ON MODULE CATEGORIES OVER FINITE-DIMENSIONAL HOPF ALGEBRAS

NICOLÁS ANDRUSKIEWITSCH AND JUAN MARTÍN MOMBELLI

Abstract. We show that indecomposable exact module categories over the category $\operatorname{Rep} H$ of representations of a finite-dimensional Hopf algebra H are classified by left comodule algebras, H-simple from the right and with trivial coinvariants, up to equivariant Morita equivalence. Specifically, any indecomposable exact module categories is equivalent to the category of finite-dimensional modules over a left comodule algebra. This is an alternative approach to the results of Etingof and Ostrik. For this, we study the stabilizer introduced by Yan and Zhu and show that it coincides with the internal Hom. We also describe the correspondence of module categories between $\operatorname{Rep} H$ and $\operatorname{Rep}(H^*)$.

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Introduction

The notion of fusion category is a far-reaching generalization of the notion of finite group. It has been studied in connection with different problems in conformal field theory, mechanical statistics, the theory of subfactors, and others, the common theme being "quantum symmetries". A comprehensive presentation of fusion categories is [ENO], see also references therein. We refer to *loc. cit.* for definitions and notations used in the present paper. There is a notion of "module category over a tensor category" known in category theory since the sixties [Be]. Semisimple module categories over a fusion category should play the same fundamental rôle as the representation theory of finite groups; see [O1, O2]. In the beautiful paper [EO], the notion of *finite tensor category* was introduced and several properties of fusion categories were extended to finite tensor categories. Finite

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tensor categories are like fusion categories but without semisimplicity; a basic example is the category of finite-dimensional representations of a finite-dimensional Hopf algebra. The natural class of module categories over a finite tensor category is the class of exact tensor categories [EO]. Let $\mathcal{C}$ be a finite tensor category. Then

- Any exact module category over $\mathcal{C}$ is a finite direct product of *indecomposable* exact module categories.
- An indecomposable module category $\mathcal{M}$ over $\mathcal{C}$ is naturally equivalent– as a module category– to the category of right A-modules in $\mathcal{C}$, where A is an algebra in $\mathcal{C}$ [O2, EO]. Explicitly, A can be chosen as the internal End of any non-zero object in $\mathcal{M}$, *cf.* the proof of [O2, Thm. 1].

The purpose of this paper is to study exact module categories over the finite tensor category $\mathcal{C} = \operatorname{Rep} H$, where H is a finite-dimensional Hopf algebra over algebraically closed field $\Bbbk$ of characteristic 0. In [EO], the authors propose to classify indecomposable exact module categories over $\mathcal{C}$ by classifying first simple from the right exact H-module algebras. Indeed, any indecomposable exact module category over $\operatorname{Rep} H$ is equivalent to the category ${}_H\mathcal{M}_R$ of Hopf (H, R)-bimodules for some H-module algebra R. Instead, our approach is through left comodule algebras: any indecomposable exact module category over $\operatorname{Rep} H$ is also equivalent to the category ${}_K\mathcal{M}$ of left modules over some H-comodule algebra K. We feel that this approach is intuitively clearer. Both approaches are related because of the correspondence between module categories over $\operatorname{Rep} H$ and $\operatorname{Rep}(H^*)$.

Section 1 contains a general discussion of module categories arising from comodule algebras. We show that indecomposable exact module category are classified by exact indecomposable left H-comodule algebras up to a suitable "equivariant" Morita equivalence, see Theorem 1.25. Examples of indecomposable left H-comodule algebras are the coideal subalgebras, thanks to a recent result of Skryabin, see Proposition 1.20.

Section 2 is devoted to the stabilizer introduced by Yan and Zhu and a generalization thereof. We study this construction and prove a general version of the duality, Theorem 2.14, answering a question of Yan and Zhu [YZ, p. 3897]. We also generalize a formula for the dimension of the stabilizer obtained by Zhu for semisimple Hopf algebras [Z].

In Section 3, we show that the Yan-Zhu stabilizer is the internal Hom and apply our version of the Yan-Zhu duality to prove a finer classification of indecomposable exact module categories in terms of comodule algebras, see Theorem 3.3. We also describe explicitly the correspondence between module categories over $\operatorname{Rep} H$ and $\operatorname{Rep}(H^*)$ using the Yan-Zhu stabilizer, see Theorem 3.10.

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1. Module categories

1.1. **Preliminaries.** Let $\Bbbk$ be an algebraically closed field of characteristic 0. All vector spaces, algebras, unadorned $\otimes$ and Hom, are over $\Bbbk$. As usual, $V^* = \operatorname{Hom}(V, \Bbbk)$ is the dual vector space of a vector space V, and $\langle\, , \rangle : V^* \times V \to \Bbbk$ is the evaluation. If V is finite dimensional, we identify $(V \otimes V)^*$ with $V^* \otimes V^*$ via

$$\langle \alpha \otimes \beta, v \otimes w \rangle = \langle \alpha, v \rangle \langle \beta, w \rangle,$$

$\alpha, \beta \in V^*$, $v, w \in V$. If A is an algebra, then ${}_A\mathcal{M}$, resp. $\mathcal{M}_A$ denotes the category of finite-dimensional left, resp. right, A-modules. If C is a coalgebra, then ${}^C\mathcal{M}$, resp. $\mathcal{M}^C$ denotes the category of finite-dimensional left, resp. right, C-comodules. The kernel of the counit of C is denoted C^+. We shall use Sweedler's notation for the coproduct and coactions: $\Delta(c) = c_{(1)} \otimes c_{(2)}$ if $c \in C$. Also, if $\lambda : M \to C \otimes M$ is a left coaction, $\lambda(m) = m_{(-1)} \otimes m_{(0)}$.

Let H be a finite-dimensional Hopf algebra with multiplication μ, comultiplication Δ, counit ε, antipode $\mathcal{S}$. We denote $\operatorname{Rep} H$ instead of ${}_H\mathcal{M}$ to emphasize the presence of the tensor structure. We denote by $L : H \to \operatorname{End} H$ and by $R : H \to \operatorname{End} H$ the left regular representation, resp. the righ regular representation, that is $L_a(b) = ab$, $R_a(b) = ba$ for $a, b \in H$. Recall that H^{op}, resp. H^{cop} is the Hopf algebra with opposite multiplication, resp. opposite comultiplication, and $H^{\mathrm{bop}} = (H^{\mathrm{op}})^{\mathrm{cop}}$. Clearly, $(H^*)^{\mathrm{cop}} = (H^{\mathrm{op}})^*$.

We denote by $\rightharpoonup : H \otimes H^* \to H^*$ and $\leftharpoonup : H^* \otimes H \to H^*$ the actions obtained by transposition of the right and left multiplications, and by $\leftharpoondown : H^* \otimes H \to H^*$ and $\rightharpoondown : H \otimes H^* \to H^*$ the corresponding compositions with the inverse of the antipode. That is,

$$\langle a \rightharpoonup \alpha, b \rangle = \langle \alpha, ba \rangle = \langle \alpha \leftharpoonup b, a \rangle, \tag{1.1}$$

$$\langle b \rightharpoondown \alpha, a \rangle = \langle \alpha, \mathcal{S}^{-1}(b) a \rangle, \qquad \langle \alpha \leftharpoondown a, b \rangle = \langle \alpha, b \mathcal{S}^{-1}(a) \rangle, \tag{1.2}$$

$a, b \in H$, $\alpha \in H^*$. We denote by $\underline{L} : H \to \operatorname{End}(H^*)$, respectively by $\overline{L} : H \to \operatorname{End}(H^*)$, the representation afforded by $\rightharpoonup$, respectively by $\rightharpoondown$. The analogous actions (and representations) of H^* on H are denoted by the same symbols. With respect to $\rightharpoonup$, H^* is a H-module algebra. Note that

$$\alpha \rightharpoonup h = \langle \alpha, h_{(2)} \rangle\, h_{(1)}, \qquad h \leftharpoonup \alpha = \langle \alpha, h_{(1)} \rangle\, h_{(2)},$$
$$\alpha \rightharpoondown h = \langle \alpha, \mathcal{S}^{-1}(h_{(1)}) \rangle\, h_{(2)}, \qquad h \leftharpoondown \alpha = \langle \alpha, \mathcal{S}^{-1}(h_{(2)}) \rangle\, h_{(1)},$$

$\alpha \in H^*$, $h \in H$. Notice that

$$(ht) \leftharpoondown \alpha = (h \leftharpoondown \alpha_{(2)})(t \leftharpoondown \alpha_{(1)}). \tag{1.3}$$

If $X, Y \in {}_H\mathcal{M}$ then $\operatorname{Hom}(X, Y)$ is again a left H-module via

$$(h \cdot T)(x) = h_{(1)} \cdot T(\mathcal{S}(h_{(2)}) \cdot x), \qquad x \in X,\ h \in H,\ T \in \operatorname{Hom}(X, Y). \tag{1.4}$$

Similarly, if $W, Z \in \mathcal{M}_H$ then $\operatorname{Hom}(W, Z)$ is again a right H-module via

$$(T \cdot h)(w) = T(w \cdot \mathcal{S}^{-1}(h_{(2)})) \cdot h_{(1)}, \qquad w \in W,\ h \in H,\ T \in \operatorname{Hom}(W, Z). \tag{1.5}$$

Lemma 1.1. [YZ]. *If $h, t \in H$, $\alpha, \beta \in H^*$ then*

$$(h \rightharpoondown \beta)\alpha = (h \leftharpoonup \mathcal{S}^{-2}(\alpha_{(1)})) \rightharpoondown (\beta \alpha_{(2)}). \quad \square \tag{1.6}$$

Let H be a Hopf algebra. A (left) H-comodule algebra is a (left) comodule that is also an associative unital algebra such that the coaction and the counit are morphisms of algebras. A (left) H-module algebra is a (left) module A that is also an associative unital algebra such that the coaction and the counit are morphisms of algebras. This means that $h\cdot(ab) = (h_{(1)}\cdot a)(h_{(2)}\cdot b)$ and $h\cdot 1 = \varepsilon(h)1$, for all $h \in H$, $a, b \in A$. Right module algebras are defined similarly. If $V \in {}_H\mathcal{M}$ (resp. $V \in \mathcal{M}_H$) then the algebra $\operatorname{End}(V)$ is a left (resp. right) H-module algebra. If H is finite-dimensional, the notions of "left H-module algebra" and "right H^*-comodule algebra" are equivalent.

Let R be an H-module algebra. An (H, R)-module is an R-module M inside the monoidal category ${}_H\mathcal{M}$; in other words, M is a left H-module and a right R-module, and the right R-action $M\otimes R \to M$ is an H-module map. We denote by ${}_H\mathcal{M}_R$ the category of (H, R)-modules with morphisms the maps preserving both actions.

Let K be a left H-comodule algebra. In the same vein as before, we denote by ${}^H_K\mathcal{M}$ the category of left H-comodules, left K-modules M such that the left K-module structure $K\otimes M \to M$ is an H-comodule map. Analogously, if S is a right H-comodule algebra, then there is a category ${}_S\mathcal{M}^H$ of right H-comodules, left S-modules with action being a morphism of H-comodules.

Let K be a left H-comodule algebra. A left H-ideal I of K is a left ideal that is also an H-comodule. Right and two-sided H-ideals are defined similarly. The following natural notions are discussed in [EO, S].

Definition 1.2. We shall say that K is *H-simple from the left*, (resp. *H-simple from the right*, resp. *H-simple*) if K has no non-trivial left (resp. right, resp. two-sided) H-ideal. We shall say that K is *H-indecomposable* if there are no nontrivial two-sided H-ideals I and J such that $K = I \oplus J$.

We shall need later the following result.

Lemma 1.3. *Assume that K is H-simple. If $W \neq 0$ is a left K-module, then the representation*

$$\rho : K \to \operatorname{End}(H^*\otimes W), \quad \rho(k)(\alpha\otimes w) = k_{(-1)} \rightharpoonup \alpha\otimes k_{(0)}\cdot w, \tag{1.7}$$

is faithful. Similar for module algebras.

Proof. Let I be the two-sided ideal $\operatorname{Ker}(\rho)$. We shall prove that I is a H-ideal. First, notice that if $k \in I$ then

$$<\gamma, \mathcal{S}^{-1}(k_{(-1)})> \; k_{(0)}\cdot w = 0, \tag{1.8}$$

for all $\gamma \in H^*$, $w \in W$.

Let $k \in I$. Then $\lambda(k) \in H\otimes I$ iff $(\operatorname{id}\otimes\rho)\lambda(k) = 0$ iff $(\mathcal{S}^{-1}\otimes\rho)\lambda(k) = 0$. Let $\alpha, \beta \in H^*, w \in W$. Evaluating $(\mathcal{S}^{-1}\otimes\rho)\lambda(k)$ in $(\alpha\otimes w)$ and the applying $\beta\otimes\operatorname{id}$ we get

$$\begin{aligned}<\beta, \mathcal{S}^{-1}(k_{(-1)})> \rho(k_{(0)})(\alpha\otimes w) &= <\beta, \mathcal{S}^{-1}(k_{(-2)})><\alpha_{(1)}, \mathcal{S}^{-1}(k_{(-1)})> \alpha_{(2)}\otimes k_{(0)}\cdot w\\ &= <\mathcal{S}^{-1}(\alpha_{(1)}\beta), k_{(-1)}> \; \alpha_{(2)}\otimes k_{(0)}\cdot w.\end{aligned}$$

Evaluating the last expression in $h \otimes \mathrm{id}$, $h \in H$ we obtain

$$\begin{aligned} \lt \mathcal{S}^{-1}(\alpha_{(1)}\beta), k_{(-1)} \gt\lt \alpha_{(2)}, h \gt k_{(0)} \cdot w &= \lt \mathcal{S}^{-1}((h \rightharpoonup \alpha)\beta), k_{(-1)} \gt k_{(0)} \cdot w \\ &= \lt (h \rightharpoonup \alpha)\beta, \mathcal{S}^{-1}(k_{(-1)}) \gt k_{(0)} \cdot w. \end{aligned}$$

The last expression is 0 by (1.8). Since α, β, w and h are arbitrary, $(\mathcal{S}^{-1} \otimes \rho)\lambda(k) = 0$. □

1.2. **Freeness over comodule algebras.** Let H be a finite dimensional Hopf algebra. In this subsection we recall some important results obtained recently by S. Skryabin [S].

Theorem 1.4. *Let A be a finite dimensional right H-comodule algebra. Assume that A is H-simple. Then all objects in $\mathcal{M}_A^H$ are projective A-modules. $M \in \mathcal{M}_A^H$ is a free A-module if and only if M/MQ is a free A/Q-module for at least one two-sided maximal ideal Q of A.*

Proof. This is a particular case of [S, Theorem 4.2]. □

Clearly, similar statements hold also for other combinations like ${}^H_K\mathcal{M}$.

Theorem 1.5. *Let A be a finite dimensional H-simple right H-comodule algebra. Let $M \in \mathcal{M}_A^H$. Then there exists $t \in \mathbb{N}$ such that M^t is a free A-module.*

Proof. This follows from the proof of [S, Theorem 3.5]. □

Proposition 1.6. *If $K \subseteq H$ is a left coideal subalgebra then K is H-simple.*

Proof. See the proof of [S, Theorem 6.1]. □

1.3. **Tensor and module categories.** In this paper, we stick to the following terminology. We refer to [BK, O2] for more details.

- A *monoidal category* is a category $\mathcal{C}$ provided with a "tensor" functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbf{1} \in \mathcal{C}$, associativity constraint a, left and right unit constraints l and r, all these subject to the pentagon and the triangle axioms.
- A monoidal category $\mathcal{C}$ is *rigid* if any object in $\mathcal{C}$ has right and left duals.
- A *tensor category* is a rigid monoidal category with $\mathcal{C}$ abelian and $\otimes$ additive in each variable (that is, $\otimes$ is a bifunctor). We shall assume hereafter that both $\mathcal{C}$ and $\otimes$ are $\Bbbk$-linear.
- The *opposite monoidal category* $\mathcal{C}^{\mathrm{op}}$ to a monoidal category $\mathcal{C}$ is the same category $\mathcal{C}$ but with $X \otimes^{\mathrm{op}} Y = Y \otimes X$, $a^{\mathrm{op}}_{X,Y,Z} = a^{-1}_{Z,Y,X}$, $l^{\mathrm{op}} = r$, $r^{\mathrm{op}} = l$, and the same unit. If $\mathcal{C}$ is rigid, resp. tensor, then so is $\mathcal{C}^{\mathrm{op}}$.
- A *monoidal functor* between monoidal categories $\mathcal{C}$ and $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$, provided with a natural isomorphism $b_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$ and an isomorphism $u : F(\mathbf{1}) \to \mathbf{1}$ satisfying two natural axioms, namely

$$a_{F(X),F(Y),F(Z)}(b_{X,Y} \otimes \mathrm{id})b_{X \otimes Y,Z} = (\mathrm{id} \otimes b_{Y,Z})b_{X,Y \otimes Z}F(a_{X,Y,Z})$$
$$F(l_X) = l_{F(X)}(u \otimes \mathrm{id})b_{\mathbf{1},X}, \qquad F(r_X) = r_{F(X)}(\mathrm{id} \otimes u)b_{X,\mathbf{1}}.$$

- A *tensor functor* between tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is a monoidal functor $F : \mathcal{C} \to \mathcal{C}'$ which is $\Bbbk$-linear.

