

CONSTRUCTING DYNAMICAL TWISTS OVER A NON-ABELIAN BASE

JUAN MARTÍN MOMBELLI

ABSTRACT. We give examples of dynamical twists in finite-dimensional Hopf algebras over an arbitrary Hopf subalgebra. The construction is based on the categorical approach of dynamical twists introduced by Donin and Mudrov [DM1].

INTRODUCTION

The theory of *dynamical quantum groups* initiated by G. Felder [F1] is nowadays an active branch of mathematics. This theory arose from the notion of *dynamical Yang-Baxter equation*, also known as the *Gervais-Neveu-Felder equation* in connection with integrable models of conformal field theories and Liouville theory, see for example [F2], [GN], [ABB]. For a detailed review and bibliography on the dynamical Yang-Baxter equation the reader is referred to [E], [ES], [Fh].

In [B], [BBB], the notion of Drinfeld's twist for Hopf algebras was generalized in the dynamical setting. A *dynamical twist* for a Hopf algebra H over A , where A is an abelian subgroup of the group of grouplike elements in H , is a function $J(\lambda) : \widehat{A} \rightarrow (H \otimes H)^\times$ satisfying certain non-linear equations. When the group A is trivial we recover the notion of Drinfeld's twist. When H is a quasi-triangular Hopf algebra with R -matrix R then $\mathcal{R}(\lambda) = J^{-1}(\lambda)^{21} R J(\lambda)$ satisfies the dynamical quantum Yang-Baxter equation. See [EN1].

In the finite-dimensional case, dynamical twists were studied first in [EN1] and later in [M]. In the first paper dynamical twists over an abelian group for the group algebra of a finite group are classified. Following closely this work, in [M], dynamical twists over an abelian group for any finite-dimensional Hopf algebra are described.

In this work we extend the results of [M] in the case where A is an arbitrary Hopf subalgebra of H . To this end we rely on the definition and categorical construction of dynamical twists introduced in [DM1] from dynamical adjoint functors. Here, the Hopf subalgebra A plays the same role

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as the Levi subalgebra of a reductive Lie algebra in [DM1]. As in [M], the language of *module categories* has been used with profit.

The contents of the paper are organized as follows: in Section 1 we recall the notion of stabilizers for Hopf algebra actions introduced by Yan and Zhu [YZ], and the definition of module categories over a tensor category.

In Section 2 we give a brief account of the results and definitions appeared in [DM1]. We explain the definition of dynamical extension of a tensor category, dynamical twist and dynamical adjoint functors. We also recall the construction of a dynamical twist coming from a pair of dynamical functors.

Section 3 is devoted to the construction of dynamical twists for a finite-dimensional Hopf algebra H over a Hopf subalgebra A . Following [EN1] we shall say that a *dynamical datum* for (H, A) is a pair (K, T) , where

- K is a H -simple left H -comodule algebra with $K^{\text{co}H} = \mathbb{k}$,
- and $T : \text{Rep}(A) \rightarrow {}_K\mathcal{M}$ is a functor such that

for any $V, W \in \text{Rep}(A)$ there are natural isomorphisms

$$\text{Stab}_K(T(V), T(W)) \simeq (\text{Ind}_A^H (V \otimes_{\mathbb{k}} W^*))^*.$$

For any dynamical datum (K, T) we shall show that the pair (T, R) , where $R : \text{Rep}(H) \rightarrow \text{Rep}(A)$ is the restriction functor, is a pair of dynamical adjoint functors. Therefore, applying the tools explained in Section 2, we obtain a dynamical twist over the base A . This construction generalizes the procedure appeared in [M] when A is a commutative cocommutative Hopf subalgebra. Also, we shall prove that any dynamical twist based in A for H comes from a dynamical datum. Finally we show some explicit examples of dynamical data.

1. PRELIMINARIES AND NOTATION

Throughout this paper \mathbb{k} will denote an arbitrary field. All categories and functors are assumed to be \mathbb{k} -linear. All vector spaces and algebras are assumed to be over \mathbb{k} . If K is an algebra, we shall denote by ${}_K\mathcal{M}$ the category of finite-dimensional left K -modules.

