DYNAMICAL TWISTS IN HOPF ALGEBRAS

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Abstract. We establish a bijective correspondence between gauge equivalence classes of dynamical twists in a finite-dimensional Hopf algebra H based on a finite abelian group A and equivalence classes of pairs $(K, \{V_\lambda\}_{\lambda\in\widehat{A}})$, where K is an H-simple left H-comodule semisimple algebra and $\{V_\lambda\}_{\lambda\in\widehat{A}}$ is a family of irreducible representations satisfying certain conditions. Our results generalize the results obtained by Etingof-Nikshych on the classification of dynamical twists in group algebras.

1. Introduction

The notion of dynamical twist introduced in [B], see also [BBB], is a generalization of Drinfeld's notion of twist to the dynamical setting. More precisely, if A is a finite Abelian subgroup of the group of group-like elements of a Hopf algebra H a dynamical twist for the pair (H, A) is a function $J : \widehat{A} \to H \otimes H$ satisfying certain equations. If A is trivial then a dynamical twist is just a usual twist. In [EN2] for any dynamical twist the authors endowed the product $H \otimes_{\Bbbk} \mathrm{End}_{\Bbbk}(A)$ with a nontrivial weak Hopf algebra structure. One of the main properties that they prove is that if H is quasitriangular with R-matrix R then $\mathcal{R}(\lambda) = J^{-1}(\lambda)^{21} R J(\lambda)$ satisfies the dynamical quantum Yang-Baxter equation. See [E] for a comprehensive presentation of the dynamical quantum Yang-Baxter equation.

Etingof and Nikshych [EN1] classify dynamical twists for group algebras of finite groups in terms of group data. In this paper we are concerned with the classification of dynamical twists over any finite-dimensional Hopf algebras. The approach of this work owns a lot to [EN1], however there are some differences; the langauge of module categories is used with profit, and we make use of the stabilizers for Hopf algebra actions introduced by M. Yan and Y. Zhu [YZ].

The paper is organized as follows. In subsection 2.1 we introduce the basic notation and conventions, also the main tools that will be needed further. We recall the definition of modules over a tensor category and some results from [AM] on modules over the category of representation of a Hopf algebra. We briefly explain the stabilizers for Hopf algebra actions and some of their properties.

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In section 3 we begin with the definition of dynamical twists over a Hopf algebra and gauge equivalence between them. In subsection 3.1 from a dynamical twist over a Hopf algebra H we construct a module category over $\mathrm{Rep}(H)$. This module category will be important to understand dynamical twists.

In subsection 3.2 following the spirit of the work [EN1] we introduce the definition of dynamical datum. The main new ingredient that appear is the Yan-Zhu stabilizer. In subsection 3.3 we show how to construct a dynamical twists from a dynamical datum in such a way that equivalence classes of dynamical data give the same gauge equivalence of dynamical twists. A converse of this result is proved in Proposition 3.18. Finally in subsection 3.5 we prove that the constructions explained above are one the inverse of each other. This is our main result stated in Theorem 3.19.

In section 4 we compute some examples. Specifically, in subsection 4.1 we compute the dynamical twist in the case when K is the group algebra of the group A, and in subsection 4.2 we show a one-parameter family of dynamical twists for the Taft Hopf algebras.

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2. Preliminaries

2.1. **Notation.** Throughout this paper $\Bbbk$ will denote an algebraically closed field of characteristic 0, and H shall denote a finite-dimensional Hopf algebra over $\Bbbk$ with counit ε and antipode $\mathcal{S}$. We shall denote by $\mathrm{Rep}(H)$ the tensor category of finite-dimensional left H-modules.

If A is a finite Abelian group the group of characters of A will be denoted by $\widehat{A}$. For any $\lambda \in \widehat{A}$ we sometimes denote by $\Bbbk_\lambda$ the one-dimensional representation afforded by λ.

Let K be a left H-comodule algebra with coaction $\delta : K \to H{\otimes}K$. An H-*costable ideal* of K is a two-sided ideal I of K such that $\delta(I) \subseteq H{\otimes}I$. We shall say that K is H-*simple* if there exists no non-trivial H-costable ideal of K.

We denote by ${}^H\mathcal{M}_K$ the category of left H-comodules, right K-modules M such that the left K-module structure $M{\otimes}_\Bbbk K \to M$ is an H-comodule map. If S is another left H-comodule algebra, the category ${}^H_S\mathcal{M}_K$ will denote the category of (S,K)-bimodules M with a left H-comodule structure such that both actions are morphisms of H-comodules.

Lemma 2.1. *Let* $P \in {}^H\mathcal{M}_K$. *Then* $\mathrm{End}_K(P)$ *is a left* H*-comodule algebra via* $\delta : \mathrm{End}_K(P) \to H{\otimes}_\Bbbk \mathrm{End}_K(P)$, $T \mapsto T_{(-1)}{\otimes}T_{(0)}$, *determined by*

$$\langle \alpha, T_{(-1)}\rangle\, T_0(p) = \langle \alpha, T(p_{(0)})_{(-1)}\mathcal{S}^{-1}(p_{(-1)})\rangle\, T(p_{(0)})_{(0)}, \tag{2.1}$$

$T \in \mathrm{End}_K(P)$, $p \in P$, $\alpha \in H^*$. *Moreover* $P \in^H_S \mathcal{M}_K$, *where* $S = \mathrm{End}_K(P)$.

Proof. See [AM, Lemma 1.26]. □

We shall need later the following Frobenius reciprocity.

Lemma 2.2. *Let R be a subalgebra of H. For every left R-module W and a left H-module V there is a natural isomorphism*

$$\mathrm{Hom}_R(W, \mathrm{Res}\,^H_R V) \simeq \mathrm{Hom}_H(\mathrm{Ind}\,^H_R W, V).$$

Proof. See, for example, [AN, Lemma 3.1]. □

2.2. **Module categories.** We briefly recall the definition of module category and the definition introduced by Etingof-Ostrik of *exact* module categories. We refer to [O1], [O2], [EO].

Let us fix $\mathcal{C}$ a finite tensor category. A *module category* over $\mathcal{C}$ is a collection $(\mathcal{M}, \overline{\otimes}, m, l)$ where $\mathcal{M}$ is an Abelian category, $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is an exact bifunctor, associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y) \overline{\otimes} M \to X \overline{\otimes} (Y \overline{\otimes} M)$, $l_M : \mathbf{1} \overline{\otimes} M \to M$, $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$, such that

$$m_{X,Y,Z\overline{\otimes}M}\, m_{X\otimes Y,Z,M} = (\mathrm{id}\,_X \otimes m_{Y,Z,M})\, m_{X,Y\otimes Z,M}\, (a_{X,Y,Z} \otimes \mathrm{id}\,_M), \tag{2.2}$$

$$(\mathrm{id}\,_X \otimes l_M)\, m_{X,\mathbf{1},M} = r_X \otimes \mathrm{id}\,_M, \tag{2.3}$$

for all $X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$. Sometimes we shall simply say that $\mathcal{M}$ is a module category omitting to mention $\overline{\otimes}, m$ and l whenever no confusion arises.

In this paper we further assume that all module categories have finitely many isomorphism classes of simple objects.

Let $\mathcal{M}$, $\mathcal{M}'$ be two module categories over $\mathcal{C}$. A module functor between $\mathcal{M}$ and $\mathcal{M}'$ is a pair (F, c) where $F : \mathcal{M} \to \mathcal{M}'$ is a functor and $c_{X,M} : F(X \overline{\otimes} M) \to X \overline{\otimes} F(M)$ are natural isomorphisms such that

$$m'_{X,Y,F(M)}\, c_{X\otimes Y,M} = (\mathrm{id}\,_X \otimes c_{Y,M})\, c_{X,Y\overline{\otimes}M} F(m_{X,Y,M}), \tag{2.4}$$

$$l'_{F(M)}\, c_{\mathbf{1},M} = F(l_M), \tag{2.5}$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$. Two module categories $\mathcal{M}$ and $\mathcal{M}'$ are *equivalent* if there exists a module functor (F, c) where F is an equivalence of categories. The module structure over $\mathcal{M} \oplus \mathcal{M}'$ is defined in an obvious way. A module category is *indecomposable* if it is not equivalent to the direct sum of two non-trivial module categories.

A module category $\mathcal{M}$ is *exact* [EO] if for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$ the object $P \overline{\otimes} M$ is again projective.

We recall the definition of the internal Hom. This object is an important tool in the study of module categories. See for example [O1], see also [EO].

Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$. Let $M_1, M_2 \in \mathcal{M}$. The functor $X \mapsto \mathrm{Hom}_{\mathcal{M}}(X \overline{\otimes} M_1, M_2)$ is representable and an object $\underline{\mathrm{Hom}}(M_1, M_2)$ representing this functor is called the *internal Hom* of M_1 and M_2. See [EO, O1] for details. Thus

$$\mathrm{Hom}_{\mathcal{M}}(X \overline{\otimes} M_1, M_2) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M_1, M_2))$$

for any $X \in \mathcal{C}$, $M_1, M_2 \in \mathcal{M}$.

The internal Hom $\underline{\mathrm{Hom}}(M,M)$ of an object $M \in \mathcal{M}$ is an algebra in $\mathcal{C}$. The multiplication is constructed as follows. Denote by

$$ev_M : \underline{\mathrm{End}}(M)\otimes M \to M$$

the *evaluation map* obtained as the image of the identity under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{End}}(M), \underline{\mathrm{End}}(M)) \simeq \mathrm{Hom}_{\mathcal{M}}(\underline{\mathrm{End}}(M)\otimes M, M).$$

Thus the product $\mu : \underline{\mathrm{End}}(M)\otimes\underline{\mathrm{End}}(M) \to \underline{\mathrm{End}}(M)$ is defined as the image of the map

$$ev_M(\mathrm{id} \otimes ev_M)\, m_{\underline{\mathrm{End}}(M),\underline{\mathrm{End}}(M),M} \tag{2.6}$$

under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{End}}(M)\otimes\underline{\mathrm{End}}(M), \underline{\mathrm{End}}(M)) \simeq \mathrm{Hom}_{\mathcal{M}}((\underline{\mathrm{End}}(M)\otimes\underline{\mathrm{End}}(M))\otimes M, M).$$

For details we refer to [O1].