- [Be] A *module category* over a tensor category $\mathcal{C}$ is an abelian category $\mathcal{M}$ provided with an exact bifunctor[1] $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$, $\ell_M : \mathbf{1} \otimes M \to M$ such that for any $X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$.

$$(\mathrm{id} \otimes m_{Y,Z,M}) m_{X,Y\otimes Z,M} (a_{X,Y,Z} \otimes id) = m_{X,Y,Z\otimes M}\, m_{X\otimes Y,Z,M} \tag{1.9}$$
$$(\mathrm{id} \otimes \ell_M) m_{X,\mathbf{1},Y} = r_X \otimes \mathrm{id} \tag{1.10}$$

- A *module functor* between module categories $\mathcal{M}$ and $\mathcal{M}'$ over a tensor category $\mathcal{C}$ is a pair $(\mathcal{F}, c)$, where $\mathcal{F} : \mathcal{M} \to \mathcal{M}'$ is a $\Bbbk$-linear functor and $c_{X,M} : \mathcal{F}(X \otimes M) \to X \otimes \mathcal{F}(M)$ is a natural isomorphism such that the following diagrams are commutative, for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$:

$$(\mathrm{id}_X \otimes c_{Y,M}) c_{X,Y\otimes M} \mathcal{F}(m_{X,Y,M}) = m_{X,Y,\mathcal{F}(M)}\, c_{X\otimes Y,M} \tag{1.11}$$
$$\ell_{\mathcal{F}(M)}\, c_{\mathbf{1},M} = \mathcal{F}(\ell_M). \tag{1.12}$$

  We shall denote $(\mathcal{F}, c) : \mathcal{M} \to \mathcal{M}'$. Note that $(\mathrm{id}\,, \mathrm{id}\,) : \mathcal{M} \to \mathcal{M}$ is a module functor. There is a composition of module functors: if $\mathcal{M}''$ is another module category and $(\mathcal{G}, d) : \mathcal{M}' \to \mathcal{M}''$ is another module functor then the composition

$$(\mathcal{G} \circ \mathcal{F}, e) : \mathcal{M} \to \mathcal{M}'', \qquad \text{where } e = d \circ \mathcal{G}(c), \tag{1.13}$$

  is also a module functor.

- Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be module categories over $\mathcal{C}$. We denote by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ the category whose objects are module functors $(\mathcal{F}, c)$ from $\mathcal{M}_1$ to $\mathcal{M}_2$. A morphism between $(\mathcal{F}, c)$ and $(\mathcal{G}, d) \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is a natural transformation $\alpha : \mathcal{F} \to \mathcal{G}$ such that for any $X \in \mathcal{C}$, $M \in \mathcal{M}_1$:

$$d_{X,M} \alpha_{X\otimes M} = (\mathrm{id}_X \otimes \alpha_M) c_{X,M}. \tag{1.14}$$

- Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exist module functors $F : \mathcal{M}_1 \to \mathcal{M}_2$ and $G : \mathcal{M}_2 \to \mathcal{M}_1$ and natural isomorphisms $\mathrm{id}_{\mathcal{M}_1} \to F \circ G$, $\mathrm{id}_{\mathcal{M}_2} \to G \circ F$ that satisfy (1.14).
- The *direct sum* of two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ is the $\Bbbk$-linear category $\mathcal{M}_1 \times \mathcal{M}_2$ with coordinate-wise module structure.
- A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories.

Let $\mathrm{Irr}\,\mathcal{M}$ be the set of isomorphism classes of irreducible objects in $\mathcal{M}$ and let the *rank* of $\mathcal{M}$ be the cardinal of $\mathrm{Irr}\,\mathcal{M}$, denoted $\mathrm{rk}\,\mathcal{M}$. If $\mathcal{M}$ is a module category of finite rank then $\mathcal{M}$ is a finite direct sum of indecomposable module categories, since $\mathrm{rk}(\mathcal{M}_1 \times \mathcal{M}_2) = \mathrm{rk}(\mathcal{M}_1) + \mathrm{rk}(\mathcal{M}_2)$.

[1] that is, exact in each variable

1.4. **Exact module categories over finite tensor categories.** We are interested in the following class of tensor categories introduced by Etingof and Ostrik.

Definition 1.7. [EO] Let $\mathcal{C}$ be a $\Bbbk$-linear category. We shall say that $\mathcal{C}$ is *finite* if

- it has finitely many simple objects;
- each simple object X has a projective cover $P(X)$;
- the Hom spaces are finite-dimensional;
- each object has finite length.

A *finite* tensor category is a tensor category $\mathcal{C}$ such that the underlying abelian category is finite and the unit object $\mathbf{1}$ is simple.

The following definition seems to be part of the folklore of the subject.

Definition 1.8. A *fusion category* is a finite tensor category $\mathcal{C}$ such that the underlying abelian category is semisimple.

If H is a finite-dimensional Hopf algebra then the category $\operatorname{Rep} H$ of finite-dimensional representations of H is a finite tensor category; $\operatorname{Rep} H$ is a fusion category exactly when H is semisimple.

It is natural to expect that the study of the module categories over a finite tensor category $\mathcal{C}$ would be crucial in the understanding of $\mathcal{C}$. As explained in [EO], one has to consider a particular class of module categories.

Definition 1.9. [EO] A module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$ is *exact* if it is finite and for any projective $P \in \mathcal{C}$ and any $M \in \mathcal{M}$, $P \otimes M$ is projective in $\mathcal{M}$.

If $\mathcal{M}$ is a semisimple finite module category over a finite tensor category $\mathcal{C}$ then it is exact, and the converse is true if $\mathcal{C}$ is fusion (tensoring by the unit object).

Remark 1.10. [EO, Rmk. 3.7] Any exact module category $\mathcal{M}$ is a Frobenius category, that is any projective object of $\mathcal{M}$ is injective and vice versa.

Remark 1.11. A direct sum of finite module categories is exact (resp. semisimple) if and only if each summand is exact (resp. semisimple). Therefore, any exact (resp. finite semisimple) module category over $\mathcal{C}$ is a finite direct product of exact (resp. finite semisimple) indecomposable module categories.

Let $\mathcal{C}$ be an arbitrary tensor category. A natural way to produce module categories over $\mathcal{C}$ is as follows. Let A be an algebra in $\mathcal{C}$; then the category $\mathcal{C}_A$ of right modules in $\mathcal{C}$ is a module category over $\mathcal{C}$. The category ${}_A\mathcal{C}_A$ is monoidal with tensor product $\otimes_A$ and ${}_A\mathcal{C}_B$ is a module category over ${}_A\mathcal{C}_A$.

Remark 1.12. Let $F = (F, b, u) : \mathcal{C} \to \mathcal{C}'$ be a tensor functor and let A be an algebra in $\mathcal{C}$.

(i). $A' := (A', \mu_{A'}, 1_{A'})$, where $A' = F(A)$, $\mu_{A'} = F(m)b^{-1}$, $1_{A'} = F(1_A)u^{-1}$, is an algebra in $\mathcal{C}'$.

(ii). If $M \in \mathcal{C}_A$ is a right A-module then $F(M) \in \mathcal{C}_{A'}$ is a right A'-module, with action $\leftharpoonup' = F(\leftharpoonup)b^{-1}$.

(iii). If $\mathcal{M}$ is a module category over $\mathcal{C}'$ with associativity m and unit ℓ, then it can be regarded as a module category over $\mathcal{C}$ with tensor action $X \otimes M := F(X) \otimes M$, $X \in \mathcal{C}$, $M \in \mathcal{M}$, associativity $\widetilde{m}_{X,Y,M} := m_{F(X),F(Y),M}(b^{-1} \otimes \mathrm{id}\,)$ and unit $\ell'_M = \ell_M(u \otimes \mathrm{id}\,)$.

(iv). Let $\mathcal{F} : \mathcal{C}_A \to \mathcal{C}'_{A'}$ be the restriction of F, that makes sense by (i) and (ii). Then $(\mathcal{F}, c) : \mathcal{C}_A \to \mathcal{C}'_{A'}$ is a module functor with respect to the coaction in (iii), where $c_{X,M} = b_{X,M}$.

(v). If F is an equivalence of tensor categories then $\mathcal{F}$ is an equivalence of module categories.

The concept of internal Hom allows to state a converse of this construction of module categories.

Definition 1.13. Let $\mathcal{M}$ be a finite module category over $\mathcal{C}$. Let $M_1, M_2 \in \mathcal{M}$. Then the functor $X \mapsto \mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2)$ is representable and an object $\underline{\mathrm{Hom}}(M_1, M_2)$ representing this functor is called the *internal Hom* of M_1 and M_2. See [EO, O1] for details. Thus

$$\mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M_1, M_2))$$

for any $X \in \mathcal{C}$, $M_1, M_2 \in \mathcal{M}$.

The "internal End" $\underline{\mathrm{End}}(M) = \underline{\mathrm{Hom}}(M, M)$ of an object $M \in \mathcal{M}$ is an algebra in $\mathcal{C}$. The multiplication is constructed as follows. Denote by $ev_M : \underline{\mathrm{End}}(M) \otimes M \to M$ the *evaluation map* obtained as the image of the identity under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{End}}(M), \underline{\mathrm{End}}(M)) \simeq \mathrm{Hom}_{\mathcal{M}}(\underline{\mathrm{End}}(M) \otimes M, M).$$

Thus the product $\mu : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \to \underline{\mathrm{End}}(M)$ is defined as the image of the map

$$ev_M(\mathrm{id} \otimes ev_M)\, m_{\underline{\mathrm{End}}(M), \underline{\mathrm{End}}(M), M}$$

under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M), \underline{\mathrm{End}}(M)) \simeq \mathrm{Hom}_{\mathcal{M}}((\underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M)) \otimes M, M).$$

Recall, on the other hand, that $M \in \mathcal{M}$ generates $\mathcal{M}$ if for any $N \in \mathcal{M}$, there exists $X \in \mathcal{C}$ such that $\mathrm{Hom}(X \otimes M, N) \neq 0$. It is known that W generates $\mathcal{M}$ iff its simple subquotients represent all equivalence classes of simple objects in $\mathcal{M}$, where "equivalence" has the meaning in [EO, Lemma 3.8]. Hence all simple objects, and *a fortiori* all non-zero objects, of an indecomposable (finite) module category are generators.

Theorem 1.14. [EO, Th. 3.17]; [O2, Th. 3.1]. *Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$ and let $M \in \mathcal{M}$ that generates $\mathcal{M}$. Then the functor $\underline{\mathrm{Hom}}(M, \_\_) : \mathcal{M} \to \mathcal{C}_{\underline{\mathrm{End}}(M)}$ is an equivalence of module categories.*

In particular, if $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an indecomposable exact module category over $\mathcal{C}$ then any non-zero $M \in \mathcal{M}$ provides an equivalence of module categories $\underline{\mathrm{Hom}}(M, \_\_) : \mathcal{M} \to \mathcal{C}_{\underline{\mathrm{End}}(M)}$.

Remark 1.15. [EO, Lemma 4.2]. Keep the notation of the Theorem above. The functor $\underline{\mathrm{Hom}}$ sends subobjects of M to subobjects of $\mathcal{C}_{\underline{\mathrm{End}}(M)}$; these are the right ideals of $\underline{\mathrm{End}}(M)$. Thus, if M is simple then $\underline{\mathrm{End}}(M)$ has no non-zero right ideals.

Example 1.16. [Z], [O1, Prop. 2], [D]. Let G be a finite group, $H = \Bbbk G$. Then all H-simple semisimple H-module algebras (up to Morita equivalence) are of the form $\Bbbk G \otimes_H \operatorname{End}(V)$, where $H \subseteq G$ is a subgroup and V is a projective representation of H. The left H-action on $\operatorname{End}(V)$ is: $(h.T)(v) := hT(h^{-1}v)$ for all $T \in \operatorname{End}(V), h \in H, v \in V$.

1.5. **Module categories over finite-dimensional Hopf algebras.** Let H be a finite-dimensional Hopf algebra. Let K be a left H-comodule algebra. Then ${}_K\mathcal{M}$ is a left module category over $\operatorname{Rep} H$ via the coaction $\lambda : K \to H \otimes K$. That is, $\otimes : \operatorname{Rep} H \times {}_K\mathcal{M} \to {}_K\mathcal{M}$ is given by

$$X \otimes V := X \otimes_{\Bbbk} V,$$

for $X \in \operatorname{Rep} H$ and $V \in {}_K\mathcal{M}$ with action $k \cdot (x \otimes v) = k_{(-1)} \cdot x \otimes k_{(0)} \cdot v$, for all $k \in K$, $x \in X$, $v \in V$.

We say that K is *exact* if ${}_K\mathcal{M}$ is exact, see Definition 1.9. If K is semisimple then it is exact but the converse is not true, see for example [EO, Th. 4.10]. However, if H is semisimple then K exact implies ${}_K\mathcal{M}$ semisimple, as said; hence K semisimple.

Remark 1.17. Assume that ${}_K\mathcal{M}$ is an exact module category. Then ${}_K\mathcal{M}$ is a Frobenius category, that is any projective object of ${}_K\mathcal{M}$ is injective and vice versa, see Remark 1.10. In particular K is an injective K-module, that is K is quasi-Frobenius.

We next turn to decomposability of ${}_K\mathcal{M}$. Recall the definition of H-indecomposable, Definition 1.2; it is clear what "H-decomposable" then means.

Proposition 1.18. *The following are equivalent:*

(1) *The module category ${}_K\mathcal{M}$ is decomposable.*
(2) *The comodule algebra K is H-decomposable.*

Proof. (2) $\implies$ (1). Since I and J are H-ideals, the quotient algebras K/I and K/J are left H-comodule algebras. The module categories ${}_{K/I}\mathcal{M}$ and ${}_{K/J}\mathcal{M}$ are non-trivial submodule categories of ${}_K\mathcal{M}$. Given $M \in {}_K\mathcal{M}$, let $M_1 = \{m \in M : I.m = 0\}$, $M_2 = \{m \in M : J.m = 0\}$; clearly, $M_1 \in {}_{K/I}\mathcal{M}$ and $M_2 \in {}_{K/J}\mathcal{M}$. Decompose $1 = i + j$, $i \in I$, $j \in J$. Let $m \in M$; then $m = im + jm$. Since $IJ \subset I \cap J = 0$, $jm \in M_1$, and similarly $im \in M_2$, thus $M = M_1 + M_2$. Also, if $m \in M_1 \cap M_2$, $m = 0$. This shows that $M = M_1 \oplus M_2$, hence ${}_K\mathcal{M} \simeq {}_{K/I}\mathcal{M} \times {}_{K/J}\mathcal{M}$.

(1) $\implies$ (2). Assume that ${}_K\mathcal{M} \simeq \mathcal{M}_1 \times \mathcal{M}_2$, where $\mathcal{M}_1$ are H-module subcategories of ${}_K\mathcal{M}$. If $M \in {}_K\mathcal{M}$ then there exist $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_2$ such that $M = M_1 \oplus M_2$. If $\varphi : M \to N$ is a morphism in ${}_K\mathcal{M}$ then $\varphi(M_1) \subseteq N_1$, $\varphi(M_2) \subseteq N_2$.

Considering $K \in {}_K\mathcal{M}$, there exist $J \in \mathcal{M}_1$, $I \in \mathcal{M}_2$ such that $K = J \oplus I$. Clearly I and J are left H-ideals of K. Let $j \in J$ and let $\eta_j : K \to K$ be the expansion of j, that is $\eta_j(x) = xj$, $x \in K$. Since η_j is a morphism of K-modules, $\eta_j(I) \subseteq I$. Thus, $IJ \subseteq I$ and I is a two-sided ideal. Similarly, J is a two-sided ideal. Now given an arbitrary $M \in {}_K\mathcal{M}$ with $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_2$ such that $M = M_1 \oplus M_2$, then $IM_1 \subset M_1$ and $JM_2 \subset M_2$, by the same argument applied to the expansion of m, $m \in M$. If $I = 0$ then $1 \in J$, hence $M = M_2$ and $\mathcal{M}_1$ is trivial. Note that these constructions are inverse to each other. □

Proposition 1.19. *If $\mathcal{M}$ is an indecomposable exact module category over the finite tensor category $\mathcal{C} = \operatorname{Rep} H$, then there exists an indecomposable exact H-comodule algebra K such that $\mathcal{M}$ is equivalent to ${}_K\mathcal{M}$ as module categories.*

Proof. By Theorem 1.14 there exists an H-module algebra R such that $\mathcal{M}$ is equivalent to $(\operatorname{Rep} H)_R =: {}_H\mathcal{M}_R$. Note that R^{op} is a H^{cop}-module algebra. Let $K = R^{\text{op}} \# H^{\text{cop}}$; this is a H-comodule algebra. Explicitly, the multiplication and coaction are respectively given by

$$(r\#h)(r'\#t) = (h_{(2)} \cdot r')r\#h_{(1)}t, \qquad \lambda(r\#h) = h_{(1)}\otimes r\#h_{(2)},$$

for $r, r' \in R$, $h, t \in H$. Then

$$\begin{aligned}\lambda\big((r\#h)(s\#t)\big) &= h_{(1)}t_{(1)}\otimes(h_{(3)} \cdot s)r\#h_{(2)}t_{(1)} \\ &= \big(h_{(1)}\otimes r\#h_{(2)}\big)\big(t_{(1)}\otimes s\#t_{(2)}\big) = \lambda(r\#h)\lambda(s\#t).\end{aligned}$$

Let $\mathcal{F} : {}_H\mathcal{M}_R \to {}_K\mathcal{M}$ be the functor given by $\mathcal{F}(M) = M$, with action $(r\#h) \cdot m = (h \cdot m) \cdot r$, $h \in H$, $m \in M$, $r \in R$. This is well-defined because the action $M\otimes R \to M$ is a morphism of H-modules. Clearly, $\mathcal{F}$ is an equivalence of abelian categories. We claim that $(\mathcal{F}, c)$ is an equivalence of module categories where $c_{X,M} : \mathcal{F}(X \otimes M) \to X \otimes \mathcal{F}(M)$ is the identity. Indeed, the only point that requires some checking is that $c_{X,M}$ is a morphism of K-modules. So, let X be an H-module, $M \in {}_H\mathcal{M}_R$, $x \in X$, $m \in M$, $h \in H$, $r \in R$. Then

$$\begin{aligned}c_{X,M}\big((r\#h) \cdot (x\otimes m)\big) &= \big(h \cdot (x \otimes m)\big) \cdot r = h_{(1)} \cdot x \otimes (h_{(2)} \cdot m) \cdot r; \\ (r\#h) \cdot c_{X,M}(x \otimes m) &= h_{(1)} \cdot x \otimes (h_{(2)}\#r) \cdot m = h_{(1)} \cdot x \otimes (h_{(2)} \cdot m) \cdot r.\end{aligned}$$

□

Proposition 1.20. *(i). If K is an H-simple comodule algebra, then K is exact.*
(ii). If $K \subseteq H$ is a left coideal subalgebra then K is exact.