If V is a vector space, we shall denote by $\langle \cdot, \cdot \rangle : V^* \otimes_{\mathbb{k}} V \rightarrow \mathbb{k}$ the evaluation map.

By H we shall denote a Hopf algebra with counit ε , and antipode \mathcal{S} . We shall denote by $\text{Rep}(H)$ the category of finite-dimensional left H -modules emphasizing the canonical tensor structure.

If K is an H -comodule algebra with coaction $\delta : K \rightarrow H \otimes K$, an *H -costable ideal* of K is a two-sided ideal I of K such that $\delta(I) \subseteq H \otimes I$. We shall say that K is *H -simple* if it has no non-trivial H -costable ideal of K .

We shall denote ${}^H\mathcal{M}_K$ the category of left H -comodules, right K -modules such that the K -module structure is an H -comodule map. If $P \in {}^H\mathcal{M}_K$ then $\text{End}_K(P)$ has a natural left H -comodule algebra structure via δ :

$\text{End}_K(P) \rightarrow H \otimes_{\mathbb{k}} \text{End}_K(P)$, $T \mapsto T_{(-1)} \otimes T_{(0)}$, determined by

$$(1.1) \quad \langle \alpha, T_{(-1)} \rangle T_0(p) = \langle \alpha, T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)},$$

$T \in \text{End}_K(P)$, $p \in P$, $\alpha \in H^*$. See [AM, Lemma 1.26].

Lemma 1.1. *Let $A \subseteq H$ be a Hopf subalgebra and V an A -module. The space $\text{Hom}_A(H, V)$ has a natural H -module structure and there are natural H -module isomorphisms*

$$(\text{Ind}_A^H V)^* \simeq \text{Hom}_A(H, V^*).$$

Proof. The H -module structure on $\text{Hom}_A(H, V)$ is as follows. If $t, h \in H$, $T \in \text{Hom}_A(H, V)$ then

$$(h \cdot T)(t) = T(th).$$

It is not difficult to prove that the maps

$$\theta : (\text{Ind}_A^H V)^* \rightarrow \text{Hom}_A(H, V^*),$$

$$\tilde{\theta} : \text{Hom}_A(H, V^*) \rightarrow (\text{Ind}_A^H V)^*,$$

given by the formulas

$$\theta(\alpha)(h) = \sum_i \alpha(\overline{\mathcal{S}(h) \otimes v_i}) v^i,$$

$$\tilde{\theta}(\beta)(\overline{h \otimes v}) = \langle \beta(\mathcal{S}^{-1}(h)), v \rangle,$$

are well defined isomorphisms, one the inverse of each other, and they are H -module maps. Here $\alpha \in (\text{Ind}_A^H V)^*$, $\beta \in \text{Hom}_A(H, V^*)$, and $(v_i), (v^i)$ are dual basis for V and V^* , $h \in H$, $\overline{h \otimes v} \in \text{Ind}_A^H V$. □

1.1. Stabilizers for Hopf algebra actions. We recall very briefly the notion of Hopf algebra stabilizers introduced in [YZ], see also [AM].

Let K be a finite-dimensional left H -comodule algebra and V, W two left K -modules. The *Yan-Zhu stabilizer* $\text{Stab}_K(V, W)$ is defined as the intersection

$$\text{Stab}_K(V, W) = \text{Hom}_K(H^* \otimes V, H^* \otimes W) \cap \mathcal{L}(H^* \otimes \text{Hom}(V, W)).$$

Here the map $\mathcal{L} : H^* \otimes \text{Hom}(V, W) \rightarrow \text{Hom}(H^* \otimes V, H^* \otimes W)$ is defined by $\mathcal{L}(\gamma \otimes T)(\xi \otimes v) = \gamma \xi \otimes T(v)$, for every $\gamma, \xi \in H^*$, $T \in \text{Hom}(V, W)$, $v \in V$.