2.3. **Yan-Zhu stabilizers.** In [YZ] the authors introduce a notion of stabilizers for Hopf algebra actions generalizing the existing notion for groups. In [AM] these objects and a slight extension thereof, called *Yan-Zhu stabilizers*, were used in connection with module categories over Hopf algebras. We recall the definition and some of its properties.

Let K be a finite-dimensional left H-comodule algebra an V, W two left K-modules. The *Yan-Zhu stabilizer* $\mathrm{Stab}_K(V,W)$ is defined as the intersection

$$\mathrm{Stab}_K(V,W) = \mathrm{Hom}_K(H^*\otimes V, H^*\otimes W) \cap \mathcal{L}\big(H^*\otimes \mathrm{Hom}(V,W)\big).$$

Here the map $\mathcal{L} : H^*\otimes \mathrm{Hom}(V,W) \to \mathrm{Hom}(H^*\otimes V, H^*\otimes W)$ is defined by $\mathcal{L}(\gamma\otimes T)(\xi\otimes v) = \gamma\xi\otimes T(v)$, for every $\gamma, \xi \in H^*$, $T \in \mathrm{Hom}(V,W)$, $v \in V$.

The K-action on $H^*\otimes V$ is given by

$$k \cdot (\gamma\otimes v) = k_{(-1)} \rightharpoondown \gamma \otimes k_{(0)} \cdot v,$$

for all $k \in K$, $\gamma \in H^*$, $v \in V$. Here $\rightharpoondown : H\otimes H^* \to H^*$ is the action defined by $\langle h \rightharpoondown \gamma, t\rangle = \langle \gamma, \mathcal{S}^{-1}(h)t\rangle$, for all $h, t \in H$, $\gamma \in H^*$. Also, we denote $\mathrm{Stab}_K(V) = \mathrm{Stab}_K(V,V)$. The Yan-Zhu stabilizers can also be presented as the coinvariants of a certain Hopf module, see [AM, Prop. 2.10].

Proposition 2.3. [AM, Prop. 2.7, Prop. 2.16] *The following assertions holds.*

1. *For any left K-module V the stabilizer $\mathrm{Stab}_K(V)$ is a left H-module algebra.*
2. *If K is H-simple then*

$$\dim(K)\dim(\mathrm{Stab}_K(V,W)) = \dim(V)\dim(W)\dim(H). \tag{2.7}$$

3. *For any $X \in \mathrm{Rep}(H)$ there are natural isomorphisms*

$$\mathrm{Hom}_H(X, \mathrm{Stab}_K(V,W)) \simeq \mathrm{Hom}_K(X\otimes_{\Bbbk} V, W).$$

□

In general the Yan-Zhu stabilizer is not easy object to compute, however we have the following result. Let $\widetilde{H} \subseteq H$ be a Hopf subalgebra and $K \supseteq R = K^{\mathrm{co}\, H}$ be a Hopf-Galois extension over $\widetilde{H}$.

Proposition 2.4. *For any V, W left K-modules there is an H-module isomorphism* $\mathrm{Stab}_K(V, W) \simeq \mathrm{Hom}_{\widetilde{H}}(H, \mathrm{Hom}_R(V, W))$.

Proof. See [AM, Theorem 2.23]. □

Let K be a H-simple left H-comodule algebra with coaction given by $\delta : K \to H \otimes_{\Bbbk} K$. The category of finite dimensional left K-modules ${}_K\mathcal{M}$ is an exact module category over $\mathrm{Rep}(H)$, see [AM, Prop. 1.20 (i)]. The action $\overline{\otimes} : \mathrm{Rep}(H) \times {}_K\mathcal{M} \to {}_K\mathcal{M}$ is $X \overline{\otimes} V = X \otimes_{\Bbbk} V$, where the left K-action on $X \otimes_{\Bbbk} V$ is given by δ, that is

$$k \cdot (x \otimes v) = k_{(-1)} \cdot x \otimes k_{(0)} \cdot v, \tag{2.8}$$

for all $k \in K$, $x \in X$, $v \in V$. The associativity and unit isomorphisms are canonical.

Proposition 2.3 (3) implies that the Yan-Zhu stabilizers are the internal Hom's of the module category ${}_K\mathcal{M}$. As a consequence of this observation and [EO, Th. 3.17] we have the following result.

Corollary 2.5. *If K is H-simple and V is a left K-module, then ${}_K\mathcal{M} \simeq \mathrm{Rep}(H)_{\mathrm{Stab}_K(V)}$ as module categories over* $\mathrm{Rep}(H)$.

Proof. See [AM, Corollary 3.2]. □

Let us recall a result from [AM] that will be very useful later.

Theorem 2.6. [AM, Theorem 3.3] *If $\mathcal{M}$ is an exact idecomposable module category over* $\mathrm{Rep}(H)$ *then $\mathcal{M} \simeq {}_K\mathcal{M}$ for some H-simple left comodule algebra K with $K^{\mathrm{co}\, H} = \Bbbk$.* □

Let S be another H-simple left H-comodule algebra.

Proposition 2.7. [AM, Prop. 1.24] *The module categories ${}_K\mathcal{M}$, ${}_S\mathcal{M}$ over* $\mathrm{Rep}(H)$ *are equivalent if and only if there exists $P \in {}^H\mathcal{M}_S$ such that $K \simeq \mathrm{End}_S(P)$ as H-module algebras. Moreover the equivalence is given by $F : {}_K\mathcal{M} \to {}_S\mathcal{M}$, $F(V) = P \otimes_S V$, for all $V \in {}_K\mathcal{M}$.* □

3. Dynamical twists for Hopf algebras

Hereafter H will denote a finite-dimensional Hopf algebra and $A \subseteq G(H)$ a finite Abelian subgroup of the group of group-like elements of H. We first recall the definition of dynamical twist over H, see [B], [BBB], [EN1], [EN2].

Definition 3.1. Let $J : \widehat{A} \to H \otimes H$ be a linear map with invertible values. J is a *dynamical twist for H* if for any $\lambda \in \widehat{A}$ and $a \in A$

$$J(\lambda)(a\otimes a) = (a\otimes a)J(\lambda), \tag{3.1}$$

$$\sum_{\mu\in\widehat{A}} (\Delta\otimes\mathrm{id}\,)J(\lambda)\ \big(J(\lambda\mu^{-1})\otimes P_\mu\big) = (\mathrm{id}\,\otimes\Delta)J(\lambda)\,(1\otimes J(\lambda)), \tag{3.2}$$

$$(\varepsilon\otimes\mathrm{id}\,)J(\lambda) = (\mathrm{id}\,\otimes\varepsilon)J(\lambda) = 1. \tag{3.3}$$

Here, for any $\mu \in \widehat{A}$, $P_\mu = \dfrac{1}{|A|}\displaystyle\sum_{a\in A}\mu(a^{-1})\,a \in \Bbbk A$, is the minimal idempotent corresponding to the character μ.

We will often use the notation: $J(\lambda) = J^1(\lambda)\otimes J^2(\lambda) = j^1(\lambda)\otimes j^2(\lambda)$, $J(\lambda)^{-1} = J^{-1}(\lambda)\otimes J^{-2}(\lambda)$, $\lambda\in\widehat{A}$.

Definition 3.2. Two dynamical twists $J, J' : \widehat{A} \to H\otimes H$ are *gauge equivalent* if there exists a map $t : \widehat{A} \to H$, called the gauge equivalence, with invertible values such that for all $a \in A$, $\lambda \in \widehat{A}$

$$\varepsilon(t(\lambda)) = 1, \tag{3.4}$$

$$t(\lambda)\,a \;=\; a\,t(\lambda), \tag{3.5}$$

$$J'(\lambda) = \Delta(t(\lambda)^{-1})\,J(\lambda)\sum_{\mu\in\widehat{A}}\big(t(\lambda\mu^{-1})\otimes P_\mu t(\lambda)\big)\,. \tag{3.6}$$

For any $\lambda \in \widehat{A}$ and V a left A-module we denote by $V[\lambda]$ the isotypic component of type λ, that is

$$V[\lambda] = \{v \in V : a\cdot v = \lambda(a)\,v \text{ for all } a \in A\}.$$

In particular if $X \in \mathrm{Rep}(H)$, then $X[\lambda]$ makes sense by restriction of the action to $\Bbbk[A]$.

Now we will define an H-module algebra that will be useful later. For any $\mu \in \widehat{A}$ define

$$C(\mu) = \{f \in H^* : f(ah) = \mu(a)f(h) \text{ for all } a \in A\}.$$

In another words $C(\mu) = H^*[\lambda]$. We shall consider the left action of H on $C(\mu)$ defined as follows. For $t, h \in H$ and $f \in C(\mu)$, $(h\cdot f)(t) = f(th)$.

If $J : \widehat{A} \to H\otimes H$ is a map that takes invertible elements and satisfies (3.1), for each $\lambda \in \widehat{A}$ we will define a product in $C(\varepsilon)$ as follows. Let $f, g \in C(\varepsilon)$ then define

$$(f\cdot g)(t) = f(J^{-1}(\lambda)h_{(1)})\ g(J^{-2}(\lambda)h_{(2)}), \tag{3.7}$$

for all $h \in H$. Since J commutes with $\Delta(a)$ for all $a \in A$ then $f\cdot g \in C(\varepsilon)$. We shall denote by B_λ the space $C(\varepsilon)$ with this product.