Proof. (i). Let X be a finite-dimensional projective H-module and $M \in {}_K\mathcal{M}$. We want to show that $X \otimes M$ is projective; it is enough to assume that $X = H$. But $H \otimes M \in {}^H_K\mathcal{M}$, hence it is projective as a K- module by Thm. 1.4. (ii) follows from Prop. 1.6 and (i). □

1.6. **Equivalence of module categories.** Let H be a finite-dimensional Hopf algebra. Let R and S be left H-comodule algebras. We now study module functors between the module categories ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$. For this, we first recall the following well-known theorem.

Theorem 1.21. *Let A and B be finite dimensional algebras. If $F : {}_A\mathcal{M} \to {}_B\mathcal{M}$ is a right exact additive functor, then there exists a bimodule $C \in {}_B\mathcal{M}_A$ and a natural isomorphism $F \simeq C\otimes_A$.* □

Proof. The proof goes entirely similar to the proof of [Wa, Thm 1]; or else it can be deduced from [Wa, Thm 2]. □

To adapt this result to module categories, we introduce the following notion. First, if P is a (S, R)-bimodule then $H\otimes P$ is a (S, R)-bimodule by

$$s \cdot (h\otimes p) \cdot r = s_{(-1)}hr_{(-1)}\otimes s_{(0)}.p.r_{(0)},$$

$r \in R$, $h \in H$, $p \in P$, $s \in S$.

Definition 1.22. An *equivariant* (S,R)*-bimodule* is a (S,R)-bimodule P provided with a left coaction $\lambda : P \to H \otimes_{\Bbbk} P$ that is a morphism of (S,R)-bimodules. Morphisms of equivariant bimodules are defined in the obvious way. The category of equivariant bimodules is denoted ${}^H_S\mathcal{M}_K$.

We next prove that the category of module functors $(\mathcal{F}, c) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is equivalent to the category of equivariant (S,R)-bimodules.

Proposition 1.23. *There is an equivalence of categories* ${}^H_S\mathcal{M}_K \simeq \operatorname{Hom}_{\operatorname{Rep} H}({}_R\mathcal{M}, {}_S\mathcal{M})$.

Proof. Let P be an equivariant (S,R)-bimodule. Let $\mathcal{F}_P = \mathcal{F} : {}_R\mathcal{M} \to {}_S\mathcal{M}$ be the functor defined by $\mathcal{F}(V) = P \otimes_R V$. Given $X \in \operatorname{Rep} H$, $V \in {}_R\mathcal{M}$ we set $c_{X,V} : P \otimes_R (X \otimes_{\Bbbk} V) \to X \otimes_{\Bbbk} (P \otimes_R V)$ by

$$c_{X,V}(p \otimes_R x \otimes v) = p_{(-1)} \cdot x \otimes p_{(0)} \otimes_R v, \qquad p \in P, x \in X, v \in V. \tag{1.15}$$

This is well defined: if $p \in P$, $x \in X$, $v \in V$ and $r \in R$ then

$$c_{X,V}(p \cdot r \otimes_R x \otimes v) = (p \cdot r)_{(-1)} \cdot x \otimes (p \cdot r)_{(0)} \otimes_R v = (p_{(-1)} r_{(-1)}) \cdot x \otimes p_{(0)} \cdot r_{(0)} \otimes_R v,$$
$$c_{X,V}(p \otimes_R r \cdot (x \otimes v)) = c_{X,V}(p \otimes_R r_{(-1)} \cdot x \otimes r_{(0)} \cdot v) = p_{(-1)} \cdot (r_{(-1)} \cdot x) \otimes p_{(0)} \otimes_R r_{(0)} \cdot v.$$

Similarly, $c_{X,V}$ is a morphism in ${}_S\mathcal{M}$. It is an isomorphism, with inverse given by $c^{-1}_{X,V}(x \otimes p \otimes_R v) = p_{(0)} \otimes_R \mathcal{S}^{-1}(p_{(-1)}) \cdot x \otimes v$, for all $p \in P$, $x \in X$, $v \in V$; and it is clearly natural. The identities (1.11) and (1.12) are immediate. Hence $(\mathcal{F}, c)$ is a module functor. Furthermore, if $f : P \to Q$ is a morphism of equivariant (S,R)-bimodules then define $\mathcal{F}_f : \mathcal{F}_P \to \mathcal{F}_Q$ by $\mathcal{F}_{f,V} = f \otimes \operatorname{id}_V : P \otimes_R V \to Q \otimes_R V$. Equation (1.14) holds since f is morphism of H-comodules. Thus we have an additive functor ${}^H_S\mathcal{M}_K \to \operatorname{Hom}_{\operatorname{Rep} H}({}_R\mathcal{M}, {}_S\mathcal{M})$.

Conversely, let $(\mathcal{F}, c) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ be a module functor. The functor $\mathcal{F}$ is exact, [EO, Lemma 3.21], hence by Theorem 1.21 there exists $P \in {}_S\mathcal{M}_R$ such that $\mathcal{F}(V) = P \otimes_R V$ for all $V \in {}_R\mathcal{M}$. We define $\lambda : P \to H \otimes_{\Bbbk} P$ by

$$\lambda(p) = c_{H,R}(p \otimes 1 \otimes 1) := p_{(-1)} \otimes p_{(0)}, \qquad p \in P.$$

We first show that λ determines c. Given $X \in \operatorname{Rep} H$, $V \in {}_R\mathcal{M}$, we consider the expansions of $x \in X$, and $v \in V$, namely $\eta^X_x : H \to X$, $\eta^V_v : R \to V$ given by $\eta^X_x(h) = h \cdot x$, $\eta^V_v(r) = r \cdot v$ for all $h \in H$, $r \in R$. The naturality of c implies that the following diagram is commutative:

$$\begin{array}{ccc} P \otimes_R (H \otimes_{\Bbbk} R) & \xrightarrow{c_{H,R}} & H \otimes_{\Bbbk} (P \otimes_R R) \\ {\scriptstyle \operatorname{id} \otimes \eta^X_x \otimes \eta^V_v} \downarrow & & \downarrow {\scriptstyle \eta^X_x \otimes \operatorname{id} \otimes \eta^V_v} \\ P \otimes_R (X \otimes_{\Bbbk} V) & \xrightarrow[c_{X,V}]{} & X \otimes_{\Bbbk} (P \otimes_R V). \end{array}$$

Hence for all $p \in P$

$$c_{X,V}(p \otimes_R x \otimes v) = (\eta^X_x \otimes \operatorname{id}_P \otimes \eta^V_v)\, c_{H,R}(p \otimes_R 1 \otimes 1) = p_{(-1)} \cdot x \otimes p_{(0)} \otimes_R v. \tag{1.16}$$

We claim that P is an equivariant bimodule. For this, we first check that (P,λ) is a left H-comodule. By naturality of c in the first variable, the following diagram is commutative:

$$\begin{array}{ccc} P\otimes_R(H\otimes_{\Bbbk}R) & \xrightarrow{c_{H,R}} & H\otimes_{\Bbbk}(P\otimes_R R) \\ {\scriptstyle \mathrm{id}_P\otimes\varepsilon\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varepsilon\otimes\mathrm{id}\otimes\mathrm{id}} \\ P\otimes_R(\Bbbk\otimes_{\Bbbk}R) & \xrightarrow[c_{\Bbbk,R}]{} & \Bbbk\otimes_{\Bbbk}(P\otimes_R R). \end{array}$$

The axiom (1.12) says that $c_{\Bbbk,R}=\mathrm{id}_P$. This shows that λ is counitary.

Again by naturality of c in the first variable, the following diagram is commutative:

$$\begin{array}{ccc} P\otimes_R(H\otimes_{\Bbbk}R) & \xrightarrow{c_{H,R}} & H\otimes_{\Bbbk}(P\otimes_R R) \\ {\scriptstyle \mathrm{id}_P\otimes\Delta\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Delta\otimes\mathrm{id}\otimes\mathrm{id}} \\ P\otimes_R(H\otimes_{\Bbbk}H\otimes_{\Bbbk}R) & \xrightarrow[c_{H\otimes H,R}]{} & (H\otimes_{\Bbbk}H)\otimes_{\Bbbk}(P\otimes_R R). \end{array} \tag{1.17}$$

If $p\in P$ then

$$\begin{aligned}(\Delta\otimes\mathrm{id}_P)\lambda(p) &= (\Delta\otimes\mathrm{id}_P)\,c_{H,R}(p\otimes 1\otimes 1)\\ &= c_{H\otimes H,R}(\mathrm{id}_P\otimes\Delta\otimes\mathrm{id}_R)(p\otimes 1\otimes 1)\\ &= (\mathrm{id}_H\otimes c_{H,R})c_{H,H\otimes R}(p\otimes 1\otimes 1\otimes 1)\\ &= (\mathrm{id}_H\otimes c_{H,R})\lambda(p)\otimes 1\otimes 1\\ &= (\mathrm{id}_H\otimes\lambda)\lambda(p).\end{aligned}$$

The first equality by definition of λ, the second by commutativity of diagram (1.17), the third by (1.11) and the fourth by (1.16). That is λ is coassociative.

We finally check that λ is a morphism of (S,R)-modules. If $p\in P$, $s\in S$, $r\in R$, then

$$\lambda(s\cdot p) = c_{H,R}(s\cdot(p\otimes 1\otimes 1)) = s\cdot c_{H,R}(p\otimes 1\otimes 1) = s_{(-1)}p_{(-1)}\otimes s_{(0)}\cdot p_{(-1)}$$
$$\lambda(p\cdot r) = c_{H,R}(p\cdot r\otimes_R(1\otimes 1)) = c_{H,R}(p\otimes_R r_{(-1)}\otimes r_{(0)}) = p_{(-1)}r_{(-1)}\otimes p_{(0)}\otimes_R r_{(0)}.$$

Here, in the first line we have used that $c_{H,R}$ is a morphism of S-modules; and the last equality of the second line comes from (1.16). □

As a consequence of this result we describe the equivalences of module categories between ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$. Recall that a *Morita context* for R and S, is a collection (P,Q,f,g) where $P\in {}_S\mathcal{M}_R$, $Q\in {}_R\mathcal{M}_S$, $f: P\otimes_R Q\xrightarrow{\simeq} S$ is an isomorphism of S-bimodules and $g: Q\otimes_S P\xrightarrow{\simeq} R$ is an isomorphism of R-bimodules such that $f(p\otimes q)p' = pg(q\otimes p')$, $g(q\otimes p)q' = qf(p\otimes q')$ for all $p,p'\in P$, $q,q'\in Q$.

We shall say that a Morita context (P,Q,f,g) is *equivariant* if P is an equivariant bimodule. We shall see that Q turns out to be equivariant too. In this case we shall say that R and S are *equivariantly Morita equivalent.*

Combining Morita theory with the Proposition 1.23, we get:

Proposition 1.24. *The equivalences of module categories between ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ are in bijective correspondence with equivariant Morita contexts for R and S.*

Proof. Let $F : {}_R\mathcal{M} \to {}_S\mathcal{M}$ be an equivalence of module categories. Then F is, in particular, an equivalence of abelian $\Bbbk$-linear categories and gives rise to a Morita context (P, Q, f, g) where $F(M) = P \otimes_R M$; furthermore P is an equivariant bimodule by Proposition 1.23. Conversely, let (P, Q, f, g) be an equivariant Morita context. Recall that $Q \simeq \operatorname{Hom}_R(P, R) \simeq \operatorname{Hom}_S(P, S)$ and $P \simeq \operatorname{Hom}_R(Q, R) \simeq \operatorname{Hom}_S(Q, S)$. Then $F : {}_R\mathcal{M} \to {}_S\mathcal{M}$, $F(M) = P \otimes_R M$, is an equivalence of $\Bbbk$-linear categories; its inverse is $G : {}_S\mathcal{M} \to {}_R\mathcal{M}$, $G(N) = Q \otimes_S N$ and the natural isomorphisms $\alpha : G \circ F \to \operatorname{id}_{{}_R\mathcal{M}}$, $\beta : F \circ G \to \operatorname{id}_{{}_S\mathcal{M}}$, are given by

$$\begin{aligned} &\alpha : Q \otimes_S P \otimes_R M \to M, && \alpha(q \otimes p \otimes m) = q(p)m, && M \in {}_R\mathcal{M},\ m \in M; \\ &\beta : P \otimes_R Q \otimes_S N \to N, && \beta(p \otimes q \otimes n) = q(p)n, && N \in {}_S\mathcal{M},\ n \in N. \end{aligned}$$

Here in the first line we have used the identification $Q \simeq \operatorname{Hom}_R(P, R)$ and in the second, $P \simeq \operatorname{Hom}_S(Q, S)$. We next consider the left H-coaction on $Q \simeq \operatorname{Hom}_R(P, R)$ corresponding to the right H^*-action given by

$$(q \leftharpoonup \gamma)(p) = q(p \leftharpoonup \mathcal{S}^{-1}(\gamma_{(2)})) \leftharpoonup \gamma_{(1)}, \qquad q \in Q, p \in P, \gamma \in H^*. \tag{1.18}$$

We claim that Q with this coaction is an equivariant (R, S)-bimodule, which amounts to

$$(rqs) \leftharpoonup \gamma = (r \leftharpoonup \gamma_{(1)})(q \leftharpoonup \gamma_{(2)})(s \leftharpoonup \gamma_{(3)}), \qquad q \in Q, r \in R, s \in S, \gamma \in H^*. \tag{1.19}$$

Evaluating both sides at $p \in P$, we have

$$\begin{aligned} \text{LHS of } (1.19)(p) &= (rqs)(p \leftharpoonup \mathcal{S}^{-1}(\gamma_{(2)})) \leftharpoonup \gamma_{(1)} = [rq(s(p \leftharpoonup \mathcal{S}^{-1}(\gamma_{(2)})))] \leftharpoonup \gamma_{(1)} \\ &= (r \leftharpoonup \gamma_{(1)})[q(s(p \leftharpoonup \mathcal{S}^{-1}(\gamma_{(3)}))) \leftharpoonup \gamma_{(2)}]. \\ \text{RHS of } (1.19)(p) &= (r \leftharpoonup \gamma_{(1)})(q \leftharpoonup \gamma_{(2)})((s \leftharpoonup \gamma_{(3)})p) \\ &= (r \leftharpoonup \gamma_{(1)})[q(((s \leftharpoonup \gamma_{(4)})p) \leftharpoonup \mathcal{S}^{-1}(\gamma_{(3)})) \leftharpoonup \gamma_{(2)}] \\ &= (r \leftharpoonup \gamma_{(1)})[q((s \leftharpoonup \gamma_{(5)}\mathcal{S}^{-1}(\gamma_{(4)}))(p \leftharpoonup \mathcal{S}^{-1}(\gamma_{(3)}))) \leftharpoonup \gamma_{(2)}]. \end{aligned}$$

Thus (1.19) holds. Finally, we show that α and β satisfy (1.14). For α, the commutativity of

$$\begin{CD} Q \otimes_S P \otimes_R (X \otimes M) @>{c_{X,M}}>> X \otimes Q \otimes_S P \otimes_R M \\ @V{\alpha_{X \otimes M}}VV @VV{\operatorname{id}_X \otimes \alpha_M}V \\ X \otimes M @>>{\operatorname{id}_{X,M}}> X \otimes M, \end{CD}$$

$$\begin{array}{ccccc} q \otimes p \otimes x \otimes m & \longmapsto & q \otimes p_{(-1)} \cdot x \otimes p_{(0)} \otimes m & \longmapsto & q_{(-1)} p_{(-1)} \cdot x \otimes q_{(0)} \otimes p_{(0)} \otimes m \\ \downarrow & & & & \downarrow \\ & & & & q(p)_{(-1)} \cdot x \otimes q(p)_{(0)} \cdot m \\ & & & & \| \, ? \\ q(p)(x \otimes m) & \longmapsto & & & q_{(-1)} p_{(-1)} \cdot x \otimes q_{(0)}(p_{(0)}) m. \end{array}$$

needs the identity $q(p)_{(-1)}\otimes q(p)_{(0)} = q_{(-1)}p_{(-1)}\otimes q_{(0)}(p_{(0)})$, that follows immediately from (1.18). For β the argument is similar once the agreement of the analog of (1.18) for the action on P and the original one is shown. $\square$

Together with Propositions 1.18 and 1.19, Proposition 1.24 implies the first approach to the classification of module categories over $\operatorname{Rep} H$.

Theorem 1.25. *Indecomposable exact module categories over a finite dimensional Hopf algebra H are classified by H-indecomposable left comodule algebras up to equivariant Morita equivalence.* $\square$

Lemma 1.26. *(i). Let P_R be a right R-module. Then $\operatorname{End}_R(P_R)$ is a left H-comodule algebra via $\lambda : \operatorname{End}_R(P_R) \to H\otimes_{\Bbbk} \operatorname{End}_R(P_R)$, $T \mapsto T_{(-1)}\otimes T_{(0)}$, determined by*

$$\langle \alpha, T_{(-1)}\rangle\, T_0(p) = \langle \alpha, T(p_{(0)})_{(-1)}\mathcal{S}^{-1}(p_{(-1)})\rangle\, T(p_{(0)})_{(0)}, \tag{1.20}$$

$T \in \operatorname{End}_R(P_R)$, $p \in P$, $\alpha \in H^*$. Furthermore, P is an equivariant (S,R)-bimodule where $S = \operatorname{End}_R(P_R)$.

