily. \square

Remark 7.9. In [Ma, Theorem A1] Masuoka proved that the Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ associated to a datum of finite Cartan type \mathcal{D} appearing in the classification of Andruskiewitsch and Schneider [AS] are cocycle deformations to the associated graded Hopf algebras $u(\mathcal{D}, 0, 0)$.

Corollaries 7.6 (1) and 7.8 provide a similar result for some families of Hopf algebras constructed from Nichols algebras not of diagonal type. It would be interesting to generalize this kind of result for larger classes of Nichols algebras.

7.2. Module categories over $\text{Rep}(\mathcal{H}(\mathcal{Q}))$.

Let A be a \mathcal{H} -comodule algebra with $\text{gr } A = \mathcal{K}_Y \# \mathbb{k}_\psi F$, for $F \leq \text{Stab } \mathcal{K}_Y$, $\psi \in Z^2(F, \mathbb{k}^*)$. Let Z be such that $X = Y \sqcup Z$ as sets. Notice that $F \leq \text{Stab } \mathcal{K}_Z$.

Lemma 7.10. *Under the above assumptions there exists a family of scalars ξ compatible with (X, F, ψ) such that $A \simeq \mathcal{B}(Y, F, \psi, \xi)$ as comodule algebras.*

Proof. The canonical projection $\pi : A_1 \rightarrow A_1/A_0 \simeq \mathcal{K}_Y(1) = \mathbb{k}Y$ is a morphism of \mathcal{A}_0 -bimodules. Let $\iota : \mathbb{k}Y \rightarrow A_1$ be a section of \mathcal{A}_0 -bimodules of π . Since elements $\{x_l : l \in Y\}$ are in the image of π we can choose elements $\{y_l : l \in Y\}$ in A_1 such that $\iota(x_l) = y_l$ for any $l \in Y$. It is straightforward to verify that

$$\lambda(y_l) = x_l \otimes 1 + g_l \otimes y_l, \quad e_f y_l = \chi_l(f) y_{f \cdot l} e_f, \quad f \in F, l \in Y.$$

Since $\text{gr } A$ is generated by elements $\{x_l, e_f : l \in Y, f \in F\}$ then A is generated as an algebra by elements $\{y_l, e_f : l \in Y, f \in F\}$.

Now, let $B = A \otimes \mathcal{K}_Z$. Then B has an comodule algebra structure for which the canonical inclusion $A \hookrightarrow A \otimes 1 \subset B$ is a homomorphism. The algebra structure is given as follows. For $i \in Y$, $j \in Z$, $f \in F$,

$$\begin{aligned} (y_i \otimes 1)(1 \otimes y_j) &= (y_i \otimes y_j); \\ (1 \otimes y_j)(y_i \otimes 1) &= \begin{cases} q_{ji}y_i \otimes y_j + \xi_C e_C \otimes 1, & \text{if } i \triangleright j = j \\ q_{ji}y_{j \triangleright i} \otimes y_j - q_{ji}q_{j \triangleright i} y_i y_{j \triangleright i} \otimes 1 + \xi_C e_C \otimes 1, & \text{if } i \triangleright j \neq j, \\ & i \triangleright j \in Y; \\ q_{ji}1 \otimes y_{j \triangleright i} y_j - q_{ji}q_{j \triangleright i} y_i \otimes y_{j \triangleright i} + \xi_C e_C \otimes 1, & \text{if } i \triangleright j \neq j, \\ & i \triangleright j \notin Y; \end{cases} \\ (e_f \otimes 1)(1 \otimes y_j) &= e_f \otimes y_j; \\ (1 \otimes y_j)(e_f \otimes 1) &= \chi_j^{-1}(f) e_f \otimes y_{f^{-1}.j}. \end{aligned}$$