The K -action on $H^* \otimes V$ is given by

$$k \cdot (\gamma \otimes v) = k_{(-1)} \rightarrow \gamma \otimes k_{(0)} \cdot v,$$

for all $k \in K$, $\gamma \in H^*$, $v \in V$. Here $\rightarrow : H \otimes H^* \rightarrow H^*$ is the action defined by $\langle h \rightarrow \gamma, t \rangle = \langle \gamma, \mathcal{S}^{-1}(h)t \rangle$, for all $h, t \in H$, $\gamma \in H^*$. Also, we denote $\text{Stab}_K(V) = \text{Stab}_K(V, V)$.

Proposition 1.2. [AM, Prop. 2.7, Prop. 2.16] *The following assertions holds.*

1. *For any left K -modules V, W, U there is a natural composition*

$$\mathrm{Stab}_K(V, W) \otimes_{\mathbb{k}} \mathrm{Stab}_K(U, V) \rightarrow \mathrm{Stab}_K(U, W)$$

making $\mathrm{Stab}_K(V)$ a left H -module algebra.

2. *If K is H -simple then*

$$(1.2) \quad \dim(K) \dim(\mathrm{Stab}_K(V, W)) = \dim(V) \dim(W) \dim(H).$$

3. *For any $X \in \mathrm{Rep}(H)$ there are natural isomorphisms*

$$\mathrm{Hom}_H(X, \mathrm{Stab}_K(V, W)) \simeq \mathrm{Hom}_K(X \otimes_{\mathbb{k}} V, W),$$

where the action on $X \otimes_{\mathbb{k}} V$ is given by the coaction of K . □

The following result concernig Yan-Zhu stabilizers will be useful later. If $A \subseteq H$ is a Hopf subalgebra, and $R = K^{\mathrm{co}A} \subseteq K$ is a left A -Hopf Galois extension then there are H -module algebra isomorphisms

$$(1.3) \quad \mathrm{Stab}_K(V, W) \simeq \mathrm{Hom}_A(H, \mathrm{Hom}_R(V, W))$$

for any left K -modules V, W . Worth to mention that the action of A on $\mathrm{Hom}_R(V, W)$ is given by

$$(a \cdot T)(v) = a^{[1]} \cdot T(a^{[2]} \cdot v),$$

for all $a \in A, T \in \mathrm{Hom}_R(V, W), v \in V$. Recall that the map $\gamma : A \rightarrow K \otimes_R K$, $\gamma(a) = a^{[1]} \otimes a^{[2]}$ is defined by $\gamma(a) = \mathrm{can}^{-1}(a \otimes 1)$, where $\mathrm{can} : K \otimes_R K \rightarrow A \otimes_{\mathbb{k}} K$ is the canonical map $\mathrm{can}(k \otimes s) = k_{(-1)} \otimes k_{(0)} s$, $k, s \in K$. For more details see [Sch, Rmk. 3.4], [AM, Thm. 2.23].

1.2. Module categories. We briefly recall the definition of module category and the definition introduced by Etingof-Ostrik of *exact* module categories. We refer to [O1], [O2], [EO].

Let us fix \mathcal{C} a finite tensor category. A *module category* over \mathcal{C} is a collection $(\mathcal{M}, \overline{\otimes}, m, l)$ where \mathcal{M} is an Abelian category, $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ is an exact bifunctor, associativity and unit isomorphisms $m_{X, Y, M} : (X \otimes Y) \overline{\otimes} M \rightarrow X \overline{\otimes} (Y \overline{\otimes} M)$, $l_M : \mathbf{1} \overline{\otimes} M \rightarrow M$, $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$, such that

$$(1.4) \quad m_{X, Y, Z \overline{\otimes} M} m_{X \otimes Y, Z, M} = (\mathrm{id}_{X \otimes Y} m_{Y, Z, M}) m_{X, Y \otimes Z, M} (a_{X, Y, Z} \otimes \mathrm{id}_M),$$

$$(1.5) \quad (\mathrm{id}_{X \otimes \mathbf{1}} l_M) m_{X, \mathbf{1}, M} = r_X \otimes \mathrm{id}_M,$$

for all $X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$. Sometimes we shall simply say that \mathcal{M} is a module category omitting to mention $\overline{\otimes}, m$ and l whenever no confusion arises.