(ii). Let (P,Q,f,g) be an equivariant Morita context. The application $\rho : S \to \operatorname{End}_R(P_R)$, $\rho(s)(p) = s\cdot p$ for all $s \in S$, $p \in P$, is an isomorphism of H-comodules.

Proof. (i). We check that λ is well-defined, i. e. that $T_{(-1)}\otimes T_{(0)} \in H\otimes_{\Bbbk} \operatorname{End}_R(P_R)$. If $\alpha \in H^*$, $r \in R$, $p \in P$, then

$$\begin{aligned}
\langle \alpha, T_{(-1)}\rangle\, T_0(r\cdot p) &= \langle \alpha, T((p\cdot r)_{(0)})_{(-1)}\mathcal{S}^{-1}((p\cdot r)_{(-1)})\rangle\, T((p\cdot r)_{(0)})_{(0)}\\
&= \langle \alpha, T(p_{(0)}\cdot r_{(0)})_{(-1)}\mathcal{S}^{-1}(p_{(-1)}r_{(-1)})\rangle\, T(p_{(0)}\cdot r_{(0)})_{(0)}\\
&= \langle \alpha, T(p_{(0)})_{(-1)}r_{(-2)}\mathcal{S}^{-1}(p_{(-1)}r_{(-1)})\rangle\, T(p_{(0)})_{(0)}\cdot r_{(0)}\\
&= \langle \alpha, T(p_{(0)})_{(-1)}\mathcal{S}^{-1}(p_{(-1)})\rangle\, T(p_{(0)})_{(0)}\cdot r\\
&= \langle \alpha, T_{(-1)}\rangle\, T_0(p)\cdot r.
\end{aligned}$$

Here the first equality holds by (1.20), the second because the coaction λ of P is a morphism of R-modules, the third because T is a morphism of right R-modules and the fourth because of properties of $\mathcal{S}^{-1}$. It is immediate that λ is coassociative and counitary. It follows directly from (1.20) that P is equivariant.

We now check that $\operatorname{End}_R(P_R)$ is a left H-comodule algebra. Let $\alpha \in H^*$, $T,U \in \operatorname{End}_R(P_R)$ and $p \in P$ then

$$\begin{aligned}
&\langle \alpha, T_{(-1)}U_{(-1)}\rangle\, T_{(0)}(U_{(0)}(p)) = \langle \alpha_{(1)}, T_{(-1)}\rangle\langle \alpha_{(2)}, U_{(-1)}\rangle\, T_{(0)}(U_{(0)}(p))\\
&= \langle \alpha_{(1)}, T_{(-1)}\rangle\langle \alpha_{(2)}, U(p_{(0)})_{(-1)}\mathcal{S}^{-1}(p_{(-1)})\rangle\, T_{(0)}(U(p_{(0)})_{(0)})\\
&= \langle \alpha, T(U(p_{(0)})_{(0)})_{(-1)}\mathcal{S}^{-1}(U(p_{(0)})_{(-2)})U(p_{(0)})_{(-1)}\mathcal{S}^{-1}(p_{(-1)})\rangle\, T(U(p_{(0)})_{(0)})_{(0)}\\
&= \langle \alpha, T(U(p_{(0)}))\mathcal{S}^{-1}(p_{(-1)})\rangle\, T(U(p_{(0)}))_{(0)}\\
&= \langle \alpha, (TU)_{(-1)}\rangle\, (TU)_{(0)}(p).
\end{aligned}$$

The fourth equality by properties of the antipode. Since this holds for arbitrary $\alpha \in H^*$ then $\lambda(TU) = \lambda(T)\lambda(U)$.

(ii). By Morita theory, the application ρ above is a linear isomorphism. Let $s \in S$, $p \in P$ and $\alpha \in H^*$. Then

$$\begin{aligned}\langle \alpha, \rho(s)(p)_{(-1)}\rangle\, \rho(s)(p)_{(0)} &= \langle \alpha, \rho(s)(p_{(0)})_{(-1)}\mathcal{S}^{-1}(p_{(-1)})\rangle\, \rho(s)(p_{(0)})_{(0)} \\ &= \langle \alpha, s_{(-1)}p_{(-1)}\mathcal{S}^{-1}(p_{(-2)})\rangle\, s_{(0)} \cdot p_{(0)} \\ &= \langle \alpha, s_{(-1)}\rangle\, s_{(0)} \cdot p.\end{aligned}$$

Here the first equality holds by (1.20) and the second because λ is a morphism of S-modules. Hence $(\mathrm{id} \otimes \rho)\lambda_S = \lambda\rho$. □

Remark 1.27. The space of coinvariants of $\mathrm{End}_R(P_R)$ is $\mathrm{End}_R(P_R)^{\mathrm{co}\,H} = \mathrm{End}_R^H(P_R)$. That is $T \in \mathrm{End}_R(P_R)$ satisfies $T_{(-1)}\otimes T_{(0)} = 1\otimes T$ if and only if T is an H-comodule map. In particular, if P is a simple object in ${}^H\mathcal{M}_R$, then $\mathrm{End}_R(P_R)$ has trivial coinvariants.

2. Yan-Zhu Stabilizers

In this section, H is a finite-dimensional Hopf algebra.

2.1. **Preliminaries.** Let K be a left H-comodule algebra, and let $W, U, Z \in {}_K\mathcal{M}$, with corresponding representations $\rho_W : K \to \mathrm{End}\, W$, etc. It is convenient to consider the linear map

$$\mathcal{L} = L\otimes \mathrm{id} : H^*\otimes \mathrm{Hom}(U, W) \to \mathrm{Hom}(H^*\otimes U, H^*\otimes W) \simeq \mathrm{End}(H^*)\otimes \mathrm{Hom}(U, W),$$

that is

$$\mathcal{L}(\alpha\otimes f)(\beta\otimes u) = \alpha\beta\otimes f(u),$$

for all $\alpha, \beta \in H^*$, $f \in \mathrm{Hom}(U, W), u \in U$. We consider the left actions of H on $H^*\otimes \mathrm{Hom}(U, W)$, $H^*\otimes U$ and $H^*\otimes W$ induced by the action $\rightharpoonup$ of H on H^* (and trivial on the second tensorand). In particular, $\mathrm{Hom}(H^*\otimes U, H^*\otimes W)$ becomes an H-module.

Lemma 2.1. *The map $\mathcal{L}$ has the following properties:*

(i) *is compatible with compositions,* i. e. *the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}(H^*\otimes U, H^*\otimes W) \times \mathrm{Hom}(H^*\otimes W, H^*\otimes Z) & \xrightarrow{\text{composition}} & \mathrm{Hom}(H^*\otimes U, H^*\otimes Z) \\ \uparrow{\scriptstyle \mathcal{L}\otimes\mathcal{L}} & & \uparrow{\scriptstyle \mathcal{L}} \\ H^*\otimes \mathrm{Hom}(U, W) \times H^*\otimes \mathrm{Hom}(W, Z) & \xrightarrow{\mu\otimes\text{composition}} & H^*\otimes \mathrm{Hom}(U, Z). \end{array} \tag{2.1}$$

(ii) $\mathcal{L} : H^*\otimes \mathrm{End}\, W \to \mathrm{End}(H^*\otimes W)$ *is a morphism of algebras.*

(iii) $\mathcal{L}$ *is an injective H-module homomorphism.*

Proof. (i) and (ii) are straightforward. Clearly $\mathcal{L}$ is injective; it preserves the action of H since L does. Indeed, for $h \in H$, $\alpha, \beta \in H^*$,

$$\begin{aligned}(h \rightharpoonup L_\alpha)(\beta) &= h_{(1)} \rightharpoonup (L_\alpha(\mathcal{S}(h_{(2)}) \rightharpoonup \beta)) = h_{(1)} \rightharpoonup (\alpha(\mathcal{S}(h_{(2)}) \rightharpoonup \beta)) \\ &= (h_{(1)} \rightharpoonup \alpha)(h_{(2)}\mathcal{S}(h_{(3)}) \rightharpoonup \beta) = L_{h\rightharpoonup\alpha}(\beta).\end{aligned}$$

□

The discussion above can be carried over for the right regular representation. Let us consider the map

$$\mathcal{R} = \mathrm{id} \otimes R : \mathrm{Hom}(U, W) \otimes H \to \mathrm{Hom}(U \otimes H, W \otimes H) \cong \mathrm{Hom}(U, W) \otimes \mathrm{End}(H)$$

defined by $\mathcal{R}(f \otimes h)(u \otimes t) = f(u) \otimes th$, where $h, t \in H$, $f \in \mathrm{Hom}(U, W)$, $u \in U$. We consider the right action of H^* on $\mathrm{Hom}(U, W) \otimes H$, $U \otimes H$, $W \otimes H$ induced by the right action $\leftharpoondown$ of H^* on H (and trivial on the first tensorand), cf. (1.5). Again $\mathrm{Hom}(U \otimes H, W \otimes H)$ is a right H^*-module.

Lemma 2.2. *The map $\mathcal{R}$ has the following properties:*

(i) *$\mathcal{R}$ is compatible with compositions,*
(ii) *$\mathcal{R} : \mathrm{Hom}(U, W) \otimes H^{\mathrm{op}} \to \mathrm{Hom}(U \otimes H, W \otimes H)$ is a morphism of algebras, and*
(iii) *$\mathcal{R}$ is an injective H^*-module map.*

Proof. (i) and (ii) are clear. The map $\mathcal{R}$ preserves the action of H^* since R does:

$$\begin{aligned}(R_h \leftharpoondown \alpha)(t) &= (R_h(t \leftharpoondown \mathcal{S}^{-1}(\alpha_{(2)})) \leftharpoondown \alpha_{(1)} = ((t \leftharpoondown \mathcal{S}^{-1}(\alpha_{(2)}))h) \leftharpoondown \alpha_{(1)} \\ &= (t \leftharpoondown \mathcal{S}^{-1}(\alpha_{(3)})\alpha_{(2)})(h \leftharpoondown \alpha_{(1)}) = t(h \leftharpoondown \alpha) = R_{h \leftharpoondown \alpha}(t),\end{aligned}$$

if $h, t \in H$, $\alpha \in H^*$. Here we have used (1.3). □

2.2. **Hopf modules.** Recall that a right Hopf module over H is a right H-comodule M with coaction $\rho : M \to M \otimes H$ provided with a right H-action such that ρ is a morphism of H-modules. Since H is finite-dimensional, a right Hopf module is the same as a vector space M provided with a right action of H and a left action of H^* such that

$$\alpha \cdot (m \cdot h) = (\alpha_{(1)} \cdot m) \cdot (\alpha_{(2)} \rightharpoonup h), \tag{2.2}$$

$h \in H$, $\alpha \in H^*$, $m \in M$. If M is a right Hopf module then the action of H induces an isomorphism $M \simeq M^{\mathrm{co}\,H} \otimes H$ by the Fundamental Theorem of Hopf modules [Mo, Th. 1.9.4, page 15]. Here $M^{\mathrm{co}\,H} = \{m \in M : \rho(m) = m \otimes 1\} = \{m \in M : \alpha.m = \langle \alpha, 1 \rangle m \quad \forall \alpha \in H^*\}$.

Lemma 2.3. $\mathrm{End}(H^*)$ *is a right Hopf module over H with actions*

$$(\alpha \cdot f)(\beta) = f(\beta \alpha_{(2)}) \mathcal{S}^{-1}(\alpha_{(1)}), \tag{2.3}$$

$$(f \cdot h)(\beta) = f(h \rightharpoonup \beta), \tag{2.4}$$

$h \in H$, $\alpha, \beta \in H^*$, $f \in \mathrm{End}(H^*)$. Furthermore the space of coinvariants $\mathrm{End}(H^*)^{\mathrm{co}\,H}$ *is the image of the left regular representation $L : H^* \to \mathrm{End}(H^*)$.*

Proof. Clearly, (2.3) and (2.4) are respectively a left and a right action; we check (2.2):

$$\begin{aligned}\big((\alpha_{(1)}\cdot f)\cdot(\alpha_{(2)}\rightharpoonup h)\big)(\beta) &= (\alpha_{(1)}\cdot f)((\alpha_{(2)}\rightharpoonup h)\rightharpoonup\beta)\\ &= f\left(\left(h_{(1)}\rightharpoonup\beta\right)\langle\alpha_{(3)},h_{(2)}\rangle\alpha_{(2)}\right)\mathcal{S}^{-1}(\alpha_{(1)})\\ &= f\left(\left(h_{(1)}\rightharpoonup\beta\right)\left(h_{(2)}\rightharpoonup\alpha_{(2)}\right)\right)\mathcal{S}^{-1}(\alpha_{(1)})\\ &= f(h\rightharpoonup(\beta\alpha_{(2)}))\mathcal{S}^{-1}(\alpha_{(1)})\\ &= (f\cdot h)(\beta\alpha_{(2)})\mathcal{S}^{-1}(\alpha_{(1)})\\ &= (\alpha\cdot(f\cdot h))(\beta).\end{aligned}$$

We prove the last statement: given $f\in\operatorname{End}(H^*)$, $f\in\operatorname{End}(H^*)^{\mathrm{co}\,H}$ iff $\alpha\cdot f=\alpha(1)f$ for all $\alpha\in H^*$ iff $f(\beta\alpha)=f(\beta)\alpha$ for all $\alpha,\beta\in H^*$ iff $f=L_\gamma$ for $\gamma=f(1)\in H^*$. □

If K is a finite-dimensional Hopf algebra, a right Hopf module over K^{cop} is the same as a vector space M provided with right actions of K and K^* such that

$$(m\cdot k)\cdot\gamma=(m\cdot\gamma_{(1)})\cdot(k\leftharpoonup\gamma_{(2)}), \tag{2.5}$$

$k\in K$, $\gamma\in K^*$, where all notations are in terms of K. We shall also need the following.

Lemma 2.4. $\operatorname{End}(H)$ *is a right Hopf module over* $H^{*\mathrm{cop}}$ *with actions*

$$(f\cdot h)(t)=\mathcal{S}(h_{(1)})f(h_{(2)}t), \tag{2.6}$$

$$(f\cdot\alpha)(t)=f(\alpha\rightharpoonup t), \tag{2.7}$$

$h,t\in H$, $\alpha\in H^*$, $f\in\operatorname{End}(H)$. *Moreover the space of coinvariants* $\operatorname{End}(H)^{\mathrm{co}\,\mathrm{H}^{*\mathrm{cop}}}$ *is the image of the right regular representation* $R:H\to\operatorname{End}(H)$. □

2.3. **The Heisenberg double.** Recall that the Heisenberg double $\mathcal{H}(H^*)$ of the Hopf algebra H^* is the vector space $H^*\otimes H$ with the multiplication $(\alpha\otimes h)(\alpha'\otimes h')=\alpha(h_{(1)}\rightharpoonup\alpha')\otimes h_{(2)}h'$. Here, and in the next proposition, $h,h',t,u\in H$, $\alpha,\alpha',\beta\in H^*$.

Proposition 2.5. *(i). There is an isomorphism of algebras* $\Psi_1:\mathcal{H}(H^*)\to\operatorname{End}H$ *given by* $\Psi_1(\alpha\otimes h)(t)=\alpha\rightharpoondown(ht)$.

(ii). There is an isomorphism of algebras $\Psi_2:\mathcal{H}(H^*)\to\operatorname{End}(H^*)$ *given by* $\Psi_2(\alpha\otimes h)(\beta)=\alpha(h\rightharpoonup\beta)$.