Here C stands for the class $C \in \mathcal{R}'$ such that $(j, i) \in C$. Recall that by definition $\xi_C = 0$ if $g_C \notin F$. Then the map

$$(7.7) \quad m : B \rightarrow \mathcal{A}(X, F, \psi, \xi), \quad a \otimes x \mapsto ax$$

is an algebra epimorphism. Now, if

$$\begin{aligned} A \ni a &\mapsto a_{(-1)} \otimes a_{(0)} \in \mathcal{H} \otimes A \quad \text{and} \\ \mathcal{K}_Z \ni x &\mapsto x_{(-1)} \otimes x_{(0)} \in \mathcal{H} \otimes \mathcal{K}_Z \end{aligned}$$

denote the corresponding coactions, define $\lambda : B \rightarrow \mathcal{H} \otimes B$ by $\lambda(a \otimes x) = a_{(-1)} x_{(-1)} \otimes a_{(0)} \otimes x_{(0)}$. It is straightforward to check that λ is well defined. We do this case by case in the definition of the multiplication of B above. For instance, if $i \triangleright j \neq j$ and $i \triangleright j \in Y$, then we have

$$\begin{aligned} \lambda(1 \otimes y_j) \lambda(y_i \otimes 1) &= (g_j \otimes (1 \otimes y_j) + x_j \otimes (1 \otimes 1))(g_i \otimes (y_i \otimes 1) + x_i \otimes (1 \otimes 1)) \\ &= (g_j \otimes (1 \otimes y_j))(g_i \otimes (y_i \otimes 1)) + (x_j \otimes (1 \otimes 1))(g_i \otimes (y_i \otimes 1)) \\ &\quad + (g_j \otimes (1 \otimes y_j))(x_i \otimes (1 \otimes 1)) + (x_j \otimes (1 \otimes 1))(x_i \otimes (1 \otimes 1)) \\ &= g_j g_i \otimes (1 \otimes y_j)(y_i \otimes 1) + x_j g_i \otimes (y_i \otimes 1) \\ &\quad + q_{ji} x_{j \triangleright i} g_j \otimes (1 \otimes y_j) + x_j x_i \otimes (1 \otimes 1) \\ &= g_j g_i \otimes (q_{ji} y_{j \triangleright i} \otimes y_j - q_{ji} q_{j \triangleright i} y_i y_{j \triangleright i} \otimes 1 + \xi_C g_C \otimes 1) \\ &\quad + x_j g_i \otimes (y_i \otimes 1) + q_{ji} x_{j \triangleright i} g_j \otimes (1 \otimes y_j) \\ &\quad + (q_{ji} x_{j \triangleright i} x_j - q_{ji} q_{j \triangleright i} x_i x_{j \triangleright i} \otimes 1) \otimes (1 \otimes 1), \end{aligned}$$

which coincides with $\lambda(q_{ji} y_{j \triangleright i} \otimes y_j - q_{ji} q_{j \triangleright i} y_i y_{j \triangleright i} \otimes 1 + \xi_C g_C \otimes 1)$.

Thus, B is an \mathcal{H} -comodule algebra with

$$\dim B = \dim A \dim \mathcal{K}_Z = \dim \mathcal{K}_Y \dim \mathcal{K}_Z |F| = \dim \mathcal{A}(X, F, \psi, \xi),$$

by Remark 6.11 and then the map m from (7.7) is an isomorphism. \square

We can now formulate the main result of the paper. For any $h \in G$, we denote $\xi_C^h = \xi_{h^{-1}.C}$. Recall that we denote by $\mathcal{B}(Y, F, \psi, \xi)$ the sub-comodule algebra of $\mathcal{A}(X, F, \psi, \xi)$ generated by $\{y_i\}_{i \in Y}$.

- Theorem 7.11.** 1. Let \mathcal{M} be an exact indecomposable module category over $\text{Rep}(\mathcal{H}(\mathcal{Q}))$, then there exists
- (i) a subgroup $F < G$, and a 2-cocycle $\psi \in Z^2(F, \mathbb{k}^\times)$,
 - (ii) a subset $Y \subset X$ such that $F \cdot Y \subset Y$,
 - (iii) a family of scalars $\{\xi_C\}_{C \in \mathcal{R}'}$ compatible with (X, F, ψ) , such that there is a module equivalence $\mathcal{M} \simeq_{\mathcal{B}(Y, F, \psi, \xi)} \mathcal{M}$.
2. Let (Y, F, ψ, ξ) , (Y', F', ψ', ξ') be two families as before. Then there is an equivalence of module categories $_{\mathcal{B}(Y, F, \psi, \xi)} \mathcal{M} \simeq_{\mathcal{B}(Y', F', \psi', \xi')} \mathcal{M}$ if and only if there exists an element $h \in G$ such that $F' = hFh^{-1}$, $\psi' = \psi^h$, $Y' = h \cdot Y$ and $\xi' = \xi^h$.