In this paper we further assume that all module categories have finitely many isomorphism classes of simple objects.

Let $\mathcal{M}, \mathcal{M}'$ be two module categories over \mathcal{C} . A module functor between \mathcal{M} and \mathcal{M}' is a pair (\mathcal{F}, c) where $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor and $c_{X,M} : \mathcal{F}(X \otimes M) \rightarrow X \otimes \mathcal{F}(M)$ is a family of natural isomorphisms such that

$$(1.6) \quad m'_{X,Y,\mathcal{F}(M)} c_{X \otimes Y, M} = (\text{id}_{X \otimes Y, M}) c_{X, Y \otimes M} \mathcal{F}(m_{X,Y,M}),$$

$$(1.7) \quad l'_{\mathcal{F}(M)} c_{1, M} = \mathcal{F}(l_M),$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$. If $(\mathcal{G}, d) : \mathcal{M} \rightarrow \mathcal{M}'$ is another module functor, a morphism between (\mathcal{F}, c) and (\mathcal{G}, d) is a natural transformation $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ such that for any $X \in \mathcal{C}$, $M \in \mathcal{M}_1$:

$$(1.8) \quad d_{X, M} \alpha_{X \otimes M} = (\text{id}_{X \otimes M}) c_{X, M}.$$

The module structure over $\mathcal{M} \oplus \mathcal{M}'$ is defined in an obvious way. A module category is *indecomposable* if it is not equivalent to the direct sum of two non-trivial module categories.

A module category \mathcal{M} is *exact* [EO] if for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$ the object $P \otimes M$ is again projective.

1.3. Module categories over Hopf algebras. Let H be a finite-dimensional Hopf algebra. Let K be a left H -comodule algebra. Then ${}_K \mathcal{M}$ is a left module category over $\text{Rep } H$ via the coaction $\lambda : K \rightarrow H \otimes K$. That is, $\otimes : \text{Rep } H \times {}_K \mathcal{M} \rightarrow {}_K \mathcal{M}$ is given by

$$X \otimes V := X \otimes_{\mathbb{k}} V,$$

for $X \in \text{Rep } H$ and $V \in {}_K \mathcal{M}$ with action $k \cdot (x \otimes v) = k_{(-1)} \cdot x \otimes k_{(0)} \cdot v$, for all $k \in K$, $x \in X$, $v \in V$. Moreover, any exact module category is of this form.

Theorem 1.3. 1. *If K is H -simple left H -comodule algebra then ${}_K \mathcal{M}$ is an indecomposable exact module category.*
 2. *If \mathcal{M} is an indecomposable exact module category over $\text{Rep}(H)$ then there exists an H -simple left H -comodule algebra K such that $\mathcal{M} \simeq {}_K \mathcal{M}$.*

Proof. See [AM, Prop. 1.20, Th. 3.3]. □

Let S be another H -simple left H -comodule algebra.

Proposition 1.4. [AM, Prop. 1.24] *The module categories ${}_K \mathcal{M}, {}_S \mathcal{M}$ over $\text{Rep}(H)$ are equivalent if and only if there exists $P \in {}^H \mathcal{M}_K$ such that $S \simeq \text{End}_K(P_K)$ as H -module algebras. Moreover the equivalence is given by $F : {}_K \mathcal{M} \rightarrow {}_S \mathcal{M}$, $F(V) = P \otimes_K V$, for all $V \in {}_K \mathcal{M}$. □*

2. DYNAMICAL TWISTS CONSTRUCTED FROM DYNAMICAL FUNCTORS

We recall a construction due to Donin and Mudrov of dynamical twists from dynamical adjoint functors, see [DM1, §6]. There are some differences in our statements and those appeared in *loc. cit.* since we use left module categories instead of right ones.