Lemma 3.3. *If $J : \widehat{A} \to H\otimes H$ is a dynamical twist then B_λ is an H-module algebra.*

Proof. We only prove the associativity of the product. Let $f, g, j \in B_\lambda$ and $h \in H$ then

$$\begin{aligned}(f \cdot (g \cdot l))(h) &= f(J^{-1}(\lambda)h_{(1)})\ (g \cdot l)(J^{-2}(\lambda)h_{(2)}) \\ &= f(J^{-1}(\lambda)h_{(1)})\ g(j^{-1}(\lambda)J^{-2}(\lambda)_{(1)}h_{(2)})\ l(j^{-2}(\lambda)J^{-2}(\lambda)_{(2)}h_{(3)}) \\ &= f(j^{-1}(\lambda)J^{-1}(\lambda)_{(1)}h_{(1)})\, g(j^{-2}(\lambda)J^{-1}(\lambda)_{(2)}h_{(2)})\, l(J^{-2}(\lambda)h_{(3)}) \\ &= ((f \cdot g) \cdot l)(h).\end{aligned}$$

The third equality by (3.2) since $l \in C(\varepsilon)$ and $\varepsilon(P_\mu) = \delta_{\mu,\varepsilon}$ for all $\mu \in \widehat{A}$. □

A variation of the algebra B_λ was considered in [EN1]. For usual twists the algebra B_λ was first considered in [Mov] for the classification of twists in group algebras. See also [AEGN] and references therein.

Remark 3.4. There is an isomorphism of H-modules $C(\lambda^{-1}) \simeq (\mathrm{Ind}_A^H \lambda)^*$ given by $\xi : C(\lambda^{-1}) \to (\mathrm{Ind}_A^H \lambda)^*$, $\xi(f)(\overline{h \otimes 1}) = f(\mathcal{S}(h))$, for all $h \in H$. Here $\overline{h \otimes 1}$ denotes the class of $h \otimes 1$ in $H \otimes_A \Bbbk_\lambda$.

3.1. **Module categories coming from dynamical twists.** For any dynamical twist we construct a semisimple module category over $\mathrm{Rep}(H)$. The idea of relate dynamical twists with module categories is due to Ostrik, see [O1, section 4.4]. In *loc. cit.* the author interpret the classification of dynamical twists over group algebras given in [EN1] as a particular case of his classification of module categories over group algebras.

Let $J : \widehat{A} \to H \otimes H$ be a dynamical twist. Denote by $\mathcal{M}^{(J)}$ the Abelian category of left $\Bbbk[A]$-modules with the following module category structure. Define $\overline{\otimes} : \mathrm{Rep}(H) \times \mathcal{M}^{(J)} \to \mathcal{M}^{(J)}$, $X \overline{\otimes} V := X \otimes_\Bbbk V$, $X, Y \in \mathrm{Rep}(H)$, $V \in \mathcal{M}^{(J)}$, where the A-module structure over $X \otimes_\Bbbk V$ is given by the diagonal map. Define also $m_{X,Y,V} : (X \otimes Y) \otimes V \to X \overline{\otimes}(Y \overline{\otimes} V)$ by

$$m_{X,Y,V}(x \otimes y \otimes n) = J^{-1}(\lambda) \cdot x \otimes J^{-2}(\lambda) \cdot y \otimes n, \tag{3.8}$$

for any $x \in X, y \in Y, n \in V[\lambda^{-1}]$. Since J commutes with elements in A then $m_{X,Y,V}$ is an $\Bbbk[A]$-module map.

Lemma 3.5. $(\mathcal{M}^{(J)}, \overline{\otimes}, m, \mathrm{id})$ *is an indecomposable module category over* $\mathrm{Rep}(H)$.

Proof. Let $X, Y, Z \in \mathrm{Rep}(H)$, $M \in \mathcal{M}^{(J)}$ and $x \in X, y \in Y, z \in Z[\mu], n \in M[\lambda^{-1}]$. Then

$$\begin{aligned}&m_{X,Y,Z \otimes M}\, m_{X \otimes Y,Z,M}(((x \otimes y) \otimes z) \otimes n) = \\ &\qquad = m_{X,Y,Z \otimes M}(J^{-1}(\lambda) \cdot (x \otimes y) \otimes J^{-2}(\lambda) \cdot z \otimes n) \\ &\qquad = j^{-1}(\lambda\mu^{-1})J^{-1}(\lambda)_{(1)} \cdot x \otimes j^{-2}(\lambda\mu^{-1})J^{-1}(\lambda)_{(2)} \otimes J^{-2}(\lambda) \cdot z \otimes n.\end{aligned}$$

Next,

$$
\begin{aligned}
(\mathrm{id}_X\otimes m_{Y,Z,M})\, m_{X,Y\otimes Z,M}((x\otimes(y\otimes z))\otimes n) =&\\
&= (\mathrm{id}_X\otimes m_{Y,Z,M})(J^{-1}(\lambda)\cdot x\otimes J^{-2}(\lambda)\cdot(y\otimes z)\otimes n\\
&= J^{-1}(\lambda)\cdot x\otimes j^{-1}(\lambda)J^{-2}(\lambda)_{(1)}\cdot y\otimes j^{-2}(\lambda)J^{-2}(\lambda)_{(2)}\cdot z\otimes n.
\end{aligned}
$$

Thus equation (3.2) implies (2.2). Equation (2.3) follows immediately from (3.3). The indecomposability of the module category $\mathcal{M}^{(J)}$ follows as in [AM, Prop. 1.18]. □

Lemma 3.6. *If $J, \widetilde{J}$ are gauge equivalent dynamical twists, then $\mathcal{M}^{(J)} \simeq \mathcal{M}^{(\widetilde{J})}$ as module categories over* $\mathrm{Rep}(H)$*.*

Proof. Let $t : \widehat{A} \to H$ be the gauge equivalence for J and J'. Define the module functor $(F, c) : \mathcal{M}^{(J)} \to \mathcal{M}^{(\widetilde{J})}$ as follows. For any $X \in \mathrm{Rep}(H), M \in \mathcal{M}^{(J)}$, $F(M) = M$ and $c_{X,M} : X\otimes_{\Bbbk} M \to X\otimes_{\Bbbk} M$ is defined by

$$c_{X,M}(x\otimes m) = t(\lambda)\cdot x\otimes m,$$

for all $x \in X, m \in M[\lambda^{-1}]$. Since $t(\lambda)$ commutes with elements in A the map $c_{X,M}$ is an A-module morphism. Equation (2.4) follows from (3.6) and equation (2.5) follows from (3.4). Clearly (F, c) is an equivalence of module categories. □

The internal Hom of the module category $\mathcal{M}^{(J)}$ can be explicitly calculated. This is the next result.

Lemma 3.7. *There is an isomorphism $B_{\lambda^{-1}} \simeq \underline{\mathrm{Hom}}(\Bbbk_\lambda, \Bbbk_\lambda)$ as H-module algebras.*

Proof. First let us prove that there are natural isomorphisms

$$\mathrm{Hom}_H(X, B_{\lambda^{-1}}) \simeq \mathrm{Hom}_A(X\otimes_{\Bbbk}\Bbbk_\lambda, \Bbbk_\lambda)$$

for every $X \in \mathrm{Rep}(H)$. Let $\phi_X : \mathrm{Hom}_H(X, B_{\lambda^{-1}}) \to \mathrm{Hom}_A(X\otimes_{\Bbbk}\Bbbk_\lambda, \Bbbk_\lambda)$, and $\psi_X : \mathrm{Hom}_A(X\otimes_{\Bbbk}\Bbbk_\lambda, \Bbbk_\lambda) \to \mathrm{Hom}_H(X, B_{-\lambda})$ be the maps defined by

$$\phi_X(\alpha)(x\otimes 1) = \alpha(x)(1), \quad \psi_X(\beta)(x)(t) = \beta(t\cdot x\otimes 1),$$

for all $x \in X$, $t \in H$.

A straightforward computation shows that ϕ_X and ψ_X are well defined maps one the inverse of the other. Now we must prove that the algebra structure on the internal Hom described in subsection 2.2 coincides with the algebra structure of $B_{\lambda^{-1}}$ given in (3.7) via these isomorphisms.

In this case is not hard to see that the evaluation map $ev : B_{\lambda^{-1}}\otimes_{\Bbbk}\Bbbk_\lambda \to \Bbbk_\lambda$ is $ev(f\otimes 1) = \phi_{B_{\lambda^{-1}}}(\mathrm{id})(f\otimes 1) = f(1)$ for all $f \in B_{\lambda^{-1}}$. The product

$\mu : B_{\lambda^{-1}} \otimes_{\Bbbk} B_{\lambda^{-1}} \to B_{\lambda^{-1}}$ according to 2.6 is

$$\begin{aligned}\mu(f\otimes g)(t) &= \psi_{B_{\lambda^{-1}}\otimes_{\Bbbk} B_{\lambda^{-1}}}\big(ev\,(\mathrm{id}\otimes ev)\, m_{B_{\lambda^{-1}},B_{\lambda^{-1}},\Bbbk_\lambda}\big)(f\otimes g)(t)\\ &= \big(ev\,(\mathrm{id}\otimes ev)\, m_{B_{\lambda^{-1}},B_{\lambda^{-1}},\Bbbk_\lambda}\big)((t_{(1)}\cdot f\otimes t_{(2)}\cdot g)\otimes 1)\\ &= \big(ev\,(\mathrm{id}\otimes ev)\big)(J^{-1}(\lambda^{-1})t_{(1)}\cdot f\otimes J^{-2}(\lambda^{-1})t_{(2)}\cdot g\otimes 1)\\ &= f(J^{-1}(\lambda^{-1})t_{(1)})\, g(J^{-2}(\lambda^{-1})t_{(2)}),\end{aligned}$$

for all $f, g \in B_{\lambda^{-1}}$, $t \in H$. □

3.2. **Dynamical Datum.** We introduce the notion of dynamical datum following ideas contained in [EN1].