Proof. 1. By Corollary 7.8 we can assume that \mathcal{M} is an exact indecomposable module category over $\text{gr } \mathcal{H}(\mathcal{Q}) = \mathcal{H}$. It follows by [AM, Theorem 3.3] that there is a right \mathcal{H} -simple left \mathcal{H} -comodule algebra \mathcal{A} such that $\mathcal{M} \simeq_{\mathcal{A}} \mathcal{M}$. Theorem 3.2 implies that there is a subgroup $F < G$, a 2-cocycle $\psi \in Z^2(F, \mathbb{k}^\times)$ and a subset $Y \subset X$ with $F \cdot Y \subset Y$ such that $\text{gr } \mathcal{A} = \mathcal{K}_Y \#_{\mathbb{k}_\psi} F$. Here $\mathcal{A}_0 = \mathbb{k}_\psi F$. Then the result follows from Lemma 7.10.

2. Assume that the module categories $_{\mathcal{B}(Y, F, \psi, \xi)} \mathcal{M}, _{\mathcal{B}(Y', F', \psi', \xi')} \mathcal{M}$ are equivalent, then Theorem 4.2 implies that there exists an element $h \in G$ such that $\mathcal{B}(Y', F', \psi', \xi') \simeq h\mathcal{B}(Y, F, \psi, \xi)h^{-1}$ as H -comodule algebras.

The algebra map $\alpha : h\mathcal{B}(Y, F, \psi, \xi)h^{-1} \rightarrow \mathcal{B}(h \cdot Y, hFh^{-1}, \psi^h, \xi^h)$ defined by

$$\alpha(he_f h^{-1}) = e_{hfh^{-1}}, \quad \alpha(hy_l h^{-1}) = \chi_l(h) y_{h \cdot l},$$

for any $f \in F$, $l \in Y$, is a well-defined comodule algebra isomorphism. Whence $\mathcal{B}(Y', F', \psi', \xi') \simeq \mathcal{B}(h \cdot Y, hFh^{-1}, \psi^h, \xi^h)$ and using Theorem 7.5 (3) we get the result. \square

As a consequence of Theorem 7.11 we have the following result.

Corollary 7.12. Any \mathcal{H} -Galois object is of the form $\mathcal{A}(X, G, \psi, \xi)$.

Proof. Let A be a \mathcal{H} -Galois object. Then ${}_A \mathcal{M}$ is an exact module category over $\text{Rep } \mathcal{H}$. Moreover, ${}_A \mathcal{M}$ is indecomposable. In fact, otherwise there would exist a proper bilateral ideal $J \subset A$ \mathcal{H} -stable [AM, Proposition 1.18]. Thus, $\text{can}(A \otimes J) = \text{can}(J \otimes A)$, what contradicts the bijectivity of can . Then, by Theorem 7.11 there exists a datum (X, G, ψ, ξ) such that $A \cong \mathcal{A}(X, G, \psi, \xi)$. \square

7.3. Modules categories over $\mathfrak{B}(\mathcal{O}_2^3, -1) \# \mathbb{k}\mathbb{S}_3$. We apply Theorem 7.11 to exhibit explicitly all module categories in this particular case. In this case the rack is

$$\mathcal{O}_2^3 = \{(12), (13), (23)\}.$$

For each $i \in \mathcal{O}_2^3$ we shall denote by g_i the element i thought as an element in the group \mathbb{S}_3 . We will show in the Appendix that the algebras in the following result are not null. Then the next corollary follows from Theorem 7.11.