(iii). The isomorphism of algebras $\Psi:\operatorname{End}H\to\operatorname{End}(H^*)$, $\Psi=\Psi_2\Psi_1^{-1}$ *satisfies*

$$\Psi(\overline{L}_\alpha)=L_\alpha, \tag{2.8}$$

$$\Psi(L(H))=\underline{L}(H), \tag{2.9}$$

$$\Psi(\underline{L}(H^*))=R(H^*), \tag{2.10}$$

$$\Psi(R(H))=\overline{L}(H). \tag{2.11}$$

Proof. We check (i):

$$\begin{aligned}\Psi_1(\alpha\otimes h)\Psi_1(\alpha'\otimes h')(t) &= \Psi_1(\alpha\otimes h)(\alpha'\rightharpoondown(h't))=\alpha\rightharpoondown\big(h(\alpha'\rightharpoondown(h't))\big);\\ \Psi_1\big((\alpha\otimes h)(\alpha'\otimes h')\big)(t) &= \Psi_1(\alpha(h_{(1)}\rightharpoonup\alpha')\otimes h_{(2)}h')(t)=\big(\alpha(h_{(1)}\rightharpoonup\alpha')\big)\rightharpoondown(h_{(2)}h't)\\ &= \alpha\rightharpoondown\big((h_{(1)}\rightharpoonup\alpha')\rightharpoondown(h_{(2)}h't)\big).\end{aligned}$$

Then (i) follows from the identity $h(\alpha' \rightharpoondown u) = (h_{(1)} \rightharpoonup \alpha') \rightharpoondown (h_{(2)}u)$, that we prove next:

$$\begin{aligned}(h_{(1)} \rightharpoonup \alpha') \rightharpoondown (h_{(2)}u) &= \langle h_{(1)} \rightharpoonup \alpha', \mathcal{S}^{-1}(h_{(2)}u_{(1)})\rangle\, h_{(3)}u_{(2)}\\ &= \langle \alpha', \mathcal{S}^{-1}(u_{(1)})\mathcal{S}^{-1}(h_{(2)})h_{(1)}\rangle\, h_{(3)}u_{(2)}\\ &= h\langle \alpha', \mathcal{S}^{-1}(u_{(1)})\rangle u_{(2)} = h(\alpha' \rightharpoondown u).\end{aligned}$$

It is well-known that the Heisenberg double is a simple algebra, hence Ψ_1 is an isomorphism by a dimension argument. We check (ii):

$$\begin{aligned}\Psi_2(\alpha\otimes h)\Psi_2(\alpha'\otimes h')(\beta) &= \Psi_2(\alpha\otimes h)(\alpha'(h' \rightharpoonup \beta)) = \alpha(h \rightharpoonup (\alpha'(h' \rightharpoonup \beta)))\\ &= \alpha(h_{(1)} \rightharpoonup \alpha')(h_{(2)}h' \rightharpoonup \beta) = \Psi_2(\alpha(h_{(1)} \rightharpoonup \alpha')\otimes h_{(2)}h')(\beta)\\ &= \Psi_2\big((\alpha\otimes h)(\alpha'\otimes h')\big)(\beta).\end{aligned}$$

Since $\Psi_1(\alpha\otimes 1) = \overline{L}_\alpha$ and $\Psi_2(\alpha\otimes 1) = L_\alpha$, (2.8) holds. Since $\Psi_1(\varepsilon\otimes h) = L_h$ and $\Psi_2(\varepsilon\otimes h) = \underline{L}_h$, (2.9) holds. Equation (2.10) follows from (2.8) and $\overline{L}(H^*)' = \underline{L}(H^*)$. Similarly (2.11) follows from (2.9) and $\underline{L}(H)' = \overline{L}(H)$. Here A' means the centralizer of a subalgebra A of $\operatorname{End} V$, see page 21 below. □

2.4. **Definition of Yan-Zhu Stabilizers.** Let K be a left H-comodule algebra. We recall the construction of the stabilizer from [YZ]. Let us consider H^* as an H-module via $\rightharpoondown$, see (1.2); and correspondingly $H^*\otimes W$ as a K-module via λ. That is,

$$k \cdot (\beta\otimes w) = k_{(-1)} \rightharpoondown \beta \otimes k_{(0)} \cdot w, \tag{2.12}$$

$k \in K$, $\beta \in H^*$, $w \in W$. Recall the map $\mathcal{L}$ considered in subsection 2.1.

Definition 2.6. [YZ] The *Yan-Zhu stabilizer* of the K-modules W and U is

$$\operatorname{Stab}_K(U, W) := \operatorname{Hom}_K(H^*\otimes U, H^*\otimes W) \cap \mathcal{L}\big(H^*\otimes \operatorname{Hom}(U, W)\big) \subset \operatorname{Hom}(H^*\otimes U, H^*\otimes W).$$

In particular, the *Yan-Zhu stabilizer* of the K-module W is

$$\operatorname{Stab}_K(W) := \operatorname{Stab}_K(W, W) = \operatorname{End}_K(H^*\otimes W) \cap \mathcal{L}(H^*\otimes \operatorname{End} W).$$

The algebra $\operatorname{Stab}_K(W)$ can be identified with a subalgebra of $H^*\otimes \operatorname{End}(W)$, since $\mathcal{L}$ is injective. Similarly, $\operatorname{Stab}_K(U, W)$ can be identified with a subspace of $H^*\otimes \operatorname{Hom}(U, W)$. We shall do this without further notice.

Proposition 2.7. *$\operatorname{Stab}_K(W)$ is a right H^*-comodule algebra and W is a left $\operatorname{Stab}_K(W)$-module.*

Proof. By (2.1), the composition induces a map

$$\operatorname{Stab}_K(U, W) \times \operatorname{Stab}_K(W, Z) \longrightarrow \operatorname{Stab}_K(U, Z). \tag{2.13}$$

In particular, $\operatorname{Stab}_K(W)$ is a subalgebra of $\operatorname{End}(H^*\otimes W)$. Since $\operatorname{End}(H^*\otimes W)$ is an H-module algebra and $\mathcal{L}(H^*\otimes \operatorname{End} W)$ is an H-submodule, it remains only to show that $\operatorname{End}_K(H^*\otimes W)$ is also an H-submodule. But, more generally, $\operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)$ is an H-submodule of $\operatorname{Hom}(H^*\otimes U, H^*\otimes W)$ because $\rightharpoonup$ and $\rightharpoondown$ commute. Hence $\operatorname{Stab}_K(W)$ is a left H-module algebra, thus a right a H^*-comodule algebra. The vector space W is a left $\mathrm{Stab}_K(W)$-module, with representation given by the composition

$\mathrm{Stab}_K(W) \longrightarrow H^*\otimes \mathrm{End}(W) \xrightarrow{\varepsilon\otimes \mathrm{id}} \Bbbk\otimes \mathrm{End}(W) = \mathrm{End}(W)$. □

The following Lemma will be useful later.

Lemma 2.8. *Let $\sum_i L_{\alpha_i}\otimes f_i \in \mathcal{L}\,(H^*\otimes \mathrm{Hom}(U,W))$; we shall omit the summation symbol. Then $L_{\alpha_i}\otimes f_i \in \mathrm{Hom}_K(H^*\otimes U, H^*\otimes W)$ if and only if*

$$\sum_i \langle \alpha_i, t\rangle\, f_i(k\cdot w) = \sum_i \langle \alpha_i, \mathcal{S}^{-1}(k_{(-1)})t\rangle\; k_{(0)}\cdot f_i(w), \tag{2.14}$$

for all $t\in H$, $w\in W$, $k\in K$.

In other words, (2.14) characterizes when $\sum_i L_{\alpha_i}\otimes f_i \in \mathrm{Stab}_K(U,W)$.

Proof. If $L_{\alpha_i}\otimes f_i \in \mathrm{Hom}_K(H^*\otimes U, H^*\otimes W)$ then

$$(L_{\alpha_i}\otimes f_i)(k\cdot(\beta\otimes w)) = k\cdot (L_{\alpha_i}\otimes f_i)(\beta\otimes w) \tag{2.15}$$

for all $\beta\in H^*, k\in K, w\in W$. Equation (2.15) translates into

$$\alpha_i(k_{(-1)} \rightharpoondown \beta)\otimes f_i(k_{(0)}\cdot w) = k_{(-1)} \rightharpoondown (\alpha_i\beta)\otimes k_{(0)}\cdot f_i(w). \tag{2.16}$$

Evaluating (2.16) in $t\in H$ and choosing $\beta=\varepsilon$ we get (2.14). Conversely, (2.16) follows from (2.14) using that $h\rightharpoondown(\alpha\beta) = (h_{(2)}\rightharpoondown\alpha)\,(h_{(1)}\rightharpoondown\beta)$, for $h\in H$, $\alpha,\beta\in H^*$. □

As a consequence of the above characterization of elements in the stabilizer, we have the following result.

Corollary 2.9. *There is an isomorphism $\mathrm{Stab}_K(U,W)^{\mathrm{co}\,H^*}\cong \mathrm{Hom}_K(U,W)$. In particular the stabilizer $\mathrm{Stab}_K(W)$ has trivial coinvariants if W is an irreducible K-module.*

Proof. The maps

$$\phi: \mathrm{Stab}_K(U,W)^{\mathrm{co}\,H^*} \to \mathrm{Hom}_K(U,W) \quad \text{and} \quad \psi: \mathrm{Hom}_K(U,W) \to \mathrm{Stab}_K(U,W)^{\mathrm{co}\,H^*}$$

given by $\phi(\alpha_i\otimes f_i) = \langle\alpha_i, 1\rangle f_i$, $\psi(f) = \epsilon\otimes f$, are the desired isomorphisms. □

We now state another characterization of the stabilizer $\mathrm{Stab}_K(U,W)$ given in [YZ] in terms of Hopf modules. We consider $\mathrm{End}(H^*)\otimes \mathrm{Hom}(U,W)$ as right Hopf module over H with structure concentrated in the first tensorand, *cf.* Lemma 2.3. We stress that these are not the same actions as before.

Proposition 2.10. *Keep the notation above.*

1. *$\mathrm{Hom}_K(H^*\otimes U, H^*\otimes W)$ is a Hopf submodule of $\mathrm{End}(H^*)\otimes \mathrm{Hom}(U,W)$,*
2. *$\mathrm{Hom}_K(H^*\otimes U, H^*\otimes W)^{\mathrm{co}\,H} = \mathrm{Stab}_K(U,W)$ and*

   $$\mathrm{Hom}_K(H^*\otimes U, H^*\otimes W) = \mathrm{Stab}_K(U,W)\circ(\underline{L}(H)\otimes \mathrm{id}\,) \simeq \mathrm{Stab}_K(U,W)\otimes H.$$

   Here $\circ$ means composition. In particular,

$$\mathrm{End}_K(H^*\otimes W) = \mathrm{Stab}_K(W)\circ(\underline{L}(H)\otimes \mathrm{id}\,). \tag{2.17}$$

Proof. (1). We have to check that $\operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)$ is stable under the actions induced by (2.3), (2.4). Let $k \in K$, $\alpha, \beta \in H^*$, $u \in U$, $h \in H$. Let also $\sum_i f_i\otimes T_i \in \operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)$, where $f_i \in \operatorname{End}(H^*)$, $T_i \in \operatorname{Hom}(W, U)$; for simplicity we omit the summation symbol in the following. Then

$$\begin{aligned}
\big(\alpha\cdot(f_i\otimes T_i)\big)\big(k\cdot(\beta\otimes w)\big) &= (\alpha\cdot f_i)(k_{(-1)} \rightharpoondown \beta)\otimes T_i(k_{(0)}\cdot u)\\
&= f_i((k_{(-1)} \rightharpoondown \beta)\alpha_{(2)})\mathcal{S}^{-1}(\alpha_{(1)})\otimes T_i(k_{(0)}\cdot u)\\
&= f_i((k_{(-1)} \leftharpoonup \alpha_{(2)}) \rightharpoondown \beta\alpha_{(3)})\mathcal{S}^{-1}(\alpha_{(1)})\otimes T_i(k_{(0)}\cdot u)\\
&= f_i(\langle k_{(-2)}, \alpha_{(2)}\rangle k_{(-1)} \rightharpoondown \beta\alpha_{(3)})\mathcal{S}^{-1}(\alpha_{(1)})\otimes T_i(k_{(0)}\cdot u)\\
&= \big((k_{(-1)} \leftharpoonup \alpha_{(2)}) \rightharpoondown f_i(\beta\alpha_{(3)})\big)\mathcal{S}^{-1}(\alpha_{(1)})\otimes k_{(0)}\cdot T_i(u)\\
&= \big((k_{(-1)} \leftharpoonup \alpha_{(3)}) \leftharpoonup \mathcal{S}^{-1}(\alpha_{(2)})\big) \rightharpoondown \big(f_i(\beta\alpha_{(4)})\mathcal{S}^{-1}(\alpha_{(1)})\big)\otimes k_{(0)}\cdot T_i(u)\\
&= k_{(-1)} \rightharpoondown \big(f_i(\beta\alpha_{(2)})\mathcal{S}^{-1}(\alpha_{(1)})\big)\otimes k_{(0)}\cdot T_i(u)\\
&= k\cdot\Big(\big(\alpha\cdot(f_i\otimes T_i)\big)(\beta\otimes w)\Big).
\end{aligned}$$

Here the first two equalities, the fourth and the last are by definitions; the third and the sixth by (1.6); the fifth, because $\sum_i f_i\otimes T_i \in \operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)$; the seventh, by elementary properties of the antipode. Next, $(f_i\otimes T_i)\cdot h$ preserves the K-action since $\rightharpoondown$ and $\rightharpoonup$ commute. Thus, (1) holds.

(2). We have

$$\begin{aligned}
\operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)^{\operatorname{co} H} &= \operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)\cap\big(\operatorname{End}(H^*)\otimes\operatorname{Hom}(U,W)\big)^{\operatorname{co} H}\\
&= \operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)\cap\operatorname{End}(H^*)^{\operatorname{co} H}\otimes\operatorname{Hom}(U,W)\\
&= \operatorname{Hom}_K(H^*\otimes U, H^*\otimes W)\cap L(H^*)\otimes\operatorname{Hom}(U,W)\\
&= \operatorname{Stab}_K(U,W).
\end{aligned}$$

The last statement follows from the Fundamental Theorem of Hopf modules. □

2.5. **Yan-Zhu duality.** Let now S be a *right* H^*-comodule algebra. Let V, X, Y be left S-modules. We adapt the construction of the Yan-Zhu stabilizer in the new setting. We consider H as a left H^*-module via $\rightharpoondown$ and $V\otimes H$ as S-module via the right coaction, that is

$$s\cdot(v\otimes h) = s_{(0)}\cdot v\otimes s_{(1)} \rightharpoondown h, \tag{2.18}$$

where $s\in S, v\in V$ and $\delta: S\to S\otimes H$, $\delta(s) = s_{(0)}\otimes s_{(1)}$. Recall the map $\mathcal{R}$ considered in subsection 2.1. Then the *Yan-Zhu stabilizer* of the S-modules V and Y is

$$\operatorname{Stab}_S(V,Y) := \operatorname{Hom}_S(V\otimes H, Y\otimes H)\cap\mathcal{R}(\operatorname{Hom}(V,Y)\otimes H).$$

In particular, the *Yan-Zhu stabilizer* of the S-module V is $\operatorname{Stab}_S(V) := \operatorname{End}_S(V\otimes H)\cap\mathcal{R}(\operatorname{End}(V)\otimes H)$. We consider $\operatorname{Hom}(V,Y)\otimes\operatorname{End}(H)$ as right Hopf module over $H^{*\mathrm{cop}}$ with structure concentrated in the second tensorand, *cf.* Lemma 2.4. Adapting the proofs of Propositions 2.10 and 2.7, we have:

Proposition 2.11.

(1) $\operatorname{Hom}_S(V\otimes H, Y\otimes H)$ *is a Hopf submodule of* $\operatorname{Hom}(V,Y)\otimes\operatorname{End}(H)$,
(2) $\operatorname{Hom}_S(V\otimes H, Y\otimes H)^{\operatorname{co} H^{*\operatorname{cop}}} = \operatorname{Stab}_S(V,Y)$,
(3) $\operatorname{Hom}_S(V\otimes H, Y\otimes H) = \operatorname{Stab}_S(V,Y)\circ(\operatorname{id}\otimes\underline{L}(H^*)) \simeq \operatorname{Stab}_S(V,Y)\otimes H^*$.

Proof. (1). We have to check that $\operatorname{Hom}_S(V\otimes H, Y\otimes H)$ is stable under the actions induced by (2.6), (2.7). Let $s\in S$, $\alpha\in H^*$, $v\in V$, $h,t\in H$; and let $\sum_j T_j\otimes f_j \in \operatorname{Hom}_S(V\otimes H, Y\otimes H)$, where $f_j\in\operatorname{End}(H)$, $T_j\in\operatorname{Hom}(V,Y)$; for simplicity we omit the summation symbol in the following. First, $(T_j\otimes f_j)\cdot\alpha$ preserves the S-action since $\rightharpoondown$ and $\rightharpoonup$ commute. Next,

$$\begin{aligned}
s\cdot((T_j\otimes f_j)\cdot h)(v\otimes t) &= s_{(0)}\cdot T_j(v)\otimes s_{(1)}\rightharpoondown (f_j\cdot h)(t)\\
&= s_{(0)}\cdot T_j(v)\otimes s_{(1)}\rightharpoondown \big(\mathcal{S}(h_{(1)})f_j(h_{(2)}t)\big)\\
&= s_{(0)}\cdot T_j(v)\otimes \big(s_{(2)}\rightharpoondown \mathcal{S}(h_{(1)})\big)\big(s_{(1)}\rightharpoondown f_j(h_{(2)}t)\big)\\
&= T_j(s_{(0)}\cdot v)\otimes \big(s_{(2)}\rightharpoondown \mathcal{S}(h_{(1)})\big)f_j(s_{(1)}\rightharpoondown (h_{(2)}t))\\
&= T_j(s_{(0)}\cdot v)\otimes \big(s_{(3)}\rightharpoondown \mathcal{S}(h_{(1)})\big)f_j((s_{(2)}\rightharpoondown h_{(2)})(s_{(1)}\rightharpoondown t))\\
&= T_j(s_{(0)}\cdot v)\otimes \mathcal{S}(h_{(1)})f_j(h_{(2)}(s_{(1)}\rightharpoondown t))\\
&= T_j(s_{(0)}\cdot v)\otimes (f_j\cdot h)(s_{(1)}\rightharpoondown t)\\
&= \big((T_j\otimes f_j)\cdot h\big)\big(s\cdot(v\otimes t)\big).
\end{aligned}$$

Here the only equality that needs explanation is the sixth, which is based on the following:

$$\begin{aligned}
\alpha_{(2)}\rightharpoondown \mathcal{S}(h_{(1)})\otimes\alpha_{(1)}\rightharpoondown h_{(2)} &= \langle\alpha_{(2)},\mathcal{S}^{-1}(\mathcal{S}(h_{(2)}))\rangle\mathcal{S}(h_{(1)})\otimes\langle\alpha_{(1)},\mathcal{S}^{-1}(h_{(3)})\rangle\mathcal{S}(h_{(4)})\\
&= \langle\alpha,1\rangle\mathcal{S}(h_{(1)})\otimes h_{(2)}.
\end{aligned}$$

Thus, (1) holds. The proof of (2) is similar the the proof of proposition 2.10 part (2), and (3) follows again from (2) and the Fundamental Theorem of Hopf modules. □

To state the next result (Yan-Zhu duality), we use the following notation: if A is a subspace of $\operatorname{End}(W)$ then $A' = \operatorname{Cent}_{\operatorname{End}(W)}(A)$ is the centralizer of A in $\operatorname{End}(W)$. Clearly:

$$\text{If } A = B\circ C \text{ and } 1\in B\cap C \text{ then } A' = B'\cap C'. \tag{2.19}$$

If $\phi:\operatorname{End}(W)\to\operatorname{End}(V)$ is an algebra isomorphism, then $\phi(A') = \phi(A)'$. If A is an algebra and $\rho: A\to\operatorname{End}(W)$ is a representation then $\rho(A)'$ is nothing but $\operatorname{End}_A(W)$.