Definition 3.8. A *dynamical datum* for the pair (H, A) is a collection $(K, \{V_\lambda\}_{\lambda\in\widehat{A}})$ where K is an H-simple left H-comodule algebra, semisimple as an algebra, such that $K^{\mathrm{co}\,H} = \Bbbk$, and V_λ is a collection of irreducible K-modules, such that for every $\lambda, \mu \in \widehat{A}$ there are H-module isomorphisms

$$\mathrm{Stab}_K(V_\lambda, V_\mu) \simeq C(\lambda\,\mu^{-1}). \tag{3.9}$$

Two dynamical data $(K, \{V_\lambda\})$, $(S, \{W_\lambda\})$ are *equivalent* if and only if there exists an object $P \in {}^H\mathcal{M}_S$ such that $K \simeq \mathrm{End}_S(P_S)$ as H-comodule algebras, and a family of K-module isomorphisms

$$\{\phi_\lambda : P\otimes_S W_\lambda \to V_\lambda | \lambda \in \widehat{A}\}.$$

Remark 3.9. The H-module algebra structure on $\mathrm{End}_S(P_S)$ is given in Lemma 2.1.

Our definition of dynamical datum is equivalent with the definition given in [EN1] when $H = \Bbbk[G]$ is the group algebra of a finite group G.

Indeed if K is a $\Bbbk[G]$-simple left comodule algebra such that $K^{\mathrm{co}\,G} = \Bbbk$ then $K = \Bbbk^\psi[F]$ is the twisted group algebra of a subgroup F of G and $\psi \in Z^2(F, \Bbbk^\times)$ is a normalized 2-cocycle.

If $(K, \{V_\lambda\})$ is dynamical datum in our sense then $\mathrm{Stab}_{\Bbbk^\psi[F]}(V_\lambda, V_\mu) \simeq C(\lambda\,\mu^{-1})$. But we know that $\mathrm{Stab}_{\Bbbk^\psi F}(V_\lambda, V_\mu) \simeq \Bbbk[G]\otimes_F \mathrm{Hom}(V_\lambda, V_\mu)$, see [AM, Ex. 2.18]. Also, $\mathrm{Ind}_A^G\,\lambda \simeq C(\lambda)$ as $\Bbbk[G]$-modules, where the isomorphism is given by $\beta : \mathrm{Ind}_A^G\,\lambda \to C(\lambda)$, $\beta(\overline{g\otimes 1})(h) = \delta_{g^{-1}}(P_\lambda\, h)$ for all $g, h \in G$. Altogether implies that $\Bbbk G\otimes_{\Bbbk F}(V_\mu\otimes_{\Bbbk} V_\lambda^*) \simeq \mathrm{Ind}_A^G\,(\lambda\,\mu^{-1})$, thus $(\Bbbk^{\hat\psi}[F], \{V_\lambda^*\})$ is a dynamical datum according to the definition given in [EN1]. Here $\hat\psi(g, h) = \psi(h^{-1}, g^{-1})$ for all $g, h \in F$.

Is not hard to prove also that in this case two dynamical data $(\Bbbk^\psi[F], \{V_\lambda\})$, $(\Bbbk^\psi[F'], \{V'_\lambda\})$ are equivalent, according our definition, if and only if there exists $g \in G$ such that $F' = ad_g F$ and the corresponding representations are conjugated by g. This follows from the fact that $\Bbbk[G]$ is pointed and the quotient $\Bbbk^{G/F}$ is a cosemisimple coalgebra.

Lemma 3.10. *Let* $(K, \{V_\lambda\}_{\lambda\in\widehat{A}})$ *be a dynamical datum. Then*

1. V_λ *and* V_μ *are non-isomorphic* K*-modules if* $\lambda \neq \mu$.
2. *For all* $\lambda \in \widehat{A}$

$$(\dim V_\lambda)^2 \mid A \mid = \dim K. \tag{3.10}$$

Proof. (1) Assume that $\lambda \neq \mu$. Then

$$\begin{aligned}\mathrm{Hom}_K(V_\lambda, V_\mu) &\simeq \mathrm{Hom}_H(\Bbbk, \mathrm{Stab}_K(V_\lambda, V_\mu)) \simeq \mathrm{Hom}_H(\Bbbk, C(\lambda\,\mu^{-1})) \\ &\simeq \mathrm{Hom}_H(\Bbbk, (\mathrm{Ind}\,_A^H(\mu\,\lambda^{-1}))^*) \simeq \mathrm{Hom}_{\Bbbk A}(\mu\,\lambda^{-1}, \Bbbk) = 0.\end{aligned}$$

The first isomorphism is a particular case of Proposition 2.3 (3) and the third is Remark 3.4.

(2) By the definition of dynamical datum we have that $\mathrm{Stab}_K(V_\lambda) \simeq C(\varepsilon)$. Thus

$$\dim \mathrm{Stab}_K(V_\lambda) = \frac{\dim H}{\mid A \mid}.$$

Now equation (3.10) follows from (2.7). □

Remark 3.11. Observe that (3.10) implies that the set $\{V_\lambda\}_{\lambda\in\widehat{A}}$ is a complete set of representatives of isomorphism classes of irreducible representations of K.

3.3. **Dynamical twists constructed from a dynamical datum.** Let $(K, \{V_\lambda\}_{\lambda\in\widehat{A}})$ be a dynamical datum. We shall construct a dynamical twist associated to $(K, \{V_\lambda\}_{\lambda\in\widehat{A}})$. This procedure is called the *exchange construction* in [EN1], see also [EV].

If $X \in \mathrm{Rep}(H)$ and $V \in {}_K\mathcal{M}$, we always assume that the vector space $X \otimes_\Bbbk V$ carries the left K-action described in (2.8).

For any pair $\lambda, \mu \in \widehat{A}$ choose H-module isomorphisms

$$\omega_{\lambda,\mu} : \mathrm{Stab}_K(V_\lambda, V_\mu) \simeq C(\lambda\,\mu^{-1})$$

such that for $\lambda = \mu$ the identity of $\mathrm{Stab}_K(V_\lambda)$ is mapped to ε.

For any $\lambda, \mu \in \widehat{A}$, $x \in X[\mu]$ denote by $\Psi(\lambda, x) : X^* \otimes_\Bbbk V_\lambda \to V_{\lambda\,\mu}$ the K-map obtained as the image of x under the (natural) isomorphisms

$$\begin{aligned}X[\mu] &\simeq \mathrm{Hom}_A(\mu, \mathrm{Res}\,_A^H X) \simeq \mathrm{Hom}_H(\mathrm{Ind}\,_A^H\,\mu, X) \\ &\simeq \mathrm{Hom}_H(X^*, (\mathrm{Ind}\,_A^H\,\mu)^*) \simeq \mathrm{Hom}_H(X^*, C(\mu^{-1})) \\ &\simeq \mathrm{Hom}_H(X^*, \mathrm{Stab}_K(V_\lambda, V_{\lambda\,\mu})) \simeq \mathrm{Hom}_K(X^* \otimes_\Bbbk V_\lambda, V_{\lambda\,\mu}).\end{aligned} \tag{3.11}$$

The second isomorphism by Frobenius reciprocity 2.2, and the fourth is remark 3.4, the fifth isomorphism is defined by $\omega_{\lambda,\lambda\,\mu}$ and the last isomorphism comes from Proposition 2.3 (3).

More explicitly if $f \in X^*$, $v \in V_\lambda$ then

$$\Psi(\lambda, x)(f \otimes v) = \omega^{-1}_{\lambda, \lambda\mu}(\widetilde{f}^x)(1 \otimes v), \tag{3.12}$$

where $\widetilde{f}^x$ is the element in $C(\mu^{-1})$ defined by $\widetilde{f}^x(h) = f(\mathcal{S}(h) \cdot x)$, for all $h \in H$.

Lemma 3.12. *Let $X, Y \in \operatorname{Rep}(H)$ and $f : X \to Y$ an H-module map. If $\lambda, \mu \in \widehat{A}$ and $x \in X[\mu]$ then*

$$\Psi(\lambda, f(x)) = \Psi(\lambda, x)(f^* \otimes \operatorname{id}_{V_\lambda}). \tag{3.13}$$

Proof. Straightforward. It follows from the naturality of the isomorphisms (3.11) or directly from 3.12. □

For any $X, Y \in \operatorname{Rep}(H)$, $V \in {}_K\mathcal{M}$ we shall denote by $\phi_{XY} : (X \otimes_{\Bbbk} Y)^* \to Y^* \otimes_{\Bbbk} X^*$ the canonical isomorphism and $m_{XYV} : (X \otimes_{\Bbbk} Y) \otimes_{\Bbbk} V \to X \otimes_{\Bbbk} (Y \otimes_{\Bbbk} V)$ the canonical associativity isomorphism.

Let Y be another H-module and $y \in Y([\eta]$. The composition

$$(X \otimes_{\Bbbk} Y)^* \otimes_{\Bbbk} V_\lambda \xrightarrow{\phi_{XY} \otimes \mathrm{id}} (Y^* \otimes_{\Bbbk} X^*) \otimes_{\Bbbk} V_\lambda \xrightarrow{m_{Y^*, X^*, V_\lambda}} Y^* \otimes_{\Bbbk} (X^* \otimes_{\Bbbk} V_\lambda) \longrightarrow$$
$$\xrightarrow{\mathrm{id} \otimes \Psi(\lambda, x)} Y^* \otimes_{\Bbbk} V_{\lambda\mu} \xrightarrow{\Psi(\lambda\mu, y)} V_{\lambda\mu\eta}$$

determines a unique element in $(X \otimes_{\Bbbk} Y)[\mu\eta]$, that we denote by $I_{XY}(\lambda)(x \otimes y)$. That is, we have defined a map $I_{XY}(\lambda) : X \otimes_{\Bbbk} Y \to X \otimes_{\Bbbk} Y$ by

$$\Psi(\lambda, I_{XY}(\lambda)(x \otimes y)) = \Psi(\lambda\mu, y)\,(\mathrm{id} \otimes \Psi(\lambda, x))\, m_{Y^*, X^*, V_\lambda}(\phi_{XY} \otimes \mathrm{id}). \tag{3.14}$$

By (3.13) the maps I_{XY} are natural, in particular, there is an element $I(\lambda) \in H \otimes_{\Bbbk} H$ such that

$$I_{XY}(\lambda)(x \otimes y) = I(\lambda)(x \otimes y), \tag{3.15}$$

for all $\lambda \in \widehat{A}$, $x \in X$, $y \in Y$.