Corollary 7.13. *Let \mathcal{M} be an indecomposable exact module category over $\text{Rep}(\mathfrak{B}(\mathcal{O}_2^3, -1) \# \mathbb{k}\mathbb{S}_3)$. Then there is a module equivalence $\mathcal{M} \simeq_{\mathcal{A}} \mathcal{M}$ where \mathcal{A} is one (and only one) of the comodule algebras in following list. In the following i, j, k denote elements in \mathcal{O}_2^3 and $\xi, \mu, \eta \in \mathbb{k}$.*

1. *For any subgroup $F \subseteq \mathbb{S}_3$, $\psi \in Z^2(F, \mathbb{k}^\times)$, the twisted group algebra $\mathbb{k}_\psi F$.*
2. *The algebra $\mathcal{A}(\{i\}, \xi, 1) = \langle y_i : y_i^2 = \xi 1 \rangle$, with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$.*
3. *The algebra $\mathcal{A}(\{i\}, \xi, \mathbb{Z}_2) = \langle y_i, h : y_i^2 = \xi 1, h^2 = 1, hy_i = -y_i h \rangle$ with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(h) = g_i \otimes h$.*
4. *The algebra $\mathcal{A}(\{i, j\}, 1) = \langle y_i, y_j : y_i^2 = y_j^2 = 0, y_i y_j y_i = y_j y_i y_j \rangle$ with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$.*
5. *The algebra $\mathcal{A}(\{i, j\}, \mathbb{Z}_2) = \langle y_i, y_j, h : y_i^2 = y_j^2 = 0, h^2 = 1, hy_i = -y_j h, y_i y_j y_i = y_j y_i y_j \rangle$ with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$, $\lambda(h) = g_k \otimes h$, where $k \neq i, j$.*
6. *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, 1)$, generated by $\{y_{(12)}, y_{(13)}, y_{(23)}\}$ subject to relations*

$$\begin{aligned} y_{(12)}^2 &= y_{(13)}^2 = y_{(23)}^2 = \xi 1, \\ y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} &= 0, \\ y_{(13)} y_{(12)} + y_{(23)} y_{(13)} + y_{(12)} y_{(23)} &= 0. \end{aligned}$$

The coaction is determined by $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

7. *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mathbb{Z}_2)$, generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ subject to relations*

$$\begin{aligned} y_{(12)}^2 &= y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^2 = 1, \\ hy_{(12)} &= -y_{(12)} h, \quad hy_{(13)} = -y_{(23)} h, \\ y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} &= 0. \end{aligned}$$

The coaction is determined by $\lambda(h) = g_{(12)} \otimes h$, $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

8. *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mu, \eta, \mathbb{Z}_3)$, generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ subject to relations*

$$\begin{aligned} y_{(12)}^2 &= y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^3 = 1, \\ hy_{(12)} &= y_{(13)} h, \quad hy_{(13)} = y_{(23)} h, \quad hy_{(23)} = y_{(12)} h, \\ y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} &= \mu h, \\ y_{(13)} y_{(12)} + y_{(23)} y_{(13)} + y_{(12)} y_{(23)} &= \eta h^2. \end{aligned}$$

The coaction is determined by $\lambda(h) = g_{(132)} \otimes h$, $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$, for any $s \in \mathcal{O}_2^3$.

The action of $e_{(23)}$ is given by $e_{(12)}e_{(13)}e_{(12)}$. Finally, we use computer program `Mathematica`© to check that these matrices satisfy the relations defining the algebra on each case.

We deal now with a generic 2-cocycle $\psi \in Z^2(\mathbb{S}_3, \mathbb{k}^\times)$. Let us fix $\mathcal{A} = \mathcal{A}(Y, F, 1, \xi)$, $\mathcal{A}' = \mathcal{A}(Y, F, \psi, \xi)$. Also, set $U = \mathcal{K}_Y \# \mathbb{k}F$, $U' = \mathcal{K}_Y \# \mathbb{k}_\psi F$. If $\bar{\psi} \in Z^2(U)$ is the 2-cocycle such that $\bar{\psi}_{F \times F} = \psi$, see Lemma 4.1, it follows that $U' = U^{\bar{\psi}}$. Now, as \mathcal{A} is an U -comodule algebra, isomorphic to U as U -comodules, it follows that there exists a 2-cocycle $\gamma \in Z^2(U)$ such that $\mathcal{A} \cong \gamma U$, see [Mo, Sections 7 & 8]. It is easy to check then that $\mathcal{A}' = \gamma U'$, by computing the multiplication on the generators, and thus $\mathcal{A}' \neq 0$. \square

To finish the proof of Corollary 7.8, we present three families of non trivial algebras $\mathcal{A}(X, G, 1, \xi)$, for $X = \mathcal{O}_2^4$, $G = \mathbb{S}_4$ and certain collections of scalars $\{\xi_C\}_{C \in \mathcal{R}'}$ satisfying (7.6). We show $\mathcal{A}(X, G, 1, \xi) \neq 0$ in Proposition 8.5.