2.1. Dynamical extensions of tensor categories. In [DM1] for any tensor category \mathcal{C} and a module category \mathcal{M} over \mathcal{C} the authors introduced a new tensor category, that we will denote by $\mathcal{M} \times \mathcal{C}$. This tensor category is called the *dynamical extension* of \mathcal{C} over \mathcal{M} .

Objects in the category $\mathcal{M} \times \mathcal{C}$ are functors $F_X : \mathcal{M} \rightarrow \mathcal{M}$, $F_X(M) = X \overline{\otimes} M$, for all $X \in \mathcal{C}$, $M \in \mathcal{M}$. Morphisms are natural transformations. Observe that for each $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ there is a natural transformation $\eta_f : F_X \rightarrow F_Y$, given by

$$(\eta_f)_M : X \overline{\otimes} M \rightarrow Y \overline{\otimes} M, \quad (\eta_f)_M = f \otimes \text{id}_M,$$

for all $M \in \mathcal{M}$.

We briefly recall the monoidal structure of $\mathcal{M} \times \mathcal{C}$. The tensor product is $F_X \otimes F_Y = F_{X \otimes Y}$, $X, Y \in \mathcal{C}$, and the associativity constraint is

$$\tilde{a}_{X,Y,Z} : (F_X \otimes F_Y) \otimes F_Z \rightarrow F_X \otimes (F_Y \otimes F_Z), \quad (\tilde{a}_{X,Y,Z})_M = (a_{X,Y,Z} \otimes \text{id}_M),$$

for all $M \in \mathcal{M}$. For any $X \in \mathcal{C}$ the left and right unit isomorphisms are given by

$$l_X : F_X \otimes F_1 \rightarrow F_X, \quad r_X : F_1 \otimes F_X \rightarrow F_X,$$

where $l_{X,M} = l_X \otimes \text{id}_M$ and $r_X = r_X \otimes \text{id}_M$ for all $M \in \mathcal{M}$.

If $\eta : F_X \rightarrow F_Z$, $\phi : F_Y \rightarrow F_W$ are two natural transformation the tensor product $\eta \otimes \phi : F_{X \otimes Y} \rightarrow F_{Z \otimes W}$ is given by the composition

$$(2.1) \quad (\eta \otimes \phi)_M = m_{ZWM}^{-1} \eta_{W \overline{\otimes} M} (\text{id}_{X \otimes \phi_M}) m_{XYM},$$

for all $M \in \mathcal{M}$. The unit element is F_1 .

Remark 2.1. Note that for any $X, Y, U, V \in \mathcal{C}$ and $f : X \rightarrow Y, g : U \rightarrow V$,

$$(2.2) \quad (\eta_f \otimes \eta_g)_M = (f \otimes g) \otimes \text{id}_M,$$

for all $M \in \mathcal{M}$.

Proposition 2.2. *If $\mathcal{M} \simeq \mathcal{N}$ as module categories then there is a tensor equivalence $\mathcal{M} \times \mathcal{C} \simeq \mathcal{N} \times \mathcal{C}$.*

Proof. Assume that $(\mathcal{F}, c) : \mathcal{M} \rightarrow \mathcal{N}$ and $(\mathcal{G}, d) : \mathcal{N} \rightarrow \mathcal{M}$ is a pair of equivalence of module categories. Let $\theta : \text{Id} \rightarrow \mathcal{F} \circ \mathcal{G}$ be a natural isomorphism of module functors, that is θ satisfies

$$(2.3) \quad c_{X, \mathcal{G}(N)} \mathcal{F}(d_{X,N}) \theta_{X \overline{\otimes} N} = \text{id}_{X \otimes \theta_N},$$

for all $X \in \mathcal{C}$, $N \in \mathcal{N}$.