Lemma 3.13. *$I(\lambda) \in H \otimes_{\Bbbk} H$ is an invertible element for all $\lambda \in \widehat{A}$.*

Proof. The proof is entirely similar to the proof of [EN1, Lemma 6.2]. For completeness we will write it down. By the definition (3.14) of I_{XY} the surjectivity of the map

$$I_{XY}(\lambda) : \bigoplus_\nu X[\mu\nu^{-1}] \otimes_{\Bbbk} Y[\nu\eta] \to (X \otimes_{\Bbbk} Y)[\mu\eta]$$

is equivalent to the surjectivity of the composition

$$\bigoplus_\nu \operatorname{Hom}_K(V_\nu, Y \otimes_{\Bbbk} V_\eta) \otimes_{\Bbbk} \operatorname{Hom}_K(X^* \otimes_{\Bbbk} V_\lambda, V_\nu) \to \operatorname{Hom}_K(X^* \otimes_{\Bbbk} V_\lambda, Y \otimes_{\Bbbk} V_\eta)$$

and this follows since $X^* \otimes_{\Bbbk} V_\lambda \simeq \bigoplus_\nu X[\nu\lambda^{-1}] \otimes_{\Bbbk} V_\nu$; indeed by the isomorphisms (3.11) $\operatorname{Hom}_K(X^* \otimes_{\Bbbk} V_\lambda, V_\nu) \simeq X[\nu\lambda^{-1}]$, so a copy of $\bigoplus_\nu X[\nu\lambda^{-1}] \otimes_{\Bbbk} V_\nu$

is inside of $X^*\otimes_{\Bbbk}V_\lambda$ but

$$\begin{aligned}\dim(\bigoplus_\nu X[\nu\lambda^{-1}]\otimes_{\Bbbk}V_\nu) &= \sum_\nu \dim(X[\nu\lambda^{-1}])\dim(V_\nu)\\ &= \sum_\nu \dim(X[\nu\lambda^{-1}])\dim(V_\lambda)\\ &= \dim(X)\dim(V_\lambda) = \dim(X^*\otimes_{\Bbbk}V_\lambda).\end{aligned}$$

Here the second equality follows from (3.10), that is all modules V_ν have the same dimension.

□

Let us define $J:\widehat{A}\to H\otimes_{\Bbbk}H$ by $J(\lambda)=\mathcal{S}^{-1}(I^{-2}(\lambda))\otimes\mathcal{S}^{-1}(I^{-1}(\lambda))$.

Proposition 3.14. *$J(\lambda)$ is a dynamical twist for H.*

Proof. Let $X,Y,Z\in \mathrm{Rep}(H)$, $x\in X(\mu)$, $y\in Y(\eta)$, $z\in Z(\nu)$ and $\mu,\eta,\nu,\lambda\in\widehat{A}$. If $a\in A$ then

$$I_{XY}(\lambda)(a\cdot x\otimes a\cdot y)=(\mu\eta)(a)I_{XY}(\lambda)(x\otimes y)=a\cdot I_{XY}(\lambda)(x\otimes y).$$

Hence $I(\lambda)$ and therefore $J(\lambda)$ commutes with $a\otimes a$ for all $a\in A$.

Set $m_1=m_{Z^*,(X\otimes Y)^*,V_\lambda}$, $m_2=m_{(Y\otimes Z)^*,X^*,V_\lambda}$. The associativity implies that

$$\Psi(\lambda\mu\eta,z)\left(\mathrm{id}_{Z^*}\otimes\Psi(\lambda,I_{XY}(\lambda)(x\otimes y))\right)m_1(\phi_{X\otimes Y,Z}\otimes\mathrm{id})$$

equals to

$$\Psi(\lambda\mu,I_{YZ}(\lambda\mu)(y\otimes z))\ \left(\mathrm{id}_{(Y\otimes Z)^*}\otimes\Psi(\lambda,x)\right)m_2(\phi_{X,Y\otimes Z}\otimes\mathrm{id})(a^*\otimes\mathrm{id}),$$

thus $I_{X\otimes Y,Z}(\lambda)(I_{XY}(\lambda)(x\otimes y)\otimes z) = I_{X,Y\otimes Z}(\lambda)(x\otimes I_{YZ}(\lambda\mu)(y\otimes z))$. This implies that

$$(\Delta\otimes\mathrm{id})I(\lambda)\,(I(\lambda)\otimes 1)=\sum_\mu(\mathrm{id}\otimes\Delta)I(\lambda)\,(P_\mu\otimes I(\lambda\mu)).$$

Using the definition of J, the fact that $\mathcal{S}(P_\mu)=P_{\mu^{-1}}$ and properties of the antipode we get that $J(\lambda)$ satisfies equation 3.2. Also, since we have chosen isomorphisms $w_{\lambda,\lambda}$ such that maps the identity of $\mathrm{Stab}_K(V_\lambda)$ to ε then $J(\lambda)$ verifies identity (3.3).

□

We shall say that $J:\widehat{A}\to H\otimes H$ is the dynamical twist *associated* to the dynamical datum $(K,\{V_\lambda\})$.

Proposition 3.15. *Let $(K,\{V_\lambda\})$ be a dynamical datum and $J:\widehat{A}\to H\otimes H$ the dynamical twists associated. Then*

$$_K\mathcal{M}\simeq\mathcal{M}^{(J)}$$

as module categories over $\mathrm{Rep}(H)$.

Proof. We know that for any $\lambda \in \widehat{A}$ there are module equivalences ${}_K\mathcal{M} \simeq \mathrm{Rep}(H)_{\mathrm{Stab}_K(V_\lambda)}$ and $\mathcal{M}^{(J)} \simeq \mathrm{Rep}(H)_{B_\lambda}$. The first one is Corollary 2.5 and the second follows from Lemma 3.7 and [EO, Theorem 3.17]. We shall prove that the H-module isomorphism $w_\lambda = w_{\lambda,\lambda} : \mathrm{Stab}_K(V_\lambda) \to B_\lambda$ is an algebra isomorphism. This will end the proof.

Let $\lambda \in \widehat{A}$, $v \in V_\lambda$ and $f, g \in B_\lambda$. Denote $X = H/(\Bbbk A)^+H$, thus $X^* = C(\varepsilon)$. Then using (3.12) we get

$$\begin{aligned}\Psi(\lambda,\overline{1})\,(\mathrm{id}\otimes\Psi(\lambda,\overline{1}))(f\otimes g\otimes v) &= \Psi(\lambda,\overline{1})(g\otimes\omega_\lambda^{-1}(\widetilde{f})(1\otimes v))\\ &= \omega_\lambda^{-1}(\widetilde{g})\,\omega_\lambda^{-1}(\widetilde{f})(1\otimes v)\\ &= \omega_\lambda^{-1}(g\circ\mathcal{S})\,\omega_\lambda^{-1}(f\circ\mathcal{S})(1\otimes v)\end{aligned}$$

Recall that here $\widetilde{f} \in B_\lambda$ denotes the map $\widetilde{f}(h) = f(\mathcal{S}(h))$, for all $h \in H$. On the other hand

$$\Psi(\lambda, I_{XX}(\lambda)(\overline{1}\otimes\overline{1}))(f\otimes g\otimes v) = \omega_\lambda^{-1}(\widetilde{f}^{I^1(\lambda)}\otimes\widetilde{g}^{I^2(\lambda)})(1\otimes v),$$

where, for all $h \in H$

$$\begin{aligned}(\widetilde{f}^{I^1(\lambda)}\otimes\widetilde{g}^{I^2(\lambda)})(h) &= (f\otimes g)(\mathcal{S}(h)\cdot I(\lambda))\\ &= (g\circ\mathcal{S})(J^{-1}(\lambda)h_{(1)})(f\circ\mathcal{S})(J^{-2}(\lambda)h_{(2)})\\ &= (g\circ\mathcal{S})\cdot(f\circ\mathcal{S})(h).\end{aligned}$$

The product on last equality is the product in B_λ. Therefore

$$\omega_\lambda^{-1}(g\circ\mathcal{S})\,\omega_\lambda^{-1}(f\circ\mathcal{S}) = \omega_\lambda^{-1}((g\circ\mathcal{S})\cdot(f\circ\mathcal{S})),$$

and this ends the proof. □

The construction of the dynamical twist from the dynamical datum is not canonical, however, in the following we shall prove that equivalent dynamical data defines the same gauge equivalence class of dynamical twist. First we need the next technical Lemma.

Let K, S be left H-comodule algebras and $P \in {}^H_K\mathcal{M}_S$. For any $X \in \mathrm{Rep}(H)$, $M \in {}_S\mathcal{M}$ let

$$\theta_{X,M} : X\otimes_\Bbbk(P\otimes_S M) \to P\otimes_S(X\otimes_\Bbbk M)$$

be defined by $\theta_{X,M}(x\otimes p\otimes m) = p_{(0)}\otimes\mathcal{S}^{-1}(p_{(-1)})\cdot x\otimes m$, for any $x \in X, m \in M, p \in P$.

Lemma 3.16. *Let $X, Y \in \mathrm{Rep}(H)$, $M, N \in {}_S\mathcal{M}$, then the maps $\theta_{X,M}$ are well-defined K-isomorphisms. Also if $g : M \to N$ is an S-module map and $f : X \to Y$ an H-module map we have*

$$\theta_{X\otimes_\Bbbk Y,M} = \theta_{X,Y\otimes_\Bbbk M}(\mathrm{id}_X\otimes\theta_{X,M}), \tag{3.16}$$

$$(\mathrm{id}_P\otimes\mathrm{id}_X\otimes g)\,\theta_{X,M} = \theta_{X,N}\,(\mathrm{id}_X\otimes\mathrm{id}_P\otimes g), \tag{3.17}$$

$$(\mathrm{id}_P\otimes f\otimes\mathrm{id}_M)\,\theta_{X,M} = \theta_{Y,M}(f\otimes\mathrm{id}_P\otimes\mathrm{id}_M). \tag{3.18}$$

Proof. Straightforward.