Let us fix a right H^*-comodule algebra S and a left S-module W; therefore W is a left $\operatorname{Stab}_S(W)$-module by proposition 2.7.

Proposition 2.12. *There is an isomorphism of left H-module algebras*

$$\operatorname{Stab}_{\operatorname{Stab}_S(W)}(W) \simeq \big(\rho_{W\otimes H}(S)\big)'',$$

where $\rho_{W\otimes H}$ is the representation of S explained in (2.18).

Proof. Since $\operatorname{Stab}_S(W)$ is a left H-comodule algebra and W is a $\operatorname{Stab}_S(W)$-module– see proposition 2.7– there is a representation $\rho_{H^*\otimes W}:\operatorname{Stab}_S(W)\to\operatorname{End}(H^*\otimes W)$, given by

(2.12). Recall the isomorphism of algebras $\Psi : \operatorname{End} H \to \operatorname{End}(H^*)$ given in proposition 2.5. We claim that

$$(\Psi^{-1}\otimes \mathrm{id}\,)(\rho_{H^*\otimes W} \operatorname{Stab}_S(W)) = \operatorname{Stab}_S(W). \tag{2.20}$$

This follows from the definitions and (2.11). Let now $\Upsilon = (\mathrm{id} \otimes\Psi^{-1})\tau : \operatorname{End}(H^*\otimes W) \to \operatorname{End}(W\otimes H)$, where $\tau : \operatorname{End}(H^*\otimes W) \to \operatorname{End}(W\otimes H^*)$ is the usual transposition. Then

$$\begin{aligned}
\big(\rho_{W\otimes H}(S)\big)'' &= (\operatorname{End}_S(W\otimes H))' \\
&= \big(\operatorname{Stab}_S(W) \circ (\mathrm{id} \otimes \underline{L}(H^*))\big)' \\
&= \operatorname{Stab}_S(W)' \cap (\mathrm{id} \otimes \underline{L}(H^*))' \\
&= \Upsilon\Upsilon^{-1} \left(\operatorname{Stab}_S(W)'\right) \cap \Upsilon\Upsilon^{-1} \left((\mathrm{id} \otimes \underline{L}(H^*))''\right) \\
&= \Upsilon \left(\operatorname{Cent}_{\operatorname{End}(H^*\otimes W)} \Upsilon^{-1} \operatorname{Stab}_S(W)\right) \cap \Upsilon \left(\Upsilon^{-1}(\mathrm{id} \otimes \underline{L}(H^*))\right)' \\
&= \Upsilon \left(\operatorname{End}_{\Upsilon^{-1} \operatorname{Stab}_S(W)}(H^*\otimes W)\right) \cap \Upsilon \left(\mathrm{id} \otimes R(H^*)'\right) \\
&= \Upsilon \left(\operatorname{End}_{\operatorname{Stab}_S(W)}(H^*\otimes W) \cap \mathcal{L}(H^* \operatorname{End}(W))\right) \\
&= \Upsilon \left(\operatorname{Stab}_{\operatorname{Stab}_S(W)}(W)\right).
\end{aligned}$$

Here the second equality holds by Proposition 2.11 (3); the third by (2.19); the fourth, because Υ is an algebra isomorphism; the fifth is a restatement; the sixth is by (2.10); the seventh follows from (2.20) and the last is by definition. The left H-action on $\operatorname{End}(W\otimes H)$ is given by $h \cdot (T\otimes f) = T\otimes L_{h_{(1)}} f L_{\mathcal{S}(h_{(2)})}$, for all $h \in H$, $T \in \operatorname{End}(W)$, $f \in \operatorname{End} H$. The map Υ is an H-module map since for all $h \in H$, $\alpha \in H^*$

$$\Psi^{-1}(h \cdot L_\alpha) = \Psi^{-1}(L_{h\rightharpoonup\alpha}) = \overline{L}_{h\rightharpoonup\alpha} = L_{h_{(1)}}\overline{L}_\alpha L_{\mathcal{S}(h_{(2)})} = h \cdot \Psi^{-1}(L_\alpha).$$

The second equality by (2.8), the third follows from the identity

$$(h \rightharpoonup \alpha) \rightharpoondown t = h_{(1)}(\alpha \rightharpoondown (\mathcal{S}(h_{(2)}))t), \quad h, t \in H, \alpha \in H^*.$$

□

If A is a quasi-Frobenius algebra and M is a faithful finitely generated A-module then $(A; M)$ has the double centralizer property, see [CR, Th. 15.6]. In view of this, and as a consequence of proposition 2.12, we have:

Corollary 2.13. *Assume that*

1. *S is a quasi-Frobenius algebra, and*
2. *$\rho_{W\otimes H} : S \to \operatorname{End}(W\otimes H)$ is injective.*

Then $\operatorname{Stab}_{\operatorname{Stab}_S(W)}(W)$ *is isomorphic to S as H^*-comodule algebras.* □

Here is the main result of this Section, proved in [YZ] assuming that H and K are semisimple.

Theorem 2.14. *Let S be an H-simple left module algebra. Then* $\operatorname{Stab}_{\operatorname{Stab}_S(W)}(W)$ *is isomorphic to S as H-module algebras.* □

Proof. We need to analyze the hypotheses in corollary 2.13. The injectivity of $\rho_{W\otimes H}$ is disposed with Lemma 1.3. Now S is exact by Proposition 1.20, which follows from Skryabin's Theorem 1.4. Hence, S is quasi-Frobenius by Remark 1.10. □

2.6. **Dimension of Yan-Zhu stabilizers.** We prove in this subsection a formula on the dimension of the stabilizers, generalizing [Z, Cor. 2.8]. We begin by a technical Lemma. Let K be a finite-dimensional H-simple left H-comodule algebra. Recall the action of H on H^* given by $\rightharpoondown$.

Lemma 2.15. *If W is a left K-module then $H^*\otimes W$ is an object in ${}^H_K\mathcal{M}$.*

Proof. The left H-comodule structure $\delta : H^*\otimes W \to H\otimes H^*\otimes W$ is given as follows. If $\alpha \in H^*$, $w \in W$ then $\delta(\alpha\otimes w) = \alpha_{(-1)}\otimes \alpha_{(0)}\otimes w$ where $\langle \beta, \alpha_{(-1)}\rangle\, \alpha_{(0)} = \alpha\, \mathcal{S}^2(\beta)$, for all $\beta \in H^*$. This left H-coaction corresponds to the right H^*-action given by the multiplication composed with $\mathcal{S}^2$. Let us verify that the map δ is a K-module map. Let $k \in K$, $\alpha, \beta \in H^*$ and $w \in W$. Then

$$\delta(k\cdot(\alpha\otimes w)) = (k_{(-1)} \rightharpoondown \alpha)_{(-1)}\otimes (k_{(-1)} \rightharpoondown \alpha)_{(0)}\otimes k_{(0)}\cdot w \tag{2.21}$$

Evaluating β on the first tensorand of (2.21) we obtain

$$\langle \beta, (k_{(-1)} \rightharpoondown \alpha)_{(-1)}\rangle\, (k_{(-1)} \rightharpoondown \alpha)_{(0)}\otimes k_{(0)}\cdot w = (k_{(-1)} \rightharpoondown \alpha)\, \mathcal{S}^2(\beta)\otimes k_{(0)}\cdot w. \tag{2.22}$$

On the other hand

$$k\cdot \delta(\alpha\otimes w) = k_{(-2)}\alpha_{(-1)}\otimes k_{(-1)} \rightharpoondown \alpha_{(0)}\otimes k_{(0)}\cdot w. \tag{2.23}$$

Again, evaluating β on the first tensorand of (2.23) we obtain

$$\begin{aligned}
&\langle \beta, k_{(-2)}\alpha_{(-1)}\rangle\, k_{(-1)} \rightharpoondown \alpha_{(0)}\otimes k_{(0)}\cdot w\\
&= \langle \beta_{(1)}, k_{(-2)}\rangle\langle \beta_{(2)}, \alpha_{(-1)}\rangle k_{(-1)} \rightharpoondown \alpha_{(0)}\otimes k_{(0)}\cdot w\\
&= \langle \beta_{(1)}, k_{(-2)}\rangle \big(k_{(-1)} \rightharpoondown (\alpha\, \mathcal{S}^2(\beta_{(2)}))\big)\otimes k_{(0)}\cdot w\\
&= \big(k_{(-1)} \leftharpoonup \mathcal{S}^{-2}(\mathcal{S}^2(\beta)_{(1)})\big) \rightharpoondown (\alpha\, \mathcal{S}^2(\beta)_{(2)})\otimes k_{(0)}\cdot w\\
&= (k_{(-1)} \rightharpoondown \alpha)\mathcal{S}^2(\beta)\otimes k_{(0)}\cdot w.
\end{aligned}$$

The last equality follows from (1.6). Since β is arbitrary $\delta(k\cdot(\alpha\otimes w)) = k\cdot\delta(\alpha\otimes w)$. □

The next formula was obtained in [Z, 2.8] for H a semisimple Hopf algebra and $U = W$.

Proposition 2.16. *Let K be an H-simple left H-comodule algebra and U, W two left K-modules. Then*

$$\dim K \dim \operatorname{Stab}_K(U, W) = \dim U \dim W \dim H. \tag{2.24}$$

Proof. Let $M = H^*\otimes W$, $N = H^*\otimes U$. By Theorem 1.5 and Lemma 2.15, there exists $t, s \in \mathbb{N}$ such that M^t and N^s are K-free, say $M^t \simeq K^d$ and $N^s \simeq K^c$ as left K-modules for some natural numbers d, c. Hence

$$t \dim H \dim W = d \dim K, \tag{2.25}$$

$$s \dim H \dim U = c \dim K. \tag{2.26}$$

Proposition 2.10 (2) implies that

$$\dim \operatorname{Stab}_K(U,W) \dim H = \dim \operatorname{Hom}_K(N,M). \tag{2.27}$$

Since $M^t \simeq K^d$, $N^s \simeq K^c$, it follows that $\operatorname{Hom}_K(N^s,M^t) \simeq \operatorname{Hom}_K(K^c,K^d)$. Therefore $ts \dim \operatorname{Hom}_K(N,M) = dc \dim K$. This equation combined with (2.27) implies that $\dim \operatorname{Stab}_K(W) \dim H = \dfrac{dc \dim K}{ts}$. The result now follows from (2.25) and (2.26). □

As an immediate consequence of (2.24) we obtain the following variation of [S, Prop. 5.4]. See also [Z, Corollary 2.7].

Corollary 2.17. *Let K be an H-simple left H-comodule algebra. If W, U are left K-modules then* $\dim K$ *divides* $\dim W \dim U \dim H$. □

Skryabin shows in [S, Prop. 5.4] that $\dim K \dim \operatorname{End}_K W$ divides $\dim W \dim U \dim H$ under the assumption W irreducible, with a different proof.

2.7. **Examples of Yan-Zhu stabilizers: Hopf subalgebras and left coideal subalgebras.** Let us compute the Yan-Zhu stabilizers in some examples.

Example 2.18. Let G be a finite group and H be the group algebra of G. Let F be a subgroup of G and $\sigma \in Z^2(F, \Bbbk^\times)$ be a normalized 2-cocycle. The twisted group algebra $\Bbbk_\sigma F$ is a left H-comodule algebra via $\delta : \Bbbk_\sigma F \to H \otimes \Bbbk_\sigma F$, $\delta(f) = f \otimes f$, for all $f \in F$.

For any left $\Bbbk_\sigma F$-module V the space $\operatorname{End}(V)$ is a left $\Bbbk F$-module via $(f \cdot T)(v) = f \cdot T(f^{-1} \cdot v)$, $T \in \operatorname{End}(V)$, $f \in F$, $v \in V$. The space $\Bbbk G \otimes_{\Bbbk F} \operatorname{End}(V)$ is a left H-module algebra and there is an isomorphism of left module algebras

$$\operatorname{Stab}_{\Bbbk_\sigma F}(V) \cong \Bbbk G \otimes_{\Bbbk F} \operatorname{End}(V). \tag{2.28}$$

Proof. If $g \in G, T \in \operatorname{End}(V)$ we denote by $\overline{g \otimes T}$ the class of $g \otimes T$ in $\Bbbk G \otimes_{\Bbbk F} \operatorname{End}(V)$. Let $\{x_i\}_{i \in I}$ be a complete set of representatives of the right cosets. The H-module algebra structure in $\Bbbk G \otimes_{\Bbbk F} \operatorname{End}(V)$ is as follows; the action of G is on the first tensorand, and the product is given by

$$(\overline{x_i \otimes T})(\overline{x_j \otimes U}) = \delta_{i,j}(\overline{x_i \otimes T \circ U}),$$

for all $i,j \in I, T, U \in \operatorname{End}(V)$. We claim that the maps

$$\psi : \operatorname{Stab}_{\Bbbk_\sigma F}(V) \to \Bbbk G \otimes_{\Bbbk F} \operatorname{End}(V), \quad \theta : \Bbbk G \otimes_{\Bbbk F} \operatorname{End}(V) \to \operatorname{Stab}_{\Bbbk_\sigma F}(V)$$

defined by

$$\psi(\alpha_j \otimes T_j) = \sum_i \alpha_j(x_i^{-1})\, \overline{x_i \otimes T_j}, \quad \theta(\overline{g \otimes T}) = \sum_{f \in F} \delta_{fg^{-1}} \otimes f \cdot T,$$

are well defined algebra isomorphisms, one the inverse of each other. Indeed if $h \in F$ then

$$\theta(\overline{gh \otimes T}) = \sum_{f \in F} \delta_{fh^{-1}g^{-1}} \otimes f \cdot T = \sum_{f \in F} \delta_{fg^{-1}} \otimes fh \cdot T = \theta(\overline{g \otimes h \cdot T}).$$

That is, θ is well-defined. Let $T, U \in \text{End}(V)$ and $\alpha_j \otimes T_j, \beta_k \otimes U_k \in \text{Stab}_{\Bbbk_\sigma F}(V)$ then

$$\begin{aligned}\theta(x_i \otimes T)\theta(x_j \otimes U) &= \sum_{f,h\in F} (\delta_{fx_i^{-1}} \otimes f \cdot T)(\delta_{hx_j^{-1}} \otimes h \cdot U)\\ &= \delta_{i,j} \sum_{f\in F} \delta_{fx_i^{-1}} \otimes (f \cdot T)(f \cdot U) = \theta((x_i \otimes T)(x_j \otimes U)),\end{aligned}$$

$$\begin{aligned}\psi(\alpha_j \otimes T_j)\psi(\beta_k \otimes U_k) &= \sum_{i,l} \alpha_j(x_i^{-1})\beta_k(x_l^{-1})\, (\overline{x_i \otimes T_j})(\overline{x_l \otimes U_k})\\ &= \sum_{i} \alpha_j(x_i^{-1})\beta_k(x_i^{-1})\, (\overline{x_i \otimes T_j U_k})\\ &= \psi((\alpha_j \otimes T_j)(\beta_k \otimes U_k)),\end{aligned}$$

thus θ and ψ are algebra morphisms. Now let us compute $\theta \circ \psi$ and $\psi \circ \theta$:

$$\begin{aligned}\theta(\psi(\alpha_j \otimes T_j)) &= \sum_{i} \alpha_j(x_i^{-1})\, \theta(\overline{x_i \otimes T_j}) = \sum_{i,f\in F} \alpha_j(x_i^{-1})\, (\delta_{fx_i^{-1}} \otimes f \cdot T_j)\\ &= \sum_{i,f\in F} \alpha_j(fx_i^{-1})\, (\delta_{fx_i^{-1}} \otimes T_j) = \sum_{g\in G} \alpha_j(g)\, (\delta_g \otimes T_j) = \alpha_j \otimes T_j.\end{aligned}$$

The third equality by (2.14). On the other hand if $g = x_j h, h \in F$ then

$$\psi(\theta(\overline{g \otimes T})) = \sum_{f\in F} \psi(\delta_{fg^{-1}} \otimes f \cdot T) = \sum_{i,f\in F} \delta_{fg^{-1}}(x_i^{-1})\, \overline{x_i \otimes f \cdot T} \quad = \overline{x_j \otimes h \cdot T} = \overline{g \otimes T}.$$

□

Example 2.19. Let H be a finite dimensional Hopf algebra and $K \subseteq H$ be a left coideal subalgebra, and therefore K is a left H-comodule algebra via the comultiplication. Denote $\overline{K} = H/\mathcal{S}^{-1}(K^+)H$. The canonical projection $\pi : H \to \overline{K}$ is an H-module coalgebra map. The transpose of π is an injective algebra homomorphism $\overline{K}^* \hookrightarrow H^*$. Via π^* the space $\overline{K}^*$ can be identified with the subalgebra of H^* consisting of elements $\alpha \in H^*$ such that $\alpha(x) = 0$ for all $x \in \mathcal{S}^{-1}(K^+)H$; clearly this is a right coideal subalgebra of H^*. Let $V = \Bbbk$ be the trivial K-module. Then there is an isomorphism of right H^*-comodule algebras

$$\text{Stab}_K(\Bbbk) \cong \overline{K}^* \tag{2.29}$$

Proof. Since $\text{End}(V) \simeq \Bbbk$ we will identify $\text{Stab}_K(\Bbbk)$ with a subalgebra of H^*. If $\alpha \in \text{Stab}_K(\Bbbk)$ identity (2.14) implies that $\varepsilon(k) < \alpha, t >=< \alpha, \mathcal{S}^{-1}(k)t >$, for any $t \in H, k \in K$. Thus if $\varepsilon(k) = 0$ then $< \alpha, \mathcal{S}^{-1}(k)t >= 0$, and therefore $\alpha \in \overline{K}^*$. Reciprocally, if $\alpha \in \overline{K}^*$ then $\alpha(\mathcal{S}^{-1}(k)t - \varepsilon(k)t) = 0$, since $\mathcal{S}^{-1}(k)t - \varepsilon(k)t \in \mathcal{S}^{-1}(K^+)H$, and (2.14) is fulfilled. This implies that $\alpha \in \text{Stab}_K(\Bbbk)$. □

2.8. **Examples of Yan-Zhu stabilizers: Yan-Zhu Stabilizers for Hopf Galois extensions.** In this subsection we shall give another expression for the Yan-Zhu stabilizer in the case that K is a Hopf-Galois extension over a Hopf subalgebra H' of H. First we recall the notion of Hopf-Galois extensions.