Define $\Phi : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{N} \times \mathcal{C}$ the functor $\Phi(F_X) = \tilde{F}_X$, for any $X \in \mathcal{C}$. Here, we denote $\tilde{F}_X : \mathcal{N} \rightarrow \mathcal{N}$ the functor $\tilde{F}_X(N) = X \overline{\otimes} N$, for all $N \in \mathcal{N}$. If $X, Y \in \mathcal{C}$ and $\eta : F_X \rightarrow F_Y$ is a natural transformation then $\Phi(\eta) : \tilde{F}_X \rightarrow \tilde{F}_Y$ is given by the composition

$$(2.4) \quad \begin{aligned} X \otimes N &\xrightarrow{\text{id}_{X \otimes \theta_N}} X \otimes \mathcal{F}(\mathcal{G}(N)) \xrightarrow{c_{X, \mathcal{G}(N)}^{-1}} \mathcal{F}(X \otimes \mathcal{G}(N)) \rightarrow \\ &\xrightarrow{\mathcal{F}(\eta_{\mathcal{G}(N)})} \mathcal{F}(Y \otimes \mathcal{G}(N)) \xrightarrow{\mathcal{F}(d_{Y, N}^{-1})} \mathcal{F}\mathcal{G}(Y \otimes N) \xrightarrow{\theta_{Y \otimes N}^{-1}} Y \otimes N, \end{aligned}$$

for all $N \in \mathcal{N}$. The tensor structure on Φ is given by the identity. That is, for any $X, Y \in \mathcal{C}$ the natural isomorphisms $\xi : \Phi(F_X \otimes F_Y) \rightarrow \Phi(F_X) \otimes \Phi(F_Y)$ are given

$$\xi_{X, Y, N} : \tilde{F}_{X \otimes Y}(N) \rightarrow \tilde{F}_{X \otimes Y}(N), \quad \xi_{X, Y, N} = \text{id}_{X \otimes Y} \otimes \text{id}_N,$$

for all $N \in \mathcal{N}$. We have to check that for all $X, Y, Z \in \mathcal{C}$, $N \in \mathcal{N}$ the following identity holds:

$$(2.5) \quad (a_{XYZ} \otimes \text{id}_N)(\xi_{XY} \text{id}_Z)_N \xi_{X \otimes Y, Z, N} = (\text{id}_X \otimes \xi_{YZ})_N \Phi(a_{XYZ} \otimes \text{id}_N).$$

The left hand side of (2.5) is equal to $(a_{XYZ} \otimes \text{id}_N)$, the right hand side is equal to $\Phi(a_{XYZ} \otimes \text{id}_N)$, and using (2.4), is equal to

$$\begin{aligned} &\theta_{(X \otimes (Y \otimes Z)) \otimes N}^{-1} \mathcal{F}(d_{X \otimes (Y \otimes Z), N}^{-1}) (a_{XYZ} \otimes \text{id}_{\mathcal{F}(\mathcal{G}(N))}) c_{(X \otimes Y) \otimes Z, \mathcal{G}(N)}^{-1} (\text{id} \otimes \theta_N) \\ &= (a_{XYZ} \otimes \text{id}_N) \theta_{((X \otimes Y) \otimes Z) \otimes N}^{-1} \mathcal{F}(d_{(X \otimes Y) \otimes Z, N}^{-1}) c_{(X \otimes Y) \otimes Z, \mathcal{G}(N)}^{-1} (\text{id} \otimes \theta_N) \\ &= (a_{XYZ} \otimes \text{id}_N). \end{aligned}$$

The last equality follows by (2.3). \square

The following definition seems to be well-known.

Definition 2.3. A *cocycle* in \mathcal{C} is a family of isomorphisms $J_{X, Y} \in \text{Hom}_{\mathcal{C}}(X \otimes Y)$ such that for all $X, Y, Z \in \mathcal{C}$

$$(2.6) \quad a_{XYZ} J_{X \otimes Y, Z} (J_{X, Y} \otimes \text{id}_Z) = J_{X, Y \otimes Z} (\text{id}_X \otimes J_{Y, Z}) a_{XYZ},$$

$$(2.7) \quad J_{X, \mathbf{1}} = \text{id}_{X \otimes \mathbf{1}}, \quad J_{\mathbf{1}, X} = \text{id}_{\mathbf{1} \otimes X}.$$