□

Proposition 3.17. *If* $(K,\{V_\lambda\})$ *and* $(S,\{W_\lambda\})$ *are equivalent dynamical data then the associated dynamical twists are gauge equivalent.*

Proof. Let J, J' be the dynamical twists associated to $(K,\{V_\lambda\})$ and $(S,\{W_\lambda\})$ respectively and correspondingly the maps Ψ, Ψ' and I, I'. Since $(K,\{V_\lambda\})$ and $(S,\{W_\lambda\})$ are equivalent there exists $P \in {}^H\mathcal{M}_S$ such that $K \simeq \mathrm{End}_S(P_S)$ as H-comodule algebras, and K-module isomorphisms $\phi_\lambda : P\otimes_S W_\lambda \to V_\lambda$.

Let $\mu, \lambda \in \widehat{A}$ and $X \in \mathrm{Rep}(H)$. For any $x \in X[\mu]$, define $\sigma_X(\lambda)(x)$ the element obtained as the preimage of the map

$$X^*\otimes_{\Bbbk} V_\lambda \xrightarrow{\mathrm{id}_{X^*}\otimes\phi_\lambda^{-1}} X^*\otimes_{\Bbbk}(P\otimes_S W_\lambda) \xrightarrow{\theta_{X^*,W_\lambda}} P\otimes_S(X^*\otimes_{\Bbbk} W_\lambda) \longrightarrow$$
$$\xrightarrow{\mathrm{id}_{P}\otimes\Psi'(\lambda,x)} P\otimes_S W_{\lambda\mu} \xrightarrow{\phi_{\lambda\mu}} V_{\lambda\mu}.$$

under the ismorphisms 3.11. That is

$$\Psi(\lambda,\sigma_X(\lambda)(x)) = \phi_{\lambda\mu}\,(\mathrm{id}_{P}\otimes\Psi'(\lambda,x))\,\theta_{X^*,W_\lambda}\,(\mathrm{id}_{X^*}\otimes\phi_\lambda^{-1}). \tag{3.19}$$

We claim that the maps $\sigma_X(\lambda)$ are natural isomorphisms. Indeed, if $x \in X[\mu]$, $Y \in \mathrm{Rep}(H)$ and $f : X \to Y$ is an H-module map then

$$\begin{aligned}
\Psi(\lambda,(\sigma_Y(\lambda)\circ f)(x)) &= \Psi(\lambda,\sigma_Y(\lambda)(f(x)))\\
&= \phi_{\lambda\mu}\,(\mathrm{id}_{P}\otimes\Psi'(\lambda,f(x)))\,\theta_{Y^*,W_\lambda}\,(\mathrm{id}^*_{Y}\otimes\phi_\lambda^{-1})\\
&= \phi_{\lambda\mu}\,\big(\mathrm{id}_{P}\otimes\Psi'(\lambda,x)(f^*\otimes\mathrm{id}_{W_\lambda})\big)\,\theta_{Y^*,W_\lambda}\,(\mathrm{id}^*_{Y}\otimes\phi_\lambda^{-1})\\
&= \phi_{\lambda\mu}\,\big(\mathrm{id}_{P}\otimes\Psi'(\lambda,x)\big)\theta_{X^*,W_\lambda}(f^*\otimes\phi_\lambda^{-1})\\
&= \Psi(\lambda,\sigma_X(\lambda)(x))(f^*\otimes\mathrm{id}_{V_\lambda})\\
&= \Psi(\lambda,(f\circ\sigma_X(\lambda))(x)).
\end{aligned}$$

The third equality by (3.13), the fourth by (3.18) and the sixth again by (3.13). Therefore there exists an invertible element $\sigma(\lambda) \in H$ such that

$$\sigma_X(\lambda)(x) = \sigma(\lambda)\cdot x$$

for all $\lambda \in \widehat{A}, x \in X$. Set $t(\lambda) = \mathcal{S}^{-1}(\sigma(\lambda)^{-1})$. We shall prove that $t(\lambda)$ is a gauge equivalence between J and J'. Clearly $t(\lambda)$ verifies (3.4) and (3.5). Let us prove that (3.6) holds. Let $\eta \in \widehat{A}$, $Y \in \mathrm{Rep}(H)$, $y \in Y[\eta]$ then

$$\begin{aligned}
&\Psi(\lambda,\sigma_{X\otimes Y}(\lambda)I'_{XY}(\lambda)(x\otimes y))) =\\
&= \phi_{\lambda\mu\eta}\,(\mathrm{id}_{P}\otimes\Psi'(\lambda,I'_{XY}(x\otimes y)))\,\theta_{(X\otimes Y)^*,W_\lambda}\,(\mathrm{id}_{(X\otimes Y)^*}\otimes\phi_\lambda^{-1})\\
&= \phi_{\lambda\mu\eta}\,(\mathrm{id}_{P}\otimes\Psi'(\lambda\mu,y))(\mathrm{id}_{P\otimes Y^*}\otimes\Psi'(\lambda,x))\,\theta_{(X\otimes Y)^*,W_\lambda}\,(\mathrm{id}_{(X\otimes Y)^*}\otimes\phi_\lambda^{-1})\\
&= \Psi(\lambda,\sigma_Y(\lambda\mu)(y))\big(\mathrm{id}_{Y^*}\otimes\Psi'(\lambda,\sigma_X(\lambda)(x)\big)(\mathrm{id}_{(X\otimes Y)^*}\otimes\phi_\lambda))\\
&(\mathrm{id}_{Y^*}\otimes\theta^{-1}_{X^*,W_\lambda})\,\theta^{-1}_{Y^*,W_\lambda}\,\theta_{(X\otimes Y)^*,W_\lambda}\,(\mathrm{id}_{(X\otimes Y)^*}\otimes\phi_\lambda^{-1})\\
&= \Psi(\lambda,I_{XY}(\lambda)(\sigma_X(\lambda)(x)\otimes\sigma_Y(\lambda\mu)(y)))
\end{aligned}$$

The third equality follows from (3.17) since $\Psi'(\lambda, x)$ is an S-module morphism and the fourth follows from (3.16). Thus

$$I_{XY}(\lambda)(\sigma_X(\lambda)(x)\otimes \sigma_Y(\lambda\mu)(y)) = \sigma_{X\otimes Y}(\lambda)I'_{XY}(\lambda)(x\otimes y),$$

and this implies (3.6). □

3.4. **Dynamical datum constructed from a dynamical twist.**
Let $J : \widehat{A} \to H\otimes H$ be a dynamical twist and consider the module category $\mathcal{M}^{(J)}$ explained in subsection 3.1. By Theorem 2.6 there exists an H-simple left H-comodule algebra K, with $K^{\mathrm{co}\,H} = \Bbbk$ and a module equivalence $(F, c) : \mathcal{M}^{(J)} \to {}_K\mathcal{M}$. Set $V_\lambda = F(\Bbbk_{\lambda^{-1}})$ for any $\lambda \in \widehat{A}$.

Proposition 3.18. *The pair $(K, \{V_\lambda\})$ is a dynamical datum.*

Proof. Let $X \in \mathrm{Rep}(H)$, $\lambda, \mu \in \widehat{A}$. Then

$$\begin{aligned}
&\mathrm{Hom}_H(X, \mathrm{Stab}_K(V_\lambda, V_\mu)) \simeq \mathrm{Hom}_K(X\otimes_\Bbbk V_\lambda, V_\mu) \simeq\\
&\simeq \mathrm{Hom}_K(X\otimes_\Bbbk F(\Bbbk_{\lambda^{-1}}), F(\Bbbk_{\mu^{-1}})) \simeq \mathrm{Hom}_K(F(X\otimes_\Bbbk \Bbbk_{\lambda^{-1}}), F(\Bbbk_{\mu^{-1}})) \simeq\\
&\simeq \mathrm{Hom}_A(X\otimes_\Bbbk \Bbbk_{\lambda^{-1}}, \Bbbk_{\mu^{-1}}) \simeq \mathrm{Hom}_A(\mathrm{Res}\,{}^H_A X, \Bbbk_{\lambda\mu^{-1}}) \simeq\\
&\simeq \mathrm{Hom}_A(\Bbbk_{\mu\lambda^{-1}}, \mathrm{Res}\,{}^H_A X^*) \simeq \mathrm{Hom}_H(\mathrm{Ind}\,{}^H_A(\mu\lambda^{-1}), X^*) \simeq\\
&\simeq \mathrm{Hom}_H(X, (\mathrm{Ind}\,{}^H_A(\mu\lambda^{-1})^*) \simeq \mathrm{Hom}_H(X, C(\lambda\mu^{-1})).
\end{aligned}$$

The last isomorphism by Remark 3.4. Thus by Yoneda's Lemma there is an H-module isomorphism $\mathrm{Stab}_K(V_\lambda, V_\mu) \simeq C(\lambda\mu^{-1})$. □

The equivalence class of the dynamical data constructed from a dynamical twist as above does not depend on the gauge equivalence class of the dynamical twists. This is evident from Proposition 2.7.