Let H' be a finite dimensional Hopf algebra.

Definition 2.20. Let K be a left H'-comodule algebra. Set $R = K^{\mathrm{co} H'}$. The canonical map $\beta : K \otimes_R K \to H \otimes K$ is defined by $\beta(x \otimes y) = x_{(-1)} \otimes x_{(0)} y$, for all $x, y \in K$. K is called a *Hopf-Galois extension of* R over H' if β is bijective.

Following [Sch] if $K \supseteq R$ is a Hopf-Galois extension denote $\beta^{-1}(h \otimes 1) := h^{[1]} \otimes h^{[2]} \in K \otimes_R K$, for all $h \in H'$. The next result is due to H.-J. Schneider, see [Sch, Rmk 3.4].

Lemma 2.21. *Let $K \supseteq R$ be a Hopf-Galois extension, then for all $h, t \in H'$, $k \in K$, $r \in R$ we have that*

$$r h^{[1]} \otimes h^{[2]} = h^{[1]} \otimes h^{[2]} r, \tag{2.30}$$

$$(th)^{[1]} \otimes (th)^{[2]} = t^{[1]} h^{[1]} \otimes h^{[2]} t^{[2]}, \tag{2.31}$$

$$h^{[1]} h^{[2]} = \varepsilon(h) 1_K, \tag{2.32}$$

$$h^{[1]} \otimes 1 \otimes h^{[2]} = {h_{(1)}}^{[1]} \otimes {h_{(1)}}^{[2]} {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]}, \tag{2.33}$$

$$k \otimes 1 = {k_{(-1)}}^{[1]} \otimes {k_{(-1)}}^{[2]} k_{(0)}, \tag{2.34}$$

$$1 \otimes k = k_{(0)} \mathcal{S}^{-1}(k_{(-1)})^{[1]} \otimes \mathcal{S}^{-1}(k_{(-1)})^{[2]}, \tag{2.35}$$

$$h_{(2)} \otimes {h_{(1)}}^{[1]} \otimes {h_{(1)}}^{[2]} = \mathcal{S}^{-1}({h^{[2]}}_{(-1)}) \otimes h^{[1]} \otimes {h^{[2]}}_{(0)}. \tag{2.36}$$

Proof. Equations (2.30), (2.31), (2.34) and (2.35) follows by applying β. Since $(\varepsilon \otimes \mathrm{id}\,)\beta = m$, equation (2.32) follows. To get (2.33) apply $(\mathrm{id} \otimes \beta)(\beta \otimes \mathrm{id}\,)$ on both sides. Finally, equation (2.36) follows from colinearity of β. More precisely,

$$\delta\beta = (\mathrm{id}_H \otimes \beta)\widetilde{\delta},$$

where $\delta : H \otimes K \to H \otimes K \otimes H$, $\widetilde{\delta} : K \otimes_R K \to K \otimes_R K \otimes H$ are defined by

$$\delta(h \otimes y) = h_{(1)} \otimes y_{(0)} \otimes \mathcal{S}^{-1}(y_{(-1)}) h_{(2)}, \quad \widetilde{\delta}(x \otimes y) = x \otimes y_{(0)} \otimes \mathcal{S}^{-1}(y_{(-1)}),$$

for all $x, y \in K, h \in H$. □

The following result is [Sch, Corollary 3.5].

Lemma 2.22. *Let W be a left K-module.* $\mathrm{End}_R(W)$ *is a left H'-module with respect to the action defined by*

$$(h \cdot T)(w) := h^{[1]} T(h^{[2]} w),$$

for all $T \in \mathrm{End}_R(W)$, $h \in H'$, $w \in W$. We have also that $\mathrm{End}_R(W)^{H'} = \mathrm{End}_K(W)$.

Proof. For any $h \in H'$, $T \in \mathrm{End}_R(W)$, $h \cdot T$ is an R-module map by equation (2.30). Equation (2.31) implies that it is in fact an action. Equation (2.32) and (2.33) implies that $\mathrm{End}_K(W)$ is a module algebra over H'. Using equation (2.34) one can easily prove that the invariants of $\mathrm{End}_R(W)$ are those who preserve the action of K. □

Let H' be a Hopf subalgebra of H. Consider H as a left H'-module via the left regular representation. Let $(x_j)_j \subseteq H$, $(\beta_j)_j \subseteq H^*$ be dual basis.

Theorem 2.23. *Let $K \supseteq R$ be a Hopf-Galois extension of H'. Let W be a left K-module, then*

1. $\mathrm{Hom}_{H'}(H, \mathrm{End}_R(W))$ *is a right H^*-comodule algebra, and*
2. $\mathrm{Stab}_K(W) \cong \mathrm{Hom}_{H'}(H, \mathrm{End}_R(W))$ *as right H^*-comodule algebras.*

Proof. (1) The product is given by the convolution, that is if $T, U \in \mathrm{Hom}_{H'}(H, \mathrm{End}_R(W))$ then $(TU)(h) = T(h_{(1)})U(h_{(2)})$, for all $h \in H$. The identity is given by ε. The left H-module structure $(h \cdot T)(x) = T(xh)$, $h, x \in H$, $T \in \mathrm{Hom}_{H'}(H, \mathrm{End}_R(W))$ induces a right H^*-comodule structure and becomes into a right H^*-comodule algebra.

(2) Let $\phi : \mathrm{Stab}_K(W) \to \mathrm{Hom}_{H'}(H, \mathrm{End}_R(W))$, $\psi : \mathrm{Hom}_{H'}(H, \mathrm{End}_R(W)) \to \mathrm{Stab}_K(W)$ be given by

$$\phi(\alpha_i \otimes f_i)(h)(w) = \alpha_i(h) f_i(w), \qquad \psi(T) = \sum_j \beta_j \otimes T(x_j),$$

for all $\alpha_i \otimes f_i \in \mathrm{Stab}_K(W)$, $h \in H, w \in W$. Let us verify that these maps are well defined. Let $r \in R$, $t \in H'$, $h \in H$ and $w \in W$. Then

$$\phi(\alpha_i \otimes f_i)(h)(r \cdot w) = \alpha_i(h) f_i(r \cdot w) = \alpha_i(\mathcal{S}^{-1}(r_{(-1)}h)) r_{(0)} \cdot f_i(w) = \alpha_i(h)\, r \cdot f_i(w),$$

the second equation by (2.14) and the last one because $r \in R = K^{coH'}$. This proves that $\phi(\alpha_i \otimes f_i)(h)$ is an R-module map. We have also that

$$\begin{aligned} t \cdot \phi(\alpha_i \otimes f_i)(h)(w) &= t^{[1]} \phi(\alpha_i \otimes f_i)(h)(t^{[2]} \cdot w) = \alpha_i(h)\, t^{[1]} \cdot f_i(t^{[2]} \cdot w) \\ &= \alpha_i(\mathcal{S}^{-1}({t^{[2]}}_{(-1)})h)\, t^{[1]} {t^{[2]}}_{(0)} \cdot f_i(w) \\ &= \alpha_i(t_{(2)}h)\, {t_{(1)}}^{[1]} {t_{(1)}}^{[2]} \cdot f_i(w) = \alpha_i(th) f_i(w). \end{aligned}$$

The third equation by (2.14), the fourth by (2.36) and the fifth by (2.32). This proves that $\phi(\alpha_i \otimes f_i)$ is an H'-module map and therefore ϕ is well defined. The proof that $\psi(T) \in \mathrm{Stab}_K(W)$ is done using (2.35) and (2.14). That ϕ is an algebra map and a right H^*-comodule morphism is a straightforward computation. The identities $\psi\phi = \mathrm{id}$, $\phi\psi = \mathrm{id}$ are checked without difficulties. □

3. Applications of the Yan-Zhu stabilizers to module categories

3.1. **Internal Hom.** We keep the notation of the preceding section.

Proposition 3.1. $\underline{\mathrm{Hom}}(U, W) = \mathrm{Stab}_K(U, W)$, *and the bilinear map*

$$\underline{\mathrm{Hom}}(U, W) \times \underline{\mathrm{Hom}}(W, Z) \to \underline{\mathrm{Hom}}(U, Z)$$

coincides with (2.13).

Proof. Let us identify $H^*\otimes \operatorname{Hom}(U,W)$ with $\operatorname{Hom}(H\otimes U,W)$ in the natural way. Let $X \in \operatorname{Rep} H$. There are natural linear inverse isomorphisms

$$G : \operatorname{Hom}_H(X, H^*\otimes \operatorname{Hom}(U,W)) \to \operatorname{Hom}(X \otimes U, W),$$
$$F : \operatorname{Hom}(X \otimes U, W) \to \operatorname{Hom}_H(X, H^*\otimes \operatorname{Hom}(U,W)),$$

given by $G(\psi)(x\otimes u) = \psi(x)(1\otimes u)$, $F(\phi)(x)(a\otimes u) = \phi(a \cdot x\otimes u)$. Here and in the rest of the proof, $\psi \in \operatorname{Hom}_H(X, H^*\otimes \operatorname{Hom}(U,W))$, $\phi \in \operatorname{Hom}(X \otimes U, W)$, $x \in X$, $a \in H$, $u \in U$; and also $\alpha \in H^*$, $k \in K$. Given ψ and x, we write symbolically $\psi(x) = \psi(x)_{(1)}\otimes\psi(x)_{(2)}$, with $\psi(x)_{(1)} \in \operatorname{End}(H^*)$, $\psi(x)_{(2)} \in \operatorname{Hom}(U,W)$. Note that

$$\mathcal{L}(\psi(x))(\alpha \otimes u)(a) = \langle \alpha, a_{(2)}\rangle \psi(x)(a_{(1)}\otimes u).$$

We claim that $F(\operatorname{Hom}_K(X \otimes U, W)) = \operatorname{Hom}_H(X, \operatorname{Stab}_K(U,W))$, up to identification via $\mathcal{L}$. Indeed, $\phi \in \operatorname{Hom}_K(X \otimes U, W)$ iff $\psi = G(\phi)$ satisfies

$$k \cdot (\psi(x)(a\otimes u)) = \psi(x)(k_{(-1)}a\otimes k_{(0)} \cdot u). \tag{3.1}$$

Let us denote by ρ either $\rho_{H^*\otimes W} : K \to \operatorname{End} H^*\otimes \operatorname{End} W \simeq \operatorname{End}(H^*\otimes W)$ or $\rho_{H^*\otimes U}$. We compute on one hand

$$\begin{aligned}
[\psi(x)\rho(k)(\alpha\otimes u)](a) &= \left[\psi(x)(k_{(-1)} \rightharpoondown \alpha\otimes k_{(0)} \cdot u)\right](a) \\
&= \langle k_{(-1)} \rightharpoondown \alpha, a_{(2)}\rangle \psi(x)(a_{(1)}\otimes k_{(0)} \cdot u) \\
&= \langle \alpha, \mathcal{S}^{-1}(k_{(-1)})a_{(2)}\rangle \psi(x)(a_{(1)}\otimes k_{(0)} \cdot u), \\
&= \boxtimes_1.
\end{aligned}$$

and on the other

$$\begin{aligned}
[\rho(k)\psi(x)(\alpha\otimes u)](a) &= \left[k_{(-1)} \rightharpoondown \psi(x)_{(1)}(\alpha)\otimes k_{(0)} \cdot \psi(x)_{(2)}(u)\right](a) \\
&= \left[\psi(x)_{(1)}(\alpha)\otimes k_{(0)} \cdot \psi(x)_{(2)}(u)\right](\mathcal{S}^{-1}(k_{(-1)})a) \\
&= \langle \alpha, \mathcal{S}^{-1}(k_{(-2)})a_{(2)}\rangle\, k_{(0)} \cdot \left[\psi(x)(\mathcal{S}^{-1}(k_{(-1)})a_{(1)}\otimes u)\right] \\
&= \boxtimes_2.
\end{aligned}$$

If $\boxtimes_1 = \boxtimes_2$ then taking $\alpha = \varepsilon$ we get (3.1). Conversely, it is not difficult to see that (3.1) implies $\boxtimes_1 = \boxtimes_2$.

Now the claim says that $\operatorname{Hom}_K(X \otimes U, W) \simeq \operatorname{Hom}_H(X, \operatorname{Stab}_K(U,W))$; so that the functor $X \mapsto \operatorname{Hom}_K(X \otimes U, W)$ is representable by $\operatorname{Stab}_K(U,W)$. Furthermore, it is not difficult to see that the composition (2.13) satisfies the defining property in [O1, Section 3.3], and the proposition follows. □

3.2. **Exact module categories.** We state our first application of Proposition 3.1.

Corollary 3.2. *Assume that K is exact. Let W be a generator of ${}_K\mathcal{M}$. Then ${}_K\mathcal{M}$ is equivalent to ${}_H\mathcal{M}_{\operatorname{Stab}_K(W)}$ as module categories over* $\operatorname{Rep} H$.

Proof. This follows from Theorem 1.14 by Proposition 3.1. □

We now give a refinement of Proposition 1.19.

Theorem 3.3. *Any indecomposable exact module category over* $\operatorname{Rep} H$ *is equivalent to* ${}_K\mathcal{M}$ *for some* H*-simple from the right left* H*-comodule algebra* K*, with* $K^{\operatorname{co} H} \simeq \Bbbk$.

Proof. By Theorem 1.14, there exists a H-module algebra R such that

$$\mathcal{M} \simeq {}_H\mathcal{M}_R \tag{3.2}$$

as module categories over $\operatorname{Rep} H$. Because of [EO, Lemma 4.2], see Remark 1.15, we can assume that R has no non trivial H-stable ideals. Hence ${}_R\mathcal{M}$ is exact as module category over $\operatorname{Rep}(H^*)^{\mathrm{cop}}$ by Proposition 1.20. It follows that R is quasi-Frobenius by Remark 1.17.

Let W be a simple R-module and set $K = \operatorname{Stab}_R(W)$. We know that W generates ${}_H\mathcal{M}_R$, see the remarks previous to Theorem 1.14. Also, K is H-simple from the right because of Remark 1.15. Hence ${}_K\mathcal{M}$ is indecomposable by Proposition 1.18, and exact by Proposition 1.20. Again, $W \neq 0$ is a generator of ${}_K\mathcal{M}$. Observe next that

$${}_K\mathcal{M} \simeq {}_H\mathcal{M}_{\operatorname{Stab}_K(W)} \tag{3.3}$$

by Corollary 3.2. Now $R \simeq \operatorname{Stab}_{\operatorname{Stab}_R(W)}(W)$ by Theorem 2.14 (Yan-Zhu duality). The Proposition now follows from this, (3.2) and (3.3). □

3.3. **The dual module category.**

Another important tool in the study of tensor categories is the notion of *dual tensor category* with respect to a module category. In some sense this notion is the categorification of the notion of a centralizer of an algebra. The dual tensor category has been intensively used in [ENO]. See also [O1], [O2].

Let $\mathcal{C}$ be a finite tensor category.

Definition 3.4. Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$. The *dual tensor category* (with respect to $\mathcal{M}$) is the category $\mathcal{C}^*_{\mathcal{M}} := \operatorname{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ with the tensor product given by the composition of module functors (1.13).

If $\mathcal{N}$ is a module category over $\mathcal{C}$ then $\operatorname{Hom}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ is a module category over $\mathcal{C}^*_{\mathcal{M}}$ via the composition $\operatorname{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \times \operatorname{Hom}_{\mathcal{C}}(\mathcal{N}, \mathcal{M}) \to \operatorname{Hom}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$, see (1.13) again.

Proposition 3.5. *Let* $\mathcal{N}$ *be an exact module category over* $\mathcal{C}$*. Then*

(1) $\operatorname{Hom}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ *is an exact module category over* $\mathcal{C}^*_{\mathcal{M}}$.
(2) *The map* $\mathcal{N} \mapsto \operatorname{Hom}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ *is a bijective correspondence between equivalence classes of exact module categories over* $\mathcal{C}$ *and* $\mathcal{C}^*_{\mathcal{M}}$.

Proof. See [EO, Lemma 3.30] and [EO, Theorem 3.31]. □

Lemma 3.6. *Let* $A \in \mathcal{C}$ *be an algebra and assume that* $\mathcal{M} = \mathcal{C}_A$ *is an exact module category over* $\mathcal{C}$*. Then*

(1) *The tensor categories* $\mathcal{C}^*_{\mathcal{M}}$ *and* $({}_A\mathcal{C}_A)^{\mathrm{op}}$ *are equivalent.*
(2) *The bijective correspondence between equivalence classes of exact module categories over* $\mathcal{C}$ *and* $({}_A\mathcal{C}_A)^{\mathrm{op}}$ *arising from Proposition 3.5 (2) is explicitly given by* $\mathcal{C}_B \mapsto {}_B\mathcal{C}_A$*,* B *any algebra in* $\mathcal{C}$.