Definition 8.2. Let $\psi \in Z^2(\mathbb{S}_4, \mathbb{k}^\times)$, $\alpha, \beta \in \mathbb{k}$.

1. $\mathcal{A}_\psi^{-1}(\alpha, \beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathcal{O}_2^4, g \in \mathbb{S}_4\}$ with relations

$$\begin{aligned} e_1 &= 1, & e_r e_s &= \psi(r, s) e_{rs}, & r, s &\in \mathbb{S}_4, \\ e_g y_l &= \text{sgn}(g) y_{g \cdot l} e_g, & g &\in \mathbb{S}_4, l \in \mathcal{O}_2^4, \\ y_{(12)}^2 &= \alpha 1, \\ y_{(12)} y_{(34)} + y_{(34)} y_{(12)} &= 2\alpha e_{(12)(34)}, \\ y_{(12)} y_{(23)} + y_{(23)} y_{(13)} + y_{(13)} y_{(12)} &= \beta e_{(132)}. \end{aligned}$$

2. $\mathcal{A}_\psi^4(\alpha, \beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathcal{O}_4^4, g \in \mathbb{S}_4\}$ with relations

$$\begin{aligned} e_1 &= 1, & e_r e_s &= \psi(r, s) e_{rs}, & r, s &\in \mathbb{S}_4, \\ e_g y_l &= \text{sgn}(g) y_{g \cdot l} e_g, & g &\in \mathbb{S}_4, l \in \mathcal{O}_4^4, \\ y_{(1234)}^2 &= \alpha e_{(13)(24)}, \\ y_{(1234)} y_{(1432)} + y_{(1432)} y_{(1234)} &= 2\alpha 1, \\ y_{(1234)} y_{(1243)} + y_{(1243)} y_{(1423)} + y_{(1423)} y_{(1234)} &= \beta e_{(132)}. \end{aligned}$$

3. $\mathcal{A}_\psi^\chi(\alpha, \beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathcal{O}_2^4, g \in \mathbb{S}_4\}$ with relations

$$\begin{aligned} e_1 &= 1, & e_r e_s &= \psi(r, s) e_{rs}, & r, s &\in \mathbb{S}_4, \\ e_g y_l &= \chi_l(g) y_{g \cdot l} e_g, & g &\in \mathbb{S}_4, l \in \mathcal{O}_2^4, \\ y_{(12)}^2 &= \alpha 1, \\ y_{(12)} y_{(34)} - y_{(34)} y_{(12)} &= 0, \\ y_{(12)} y_{(23)} - y_{(23)} y_{(13)} - y_{(13)} y_{(12)} &= \beta e_{(132)}. \end{aligned}$$

Remark 8.3. Let $\mathcal{Q} = \mathcal{Q}^{-1}[t]$. It is clear $\mathcal{A}_\psi^{-1}(\alpha, \beta) \cong \mathcal{A}(\mathcal{O}_2^4, \mathbb{S}_4, \psi, \xi)$ for the family $\xi = \{\xi_C\}_{C \in \mathcal{R}}$ where $\xi_C = \xi_i$, if $i = 1, 2, 3$, is constant in the classes C with the same cardinality $|C| = i$ and where in this case $\xi_1 = \alpha$, $\xi_2 = 2\alpha$, $\xi_3 = \beta$.

Analogously, if $\mathcal{Q} = \mathcal{Q}^X[t]$, $\mathcal{A}_\psi^X(\alpha, \beta)$ is the algebra $\mathcal{A}(\mathcal{O}_2^4, \mathbb{S}_4, \psi, \xi)$ for certain family ξ subject to similar conditions as in the previous paragraph. The same holds for $\mathcal{Q} = \mathcal{D}[t]$, $\mathcal{A}_\psi^4(\alpha, \beta)$ and $\mathcal{A}(\mathcal{O}_4^4, \mathbb{S}_4, \psi, \xi)$.