3.5. **Main result.** Using the same notation as in [EN1] we shall denote by T and D the maps between gauge equivalence classes of dynamical twists and equivalence classes of dynamical data described in subsections 3.3 and 3.4 respectively. That is,

$$\left\{\begin{array}{c}\text{gauge equivalence}\\ \text{classes of}\\ \text{dynamical twists}\\ J:\widehat{A}\to H\otimes H\end{array}\right\} \underset{T}{\overset{D}{\rightleftarrows}} \left\{\begin{array}{c}\text{equivalence}\\ \text{classes of}\\ \text{dynamical data}\\ (K,\ \{V_\lambda\}_{\lambda\in\widehat{A}})\end{array}\right\}.$$

Theorem 3.19. *The maps D and T are inverses of each other.*

Proof. First we shall prove that $D \circ T = \mathrm{Id}$. Let $(K, \{V_\lambda\})$ be a dynamical data and $J : \widehat{A} \to H\otimes H$ the dynamical twist coming from the exchange construction according to subsection 3.3. By Proposition 3.15 ${}_K\mathcal{M} \simeq \mathcal{M}^{(J)}$. Let $(S, \{W_\lambda\})$ be a dynamical data as constructed in subsection 3.4. By definition ${}_S\mathcal{M} \simeq \mathcal{M}^{(J)}$, then ${}_S\mathcal{M} \simeq {}_K\mathcal{M}$, therefore, using Proposition 2.7, $(S, \{W_\lambda\})$ is equivalent to $(K, \{V_\lambda\})$.

Now, let us prove that $T \circ D = \mathrm{Id}$. Let $J_1 : \widehat{A} \to H \otimes H$ be a dynamical twist and $(K, \{V_\lambda\})$ be the dynamical data constructed as in subsection 3.4. In particular this means that there is a module equivalence $(F, c) : \mathcal{M}^{(J_1)} \to {}_K\mathcal{M}$ such that $F(\Bbbk_{\lambda^{-1}}) = V_\lambda$ for all $\lambda \in \widehat{A}$. Let $J_2 : \widehat{A} \to H \otimes H$ be the dynamical data associated to $(K, \{V_\lambda\})$. Let $I_1, I_2 : \widehat{A} \to H \otimes H$ be the maps defined as

$$I_i(\lambda) = \mathcal{S}(J_i^{-2}(\lambda)) \otimes \mathcal{S}(J_i^{-1}(\lambda)),\ \ i = 1, 2.$$

Let Ψ_2 be the map defined form the dynamical data $(K, \{V_\lambda\})$ as in (3.12). By the exchange construction we know that

$$\Psi_2(\lambda, I_2(\lambda) \cdot (x \otimes y)) = \Psi_2(\lambda\mu, y)\,(\mathrm{id} \otimes \Psi_2(\lambda, x))\, m_{Y^*, X^*, V_\lambda}(\phi_{XY} \otimes \mathrm{id}\,), \tag{3.20}$$

for all $\lambda, \mu, \eta \in \widehat{A}$, $X, Y \in \mathrm{Rep}(H)$, $x \in X[\mu]$, $y \in Y[\eta]$.

For any $\mu \in \widehat{A}$, $X \in \mathrm{Rep}(H)$, $x \in X[\mu]$ we denote by $\Psi_1(\lambda, x) : X^* \otimes \Bbbk_{\lambda^{-1}} \to \Bbbk_{(\mu\lambda)^{-1}}$ the map obtained as the image of x under the composition of isomorphisms

$$X[\mu] \simeq \mathrm{Hom}_A(\Bbbk_\mu, X) \simeq \mathrm{Hom}_A(X^*, \Bbbk_{\mu^{-1}}) \simeq \mathrm{Hom}_A(X^* \otimes_\Bbbk \Bbbk_{\lambda^{-1}}, \Bbbk_{(\mu\lambda)^{-1}}).$$

An easy computation shows that for all $x \in X[\mu]$, $y \in Y[\eta]$

$$\Psi_1(\lambda, I_1(\lambda) \cdot (x \otimes y)) = \Psi_1(\lambda\mu, y)\big(\mathrm{id} \otimes \Psi_1(\lambda, x)\big)\widetilde{m}_{Y^*, X^*, \Bbbk_{-\lambda}}(\phi_{XY} \otimes \mathrm{id}\,),$$

where $\widetilde{m}$ is the associativity as in (3.8).

To prove that J_1 is gauge equivalent to J_2 we will use the same idea as in Proposition 3.17. For any $x \in X[\mu]$ define the maps $\sigma_X(\lambda) : X \to X$ by

$$\Psi_2(\lambda, \sigma_X(\lambda)(x)) = F(\Psi_1(\lambda, x))\ c^{-1}_{X^*, \Bbbk_{\lambda^{-1}}}.$$

The maps $\sigma_X(\lambda)$ are natural isomorphisms, hence there exists an invertible element $\sigma(\lambda) \in H$ such that $\sigma_X(\lambda)(x) = \sigma(\lambda) \cdot x$. Set $t(\lambda) = \mathcal{S}^{-1}(\sigma(\lambda)^{-1})$. We will prove that $t(\lambda)$ defines a gauge equivalence between J_1 and J_2. Now

$$\begin{aligned}
&\Psi_2(\lambda, \sigma_{X\otimes Y}(\lambda) I_1(\lambda) \cdot (x \otimes y)) = F(\Psi_1(\lambda, I_1(\lambda) \cdot (x \otimes y)))\, c^{-1}_{(X\otimes Y)^*, \Bbbk_{\lambda^{-1}}}\\
&= F(\Psi_1(\lambda\mu, y)) F(\mathrm{id} \otimes \Psi_1(\lambda, x)) F(\widetilde{m}_{Y^*, X^*, \Bbbk_{\lambda^{-1}}})\, c^{-1}_{Y^* \otimes X^*, \Bbbk_{\lambda^{-1}}}\\
&= F(\Psi_1(\lambda\mu, y)) F(\mathrm{id} \otimes \Psi_1(\lambda, x))\, c^{-1}_{Y^*, X^* \otimes \Bbbk_{\lambda^{-1}}} (\mathrm{id} \otimes c^{-1}_{X^*, \Bbbk_{\lambda^{-1}}})\, m_{Y^*, X^*, V_\lambda}\\
&= F(\Psi_1(\lambda\mu^{-1}, y))\, c^{-1}_{Y^*, \Bbbk_{(\mu\lambda)^{-1}}} (\mathrm{id} \otimes F(\Psi_1(\lambda, x))\, c^{-1}_{X^*, \Bbbk_{\lambda^{-1}}})\, m_{Y^*, X^*, V_\lambda}\\
&= \Psi_2(\lambda\mu, \sigma_Y(\lambda\mu)(y))(\mathrm{id} \otimes \Psi_2(\lambda, \sigma_X(\lambda)(x)))\, m_{Y^*, X^*, V_\lambda}\\
&= \Psi_2(\lambda, I_2(\lambda) \cdot (\sigma_X(\lambda)(x)) \otimes \sigma_Y(\lambda\mu)(y))).
\end{aligned}$$

The third equality by (2.4) and the fourth by the naturality of c. Therefore

$$I_2(\lambda) \cdot (\sigma_X(\lambda)(x)) \otimes \sigma_Y(\lambda\mu)(y) = \sigma_{X\otimes Y}(\lambda) I_1(\lambda) \cdot (x \otimes y),$$

and this equality implies that $t(\lambda)$ satisfies (3.6). □

Remark 3.20. As a immediate consequence of Theorem 3.19 we note that gauge equivalence classes of (usual) twists for Hopf algebras are parameterized by equivalence classes of pairs (K, V) where

- K is a semisimple H-simple left H-comodule algebra,
- $K^{\mathrm{co}\,H} = \Bbbk$, and
- $\mathrm{Stab}_K(V) \simeq H^*$ as left H-modules

If this is the case then K is simple algebra, something expected since the category ${}_K\mathcal{M}$ must have only one simple object.

4. Some examples

In this section we shall give some examples of dynamical data and we compute the corresponding dynamical twist.

4.1. Case when $K = \Bbbk[A]$.

This example is [EN1, Ex. 6.10] for an arbitrary Hopf algebra. Let $K = \Bbbk[A]$ the group algebra of the group A. Let $f : \widehat{A} \to \widehat{A}$ be a bijection. Assume that for all $\lambda, \mu \in \widehat{A}$ there exists an element $g(\lambda, \mu)$ in the normalizer $N(A)$ of A such that

$$\big(f(\lambda)f(\mu)^{-1}\big)(a) = (\mu\lambda^{-1})(g(\lambda,\mu)\, a\, g(\lambda,\mu)^{-1}) \tag{4.1}$$

for all $a \in A$. Set $V_\lambda = \Bbbk_{f(\lambda)}$, then $(K, \{V_\lambda\})$ is a dynamical datum for (H, A).

Proof. Clearly K is an H-simple left H-comodule algebra, semisimple as an algebra with trivial coinvariants. By Proposition 2.4 $\mathrm{Stab}_{\Bbbk[A]}(V_\lambda, V_\mu) \simeq \mathrm{Hom}_{\Bbbk[A]}(H, \mathrm{Hom}_{\Bbbk}(V_\lambda, V_\mu)$. Define the map $\omega_{\lambda,\mu} : \mathrm{Hom}_{\Bbbk[A]}(H, \Bbbk_{f(\mu)f(\lambda)^{-1}}) \to C(\lambda\mu^{-1})$ as follows. If $\alpha \in \mathrm{Hom}_{\Bbbk[A]}(H, \Bbbk_{f(\mu)f(\lambda)^{-1}})$, $h \in H$ then

$$\omega_{\lambda,\mu}(\alpha)(h) = \alpha(g(\lambda,\mu)^{-1}h).$$

Equation (4.1) implies that $\omega_{\lambda,\mu}(\alpha) \in C(\lambda\mu^{-1})$. □

Now we compute the corresponding dynamical twist associated to $(K, \{V_\lambda\})$. If $X \in \mathrm{Rep}(H), x \in X[\mu]$, in this case the maps $\Psi : X^* \otimes_{\Bbbk} \Bbbk_{f(\lambda)} \to \Bbbk_{f(\lambda\mu)}$ are

$$\Psi(\lambda, x)(f \otimes 1) = f(g(\lambda, \lambda\mu)^{-1} \cdot x)$$

for all $f \in X^*$. Let us compute the corresponding dynamical twist. Let $\mu, \eta \in \widehat{A}$ and $f_1, f_2 \in H^*$, then $\Psi(\lambda, I(\lambda)(P_\mu \otimes P_\eta))(f_1 \otimes f_2 \otimes 1)$ is equal to

$$f_1(g(\lambda, \lambda\mu\eta)^{-1} I^2(\lambda) P_\eta)\; f_2(g(\lambda, \lambda\mu\eta)^{-1} I^1(\lambda) P_\mu).$$