Proof. See the proof of [EO, Lemma 3.30]. □

Remark 3.7. If $A = \mathbf{1}$, then we conclude that the tensor categories $\mathcal{C}^*_{\mathcal{C}}$, $\mathcal{C}^{\rm op}$ are equivalent. Hence the correspondence of exact module categories over $\mathcal{C}$ and $\mathcal{C}^{\rm op}$ is just $\mathcal{C}_B \mapsto {}_B\mathcal{C}$, B an algebra in $\mathcal{C}$.

3.4. **Correspondence of module categories over** $\operatorname{Rep}(H^*)$ **and** $\operatorname{Rep} H$**.** Let H be a finite-dimensional Hopf algebra. In this subsection we study a bijective correspondence between equivalence classes of exact module categories over $\operatorname{Rep} H$ and $\operatorname{Rep}(H^*)$, and show that this agrees with Proposition 3.5 (2). Roughly the correspondence is as follows. If K is a H-simple left H-comodule algebra, then $K^{\rm op}$ is a left H^*-module algebra and therefore is a right H-comodule algebra. If $V \neq 0$ is a right K-module, then the stabilizer $\operatorname{Stab}_{K^{\rm op}}(V)$ is a left H^*-comodule algebra and the module category ${}_{\operatorname{Stab}_{K^{\rm op}}(V)}\mathcal{M}$ does not depend on V. Therefore we have a map

$$ {}_K\mathcal{M} \longmapsto {}_{\operatorname{Stab}_{K^{\rm op}}(V)}\mathcal{M} $$

assigning module categories over $\operatorname{Rep} H$ to module categories over $\operatorname{Rep}(H^*)$.

We begin by the following well-known Lemma. Recall that H^* is a left H-module algebra via $\rightharpoonup$.

Lemma 3.8. *There is a tensor equivalence* $\operatorname{Rep}(H^*) \simeq {}_{H^*}\operatorname{Rep} H_{H^*}$*.*

Proof. We only sketch the proof. The functors $\mathcal{F} : \operatorname{Rep}(H^*) \to {}_{H^*}\operatorname{Rep} H_{H^*}$, $\mathcal{G} : {}_{H^*}\operatorname{Rep} H_{H^*} \to \operatorname{Rep}(H^*)$ are defined by $\mathcal{G}(V) = V^H = \{v \in V : h \cdot v = \varepsilon(h)v\}$, $\mathcal{F}(X) = X \otimes H^*$ with the following structure. For all $x \in X, \alpha, \beta \in H^*, h \in H$

$$ h \cdot (x \otimes \alpha) = x \otimes h \rightharpoonup \alpha, \quad \beta \cdot (x \otimes \alpha) = \beta_{(1)} \cdot x \otimes \beta_{(2)}\alpha, \quad (x \otimes \alpha) \cdot \beta = x \otimes \alpha\beta. $$

□

The next Proposition shows that there is a correspondence between module categories over $\operatorname{Rep} H$ and $\operatorname{Rep}(H^*)$. This result was first established by Ostrik for weak Hopf algebras. See [O2, Theorem 5].

Proposition 3.9. *There is a bijective correspondence between equivalence classes of module categories over* $\operatorname{Rep} H$ *and* $\operatorname{Rep}(H^*)$*. Explicitly, if R is H-module algebra, then* $\operatorname{Rep} H_R \mapsto {}_{H^*}\operatorname{Rep} H_R$*, where* $\otimes : \operatorname{Rep}(H^*) \times {}_{H^*}\operatorname{Rep} H_R \to {}_{H^*}\operatorname{Rep} H_R$ *is given by* $X \otimes V = \mathcal{F}(X) \otimes_{H^*} V$*.*

Proof. We know that the module category $\operatorname{Vect} \simeq \operatorname{Rep} H_{H^*}$. By Lemma 3.6 applied to $A = H^*$, we have $(\operatorname{Rep} H)^*_{\operatorname{Vect}} \simeq ({}_{H^*}\operatorname{Rep} H_{H^*})^{\rm op}$. Hence the application we are looking into is just the composition

$$\begin{array}{ccc} \text{mod Rep}\, H & \xrightarrow{\hspace{4em}} & \text{mod}\ ({}_{H^*}\text{Rep}\, H_{H^*}) \\ & \searrow^{\text{Remark 3.7}} \quad \nearrow_{\text{Lemma 3.6 (2)}} & \\ & \text{mod}\ (\text{Rep}\, H)^{\text{op}} & \end{array}$$

$$\begin{array}{ccc} \text{Rep}\, H_R & \longmapsto & {}_{H^*}\text{Rep}\, H_R \\ & \searrow \quad \nearrow & \\ & {}_R\text{Rep}\, H & \end{array}$$

combined with Lemma 3.8. □

The main result of this Subsection is an explicit description of this correspondence.

Theorem 3.10. *The bijective correspondence between equivalence classes of exact module categories over* $\text{Rep}\, H$ *and* $\text{Rep}(H^*)$ *settled in Proposition 3.9 coincides with the map*

$${}_K\mathcal{M} \longmapsto {}_{\text{Stab}_{K^{\text{op}}}(V)}\mathcal{M},$$

K an H-simple left H-comodule algebra and $V \neq 0$ a right K-module.

In presence of Example 2.19, the Theorem "explains" the correspondence between coideal subalgebras of H and H^* described by Masuoka [M, 2.10 (iii)].

Proof. Let K be a H-simple left H-comodule algebra and let W be a left K-module. By Corollary 3.2 there is an equivalence of module categories $(F, c) : {}_K\mathcal{M} \to \text{Rep}\, H_{\text{Stab}_K(W)}$.

We first claim that the functor F induces an equivalence

$$\text{Rep}(H^*)_{K^{\text{op}}} \simeq {}_{H^*}\,\text{Rep}\, H_{\text{Stab}_K(W)} \tag{3.4}$$

of module categories over $\text{Rep}(H^*)$. We first observe that indeed K^{op} is a left H^*-module algebra with left action given by $\alpha \cdot x = \langle \alpha, \mathcal{S}^{-1}(x_{(-1)})\rangle\, x_{(0)}$, for every $\alpha \in H^*$, $x \in K$. Thus, we may consider the module category $\text{Rep}(H^*)_{K^{\text{op}}}$ over $\text{Rep}(H^*)$. Now, an object $M \in \text{Rep}(H^*)_{K^{\text{op}}}$ is a left K-module $\cdot : K \otimes M \to M$ provided with a left H^*-action $\triangleright : H^* \otimes M \to M$ such that

$$\alpha \triangleright (x \cdot m) = \langle \alpha_{(2)}, \mathcal{S}^{-1}(x_{(-1)})\rangle\, x_{(0)} \cdot (\alpha_{(1)} \triangleright m) \tag{3.5}$$

Recall that H^* is a left H-module via $\rightharpoonup$ and if M is a K-module then $H^* \otimes M \in {}_K\mathcal{M}$.

Step 1. *The left action $\triangleright : H^* \otimes M \to M$ is a K-module map.*

Proof. The map $\triangleright : H^* \otimes M \to M$ is a K-module map if and only if

$$k \cdot (\alpha \triangleright m) = (k_{(-1)} \rightharpoonup \alpha) \triangleright (k_{(0)} \cdot m), \tag{3.6}$$

for all $\alpha \in H^*, m \in M, k \in K$. The right hand side equals to

$$\begin{aligned}&= \langle \alpha_{(2)}, k_{(-1)}\rangle\ \alpha_{(1)} \triangleright (k_{(0)} \cdot m)\\ &= \langle \alpha_{(3)}, k_{(-2)}\rangle \langle \alpha_{(2)}, \mathcal{S}^{-1}(k_{(-1)})\rangle\ k_{(0)} \cdot (\alpha_{(1)} \triangleright m)\\ &= \langle \alpha_{(2)}, \mathcal{S}^{-1}(k_{(-1)}) k_{(-2)}\rangle\ k_{(0)} \cdot (\alpha_{(1)} \triangleright m)\\ &= k \cdot (\alpha \triangleright m).\end{aligned}$$

The second equality follows from (3.5). □

Step 1 says that the map $\triangleright : H^* \otimes M \to M$ is in the category ${}_K\mathcal{M}$. Thus, applying the functor F, we get the map $F(H^* \otimes M) \xrightarrow{F(\triangleright)} F(M)$; we can consider the composition

$$H^* \otimes F(M) \xrightarrow{c^{-1}_{H^*,M}} F(H^* \otimes M) \xrightarrow{F(\triangleright)} F(M). \tag{3.7}$$

Step 2. *Suppose $M \in \operatorname{Rep}(H^*)_{K^{\mathrm{op}}}$. Then*

1. *The composition* (3.7) *is a left H^*-action on $F(M)$.*
2. *$F(M)$ is an object in ${}_{H^*}\operatorname{Rep} H_{\operatorname{Stab}_K(W)}$.*

Proof. (2) follows from (1), since the composition (3.7) is a morphism in $\operatorname{Rep} H_{\operatorname{Stab}_K(W)}$. Let us prove (1). The associativity of the action given by (3.7) is equivalent to

$$F(\triangleright)\, c^{-1}_{H^*,M}\, (\mu \otimes \mathrm{id}_{F(M)}) = F(\triangleright)\, c^{-1}_{H^*,M} \left(\mathrm{id}_{H^*} \otimes F(\triangleright)\, c^{-1}_{H^*,M}\right),$$

where μ denotes the multiplication of H^*. Since $\mu : H^* \otimes H^* \to H^*$ is a morphism in $\operatorname{Rep} H$ the naturality of c implies that

$$(\mu \otimes \mathrm{id}_{F(M)})\, c_{H^* \otimes H^*, M} = c_{H^*,M}\, F(\mu \otimes \mathrm{id}_M). \tag{3.8}$$

Analogously, since $\triangleright : H^* \otimes M \to M$ is in ${}_K\mathcal{M}$ the naturality of c implies that

$$(\mathrm{id}_{H^*} \otimes F(\triangleright))\, c_{H^*, H^* \otimes M} = c_{H^*,M}\, F(\mathrm{id}_{H^*} \otimes \triangleright). \tag{3.9}$$

Hence

$$\begin{aligned}F(\triangleright)\, c^{-1}_{H^*,M}\, (\mu \otimes \mathrm{id}_{F(M)}) &= F(\triangleright) F(\mu \otimes \mathrm{id}_M)\, c^{-1}_{H^* \otimes H^*, M}\\ &= F(\triangleright\, (\mu \otimes \mathrm{id}_M))\, c^{-1}_{H^*, H^* \otimes M} (\mathrm{id}_{H^*} \otimes c^{-1}_{H^*,M})\\ &= F(\triangleright) F(\mathrm{id}_{H^*} \otimes \triangleright)\, c^{-1}_{H^*, H^* \otimes M} (\mathrm{id}_{H^*} \otimes c^{-1}_{H^*,M})\\ &= F(\triangleright)\, c^{-1}_{H^*,M} (\mathrm{id}_{H^*} \otimes F(\triangleright)) (\mathrm{id}_{H^*} \otimes c^{-1}_{H^*,M})\end{aligned}$$

The first equality by (3.8), the second by (1.11), the third because $\triangleright$ is an action and the last equality follows from (3.9). □

Step 3. *Let $X \in \operatorname{Rep}(H^*)$, $M \in \operatorname{Rep}(H^*)_{K^{\mathrm{op}}}$. The map $c_{X,M} : F(X \otimes M) \to X \otimes F(M)$ is a morphism in ${}_{H^*}\operatorname{Rep} H_{\operatorname{Stab}_K(W)}$. Here X is an H-module with trivial action.*

Proof. By definition the map $c_{X,M}$ is a morphism in $\operatorname{Rep} H_{\operatorname{Stab}_K(W)}$. Thus, we only must show that $c_{X,M}$ is a morphism of H^*-modules. This is equivalent to prove that

$$c_{X,M}F(\theta)c^{-1}_{H^*,X\otimes M} = (\triangleright_X\otimes F(\triangleright_M)c^{-1}_{H^*,M})(\mathrm{id}_{H^*}\otimes\tau\otimes\mathrm{id}_{F(M)})(\Delta\otimes\mathrm{id}_{X\otimes F(M)})(\mathrm{id}_{H^*}\otimes c_{X,M}). \tag{3.10}$$

Here, $\triangleright_X$ and $\triangleright_M$ are the H^*-actions on X, M respectively; $\tau : H^*\otimes X \to X\otimes H^*$ is the usual transposition; and $\theta : H^*\otimes X\otimes M \to X\otimes M$ is the left H^*-action on $X\otimes M$, that is $\theta = (\triangleright_X\otimes\triangleright_M)(\mathrm{id}_{H^*}\otimes\tau\otimes\mathrm{id}_M)(\Delta\otimes\mathrm{id}_X\otimes\mathrm{id}_M)$. Let $\phi : H^*\otimes X \to X\otimes H^*$ be the morphism of H-modules defined by $\phi = (\mathrm{id}_{H^*}\otimes\triangleright_X)(\mathrm{id}_{H^*}\otimes\tau_{H^*X})(\Delta\otimes\mathrm{id}_X)$. By the naturality of c implies that the diagram

$$\begin{array}{ccc} F(H^*\otimes X\otimes M) & \xrightarrow{\mathrm{id}\otimes c_{H^*\otimes X,M}} & H^*\otimes X\otimes F(M) \\ {\scriptstyle F(\phi\otimes\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \phi\otimes\mathrm{id}} \\ F(X\otimes H^*\otimes M) & \xrightarrow[c_{X\otimes H^*,M}]{} & X\otimes H^*\otimes F(M) \end{array} \tag{3.11}$$

is commutative. Since the map $\triangleright_M : H^*\otimes M \to M$ is a K-module map– Step 1– the naturality of c implies the commutativity of the diagram

$$\begin{array}{ccc} F(X\otimes H^*\otimes M) & \xrightarrow{c_{X,H^*\otimes M}} & X\otimes F(H^*\otimes M) \\ {\scriptstyle F(\mathrm{id}\otimes\triangleright_M)}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes F(\triangleright_M)} \\ F(X\otimes M) & \xrightarrow[c_{X*,M}]{} & X\otimes F(M). \end{array} \tag{3.12}$$

Then, (3.10) equals to

$$\begin{aligned}
&= (\mathrm{id}_X\otimes F(\triangleright_M)c^{-1}_{H^*,M})(\phi\otimes\mathrm{id})(\mathrm{id}\otimes c_{X,M})\\
&= (\mathrm{id}_X\otimes F(\triangleright_M)c^{-1}_{H^*,M})(\phi\otimes\mathrm{id})c_{H^*\otimes X,M}c^{-1}_{H^*,X\otimes M}\\
&= (\mathrm{id}_X\otimes F(\triangleright_M)c^{-1}_{H^*,M})c_{X\otimes H^*,M}F(\phi\otimes\mathrm{id})c^{-1}_{H^*,X\otimes M}\\
&= (\mathrm{id}_X\otimes F(\triangleright_M))c_{X,H^*\otimes M}F(\phi\otimes\mathrm{id})c^{-1}_{H^*,X\otimes M}\\
&= c_{X*,M}F(\mathrm{id}\otimes\triangleright_M)F(\phi\otimes\mathrm{id})c^{-1}_{H^*,X\otimes M}.
\end{aligned}$$

The second and the fourth equalities by (1.11), the third by (3.11) and the last by (3.12). □

By Step 2 (2), the restriction $F : \operatorname{Rep}(H^*)_{K^{\mathrm{op}}} \to {}_{H^*}\operatorname{Rep} H_{\operatorname{Stab}_K(W)}$ is well-defined. If $X \in \operatorname{Rep}(H^*)$, $M \in \operatorname{Rep}(H^*)_{K^{\mathrm{op}}}$, then the map $d_{X,M} : F(X\otimes M) \to (X\otimes H^*)\otimes_{H^*}F(M)$ is defined as the composition $F(X\otimes M) \xrightarrow{c_{X,M}} X\otimes F(M) \xrightarrow{\simeq} (X\otimes H^*)\otimes_{H^*}F(M)$. Here X is considered as a trivial left H-module. Step 3 implies that $d_{X,M}$ is a morphism in ${}_{H^*}\operatorname{Rep} H_{\operatorname{Stab}_K(W)}$. This finishes the proof of (3.4). The Theorem now follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} {}_K\mathcal{M} & \xrightarrow[\text{Cor. 3.2}]{\sim} & \operatorname{Rep} H_{\operatorname{Stab}_K(W)} & \xrightarrow[\text{Prop. 3.9}]{\sim} & {}_{H^*}\operatorname{Rep} H_{\operatorname{Stab}_K(W)} \\ & \overset{\sim}{\searrow} & & & {\scriptstyle (3.4)}\downarrow\wr \\ & & {}_{\operatorname{Stab}_{K^{\mathrm{op}}}(V)}\mathcal{M} & \xleftarrow[\text{Cor. 3.2}]{\sim} & \operatorname{Rep}(H^*)_{K^{\mathrm{op}}} \end{array}$$

□

Corollary 3.11. *Let K be a H-simple left H-comodule algebra. If V and W are left K-modules then $\operatorname{Stab}_K(V)$ and $\operatorname{Stab}_K(W)$ are equivariant Morita equivalent.* □

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Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba
CIEM – CONICET
Medina Allende s/n, (5000) Ciudad Universitaria, Córdoba, Argentina
E-mail address: andrus@mate.uncor.edu, *URL:* http://www.mate.uncor.edu/andrus
E-mail address: mombelli@mate.uncor.edu, *URL:* http://www.mate.uncor.edu/mombelli