Recall that there is a group epimorphism $\pi : \mathbb{S}_4 \rightarrow \mathbb{S}_3$ with kernel $H = \langle (12)(34), (13)(24), (23)(14) \rangle$. Moreover, $\pi(\mathcal{O}_2^4) = \mathcal{O}_2^3$. Let \mathcal{Q} be one of the ql-data from Subsection 5.4, for $n = 4$.

Lemma 8.4. *Let \mathcal{Q} as above. Take $\gamma = 0$ if $\mathcal{Q} = \mathcal{Q}_4^{-1}$. Then there is an epimorphism of algebras $\mathcal{H}(\mathcal{Q}) \twoheadrightarrow \mathcal{H}(\mathcal{Q}_3^{-1}[\lambda])$.*

Proof. Consider the ideal I in $\mathcal{H}(\mathcal{Q})$ generated by the element $H_{(12)}H_{(34)} - 1$, and let $\mathcal{L} = \mathcal{H}(\mathcal{Q})/I$. We have

$$\begin{aligned} H_{(14)}H_{(23)} &= \text{ad}(H_{(24)})(H_{(12)}H_{(34)}) & \text{so} & \quad H_{(14)}H_{(23)} = 1 & \text{in } \mathcal{L}, \\ a_{(34)} &= \text{ad}(H_{(14)}H_{(23)})(a_{12}) & \text{so} & \quad a_{(34)} = a_{(12)} & \text{in } \mathcal{L}. \end{aligned}$$

Analogously, $H_{(13)} = H_{(24)}$, $a_{(14)} = a_{(23)}$ and $a_{(24)} = a_{(13)}$ in \mathcal{L} . Since, for this ql-data, the action $\cdot : \mathbb{S}_4 \times X \rightarrow X$ is given by conjugation and $g : X \rightarrow \mathbb{S}_4$ is the inclusion, relations (5.6) and (5.7) in the definition of $\mathcal{H}(\mathcal{Q})$ are satisfied in the quotient. It is now easy to check that the quadratic relations (5.8) defining $\mathcal{H}(\mathcal{Q})$ become in the quotient the corresponding ones defining the algebra $\mathcal{H}(\mathcal{Q}_3^{-1}[\lambda])$. \square

Proposition 8.5. *Assume that (Y, F, ψ, ξ) satisfies*

$$(i) \quad \xi_{C_i} = \xi_{C_j}, \quad \forall i, j \in Y.$$

If $\mathcal{Q} \neq \mathcal{Q}_4^X(\lambda)$ assume in addition that

$$(ii) \quad \text{if } i, j \in Y, i \triangleright j = j \text{ and } (i, j) \in C \text{ then } \xi_C = 2\xi_i.$$

Then the algebra $\mathcal{A}(Y, F, \psi, \xi)$ is not null.

Proof. Assume first that $\psi \equiv 1$. Now, given a datum (Y, F, ψ, ξ) , $\pi(F) < \mathbb{S}_3$ and it is easy to see that $\pi(Y)$ is a subrack of \mathcal{O}_2^3 . Moreover, it follows that ξ is compatible with the triple $(\pi(Y), \pi(F), \psi)$. Then we have the algebra $\mathcal{A}(\pi(Y), \pi(F), \psi, \xi)$. As in Lemma 8.4, it is easy to see that if we quotient out by the ideal generated by $\langle e_f e_g : fg^{-1} \in N \rangle$, then we have an algebra epimorphism $\mathcal{A}(Y, F, \psi, \xi) \twoheadrightarrow \mathcal{A}(\pi(Y), \pi(F), \psi, \xi)$. As these algebras are non-zero by Proposition 8.1, so is $\mathcal{A}(Y, F, \psi, \xi)$.

Notice that in the case in which (Y, F, ψ, ξ) is associated with the ql-datum $\mathcal{Q}_4^X(\lambda)$, assumption (ii) is not needed, since the first equation in Definition 7.1 implies that, if $i, j \in Y$ are such that $i \triangleright j = i$ and $C \in \mathcal{R}'$ is the corresponding class, then $\xi_C = 0$ and this relation is contained in the ideal by which we make the quotient.

The case $\psi \neq 1$ follows now as in the proof of Proposition 8.1. \square

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