On the other hand we have that

$$\Psi(\lambda\mu, P_\eta)(\mathrm{id}_{H^*} \otimes \Psi(\lambda, P_\mu)(f_1 \otimes f_2 \otimes 1)$$

is equal to

$$f_1(g(\lambda, \lambda\mu)^{-1} P_\mu)\; f_2(g(\lambda, \lambda\eta)^{-1} P_\eta).$$

Hence

$$I^{-1}(\lambda)=\sum_{\mu,\eta} g(\lambda,\lambda\mu)\, g(\lambda,\lambda\mu\eta)^{-1}P_\mu\otimes g(\lambda,\lambda\eta)\, g(\lambda,\lambda\mu\eta)^{-1}P_\eta.$$

Thus the dynamical twist in this case is

$$J(\lambda)=\sum_{\mu,\eta} P_\mu g(\lambda,\lambda\mu^{-1}\eta)g(\lambda,\lambda\mu^{-1})^{-1}\otimes P_\eta g(\lambda,\lambda\mu^{-1}\eta)g(\lambda,\lambda\eta^{-1})^{-1}.$$

4.2. **Dynamical twists for the Taft Hopf algebras.** In this subsection for each $c\in\Bbbk^\times$ we construct a dynamical twist for the Taft Hopf algebras.

Let q be a n-primitive root of 1. Recall that the Taft algebra $T(q)$ is the algebra generate by g,x subject to the relations $x^n=0, g^n=1$, $gx=q\,xg$. The Hopf algebra structure is determined by

$$\Delta(g)=g\otimes g,\ \Delta(x)=1\otimes x+x\otimes g,\ \varepsilon(g)=1,\ \varepsilon(x)=0,$$

$$\mathcal{S}(g)=g^{-1},\ \mathcal{S}(x)=-xg^{-1}.$$

Let $d\in\mathbb{N}$ a divisor of n and set $n=dm$. For any $c\in\Bbbk$ denote by $\mathcal{A}(d,c)$ the algebra generated by h and y subject to the relations $y^n=c.1$, $h^d=1$ and $hy=q^m\,yh$.

Define $\delta:\mathcal{A}(d,c)\to T(q)\otimes_{\Bbbk}\mathcal{A}(d,c)$ by

$$\delta(h)=g^{-m}\otimes h,\quad \delta(y)=g^{-1}\otimes y-xg^{-1}\otimes 1.$$

These algebras are left $T(q)$-module algebra $T(q)$-simple. Moreover $\mathcal{A}(d,c)$ is semisimple if and only if $c\neq 0$. This algebras were considered in [MS], and also in [EO] where they classify indecomposable exact module categories over $\operatorname{Rep}(T(q))$.

Fix $c,b\in\Bbbk^\times$ such that $b^n=c$. For $i=0\dots n-1$ let V_i be the one-dimensional vector space generated by v_i together with an action of $\mathcal{A}(1,c)$ defined by $y\cdot v_i=q^i b\, v_i$. The collection $\{V_i\}$ is a complete set of representatives of isomorphism classes of irreducible modules of $\mathcal{A}(1,c)$.

We shall prove that $(\mathcal{A}(1,c),\{V_i\})$ is a dynamical datum over the abelian group $A=<g>$ and we compute the corresponding dynamical twist. First let us prove the following technical result.

Lemma 4.1. *Let $\eta\in\Bbbk^\times$. Define $\xi\in T(q)$ by*

$$\xi=1+\sum_{i=1}^{n-1} a_i\, x^i g^{n-i}, \tag{4.2}$$

where $a_l=\eta^l\prod_{j=1}^{l}\dfrac{q^{n-j+1}}{q^j-1}$ for $l=1\dots n-1$. Then ξ is invertible and $\xi(g+\eta x)=g\xi$.

Proof. The proof that $\xi(g+\eta x) = g\xi$ is done by a straightforward computation. Is easy to see that $(\sum_{i=1}^{n-1} a_i\, x^i g^{n-i})^n = 0$, this implies that ξ is invertible.

□

We shall denote $\xi_j = 1 + \sum_{i=1}^{n-1} a_i\, x^i g^{n-i}$, where $a_l = \dfrac{1}{c^l q^{lj}} \displaystyle\prod_{j=1}^{l} \dfrac{q^{n-j+1}}{q^j - 1}$.

Set χ_i the character of the group A determined by $\chi_i(g) = q^i$. So

$$C(\chi_i) = \{\alpha \in T(q)^* : \langle \alpha, gt\rangle = q^i \langle \alpha, t\rangle \quad \text{for all } t \in T(q)\}.$$

Denote $V_{\chi_i} = V_i$ for all $i = 0 \dots n-1$.

Proposition 4.2. *The collection* $(\mathcal{A}(1,c), \{V_{\chi_i}\}_{i=0\dots n-1})$ *is a dynamical datum.*

Proof. Since all representations V_i are one-dimensional we can identify the stabilizer $\mathrm{Stab}_{\mathcal{A}(1,c)}(V_i, V_j)$ with the set

$$D(i,j) = \{\alpha \in T(q)^* : q^{i-j}\, \langle \alpha, t\rangle = \langle \alpha, (g + \frac{1}{cq^j} x)t\rangle \quad \text{for all } t \in T(q)\}.$$

Indeed by [AM, Lemma 2.8] $\alpha \otimes T \in \mathrm{Stab}_{\mathcal{A}(1,c)}(V_i, V_j)$ if and only if

$$\langle \alpha, k_{(-1)} t\rangle\, T(k_{(0)} \cdot v_i) = \langle \alpha, t\rangle\, k \cdot T(v_i)$$

for all $k \in \mathcal{A}(1,c), t \in T(q)$. Since V_i are one-dimensional we can assume that $T(v_i) = v_j$, and taking $k = y$ we get the result. Now we shall prove that there is a $T(q)$-module isomorphism $D(i,j) \simeq C(\chi_{i-j})$.

Define $\omega_{i,j} : D(i,j) \to C(\chi_{i-j})$ by

$$\langle \omega_{i,j}(\alpha), t\rangle = \langle \alpha, \xi_j^{-1} t\rangle.$$

The maps $\omega_{i,j}$ are well-defined, indeed if $\alpha \in D(i,j)$ and $t \in T(q)$ then

$$\begin{aligned}\langle \omega_{i,j}(\alpha), gt\rangle &= \langle \alpha, \xi_j^{-1} gt\rangle = \langle \alpha, (g + \frac{1}{cq^j} x)\xi_j^{-1} t\rangle \\ &= q^{i-j}\, \langle \alpha, \xi_j^{-1} t\rangle = q^{i-j}\, \langle \omega_{i,j}(\alpha), t\rangle.\end{aligned}$$

The second equality by Lemma 4.1. Thus $\omega_{i,j}(\alpha) \in C(\chi_{i-j})$. Clearly $\omega_{i,j}$ is a $T(q)$-module isomorphism. □

Now we compute the dynamical twist associated to $(\mathcal{A}(1,c), \{V_{\chi_i}\})$. Let $X \in \mathrm{Rep}(T(q))$ and $x \in X[\chi_j]$, $f \in X^*$ then

$$\begin{aligned}\Psi(\chi_i, x)(f \otimes v_i) &= \omega_{i,i+j}^{-1}(\widetilde{f^x})(1 \otimes v_i) = \langle \widetilde{f^x}, \xi_{i+j}\rangle\, v_{i+j} \\ &= \langle f, \mathcal{S}(\xi_{i+j}) \cdot x\rangle\, v_{i+j}.\end{aligned}$$

For any $i = 0 \dots n-1$ denote $P_i = P_{\chi_i}$. Let $l, r = 0 \dots n-1$ $\chi_l, \chi_r \in \widehat{A}$ and $f_1, f_2 \in T(q)^*$, then $\Psi(\chi_i, I(\chi_i)(P_r \otimes P_l))(f_1 \otimes f_2 \otimes v_i)$ is equal to

$$\langle f_1, \mathcal{S}(\xi_{i+r+l})_{(2)} I^2(\chi_i) P_l\rangle\ \langle f_2, \mathcal{S}(\xi_{i+r+l})_{(1)} I^1(\chi_i) P_r\rangle\ v_{i+r+l}.$$

On the other hand

$$\Psi(\chi_{i+r}, P_l)(\mathrm{id}\otimes\Psi(\chi_i, P_r))(f_1\otimes f_2\otimes v_i)$$

is equal to

$$\langle f_1, \mathcal{S}(\xi_{i+r+l})P_l\rangle\ \langle f_2, \mathcal{S}(\xi_{i+r})P_r\rangle\ v_{i+r+l}.$$

Hence

$$\mathcal{S}(\xi_{i+r+l})_{(2)}P_l\otimes \mathcal{S}(\xi_{i+r+l})_{(1)}P_r = \mathcal{S}(\xi_{i+r+l})P_l I^{-2}(\chi_i)\otimes \mathcal{S}(\xi_{i+r})P_r I^{-1}(\chi_i),$$

so we deduce that

$$J(\chi_i) = \sum_{r,l} (\xi_{i-r-l})_{(1)}\xi_{i-r-l}^{-1}P_l\otimes (\xi_{i-r-l})_{(2)}\xi_{i-r}^{-1}P_r.$$

Remark 4.3. It would be interesting to prove that for each d divisor of n $(\mathcal{A}(d,c), \{V_i^d\})$, where V_i^d are the irreducible $\mathcal{A}(d,c)$-modules, is a dynamical datum. This result would classify all dynamical twists since the categories ${}_{\mathcal{A}(d,c)}\mathcal{M}$ are all exact indecomposable module categories over $\mathrm{Rep}(T(q))$, [EO].

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