If J is a cocycle in \mathcal{C} then there is a new monoidal category, \mathcal{C}^J defined as follows. The objects and morphisms are the same as in \mathcal{C} . The tensor product of \mathcal{C}^J coincides with the tensor product of \mathcal{C} on objects. If $f : X \rightarrow Y, g : Z \rightarrow W$ is a pair of morphisms then

$$f \tilde{\otimes} g = J_{Y, W}(f \otimes g) J_{X, Z}^{-1}.$$

Evidently if J commutes with morphisms in \mathcal{C} then the tensor category \mathcal{C}^J is equivalent to \mathcal{C} .

Let (\mathcal{M}, m, l) be a module category over \mathcal{C} . The following definition is due to Donin and Mudrov, see [DM1, Definition 5.2].

Definition 2.4. A *dynamical twist* for the extension $\mathcal{M} \times \mathcal{C}$ is a cocycle J in $\mathcal{M} \times \mathcal{C}$ such that J commutes with morphisms in \mathcal{C} , that is

$$(2.8) \quad J_{Z, W}(\eta_f \otimes \eta_g) = (\eta_f \otimes \eta_g) J_{X, Y},$$

for all $f \in \text{Hom}_{\mathcal{C}}(X, Z), g \in \text{Hom}_{\mathcal{C}}(Y, W)$.

More explicitly, a dynamical twist is a family of isomorphisms

$$J_{X,Y,M} : (X \otimes Y) \overline{\otimes} M \rightarrow (X \otimes Y) \overline{\otimes} M,$$

$X, Y \in \mathcal{C}, M \in \mathcal{M}$ such that

(2.9)

$$\begin{aligned} (a_{XYZ} \otimes \text{id}_M) J_{X \otimes Y, Z, M} m_{X \otimes Y, Z, M}^{-1} J_{X, Y, Z \overline{\otimes} M} m_{X \otimes Y, Z, M} &= \\ = J_{X, Y \otimes Z, M} m_{X, Y \otimes Z, M}^{-1} (\text{id}_X \otimes J_{Y, Z, M}) m_{X, Y \otimes Z, M} (a_{XYZ} \otimes \text{id}_M), \end{aligned}$$

$$(2.10) \quad (l_X \otimes \text{id}_M) J_{X, \mathbf{1}, M} = (l_X \otimes \text{id}_M), \quad J_{\mathbf{1}, X, M} (r_X \otimes \text{id}_M) = (r_X \otimes \text{id}_M),$$

for all $X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$.

Equation (2.8) implies that

$$J_{Z, W, M} m_{Z, W, M}^{-1} (f \otimes (g \otimes \text{id}_M)) m_{X, Y, M} = m_{Z, W, M}^{-1} (f \otimes (g \otimes \text{id}_M)) m_{X, Y, M} J_{X, Y, M},$$

for all morphisms $f : X \rightarrow Z$, $g : Y \rightarrow W$ in \mathcal{C} , and all $M \in \mathcal{M}$.

2.2. Module categories coming from dynamical twists. If J is a dynamical twist for the dynamical extension $\mathcal{M} \times \mathcal{C}$ we will denote by $\mathcal{M}^{(J)}$ the category \mathcal{M} with the following module category structure; the action is the same as in \mathcal{M} and the associativity isomorphisms are

$$\widehat{m}_{X,Y,M} : (X \otimes Y) \overline{\otimes} M \rightarrow X \overline{\otimes} (Y \overline{\otimes} M), \quad \widehat{m}_{X,Y,M} = m_{X,Y,M} J_{X,Y,M}^{-1},$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$.

Proposition 2.5. *Let J be a dynamical twist for the dynamical extension $\mathcal{M} \times \mathcal{C}$.*

1. $\mathcal{M}^{(J)}$ is a module category over \mathcal{C} . If \mathcal{M} is indecomposable then so is $\mathcal{M}^{(J)}$.
2. There is a tensor equivalence $\mathcal{M}^{(J)} \times \mathcal{C} \simeq (\mathcal{M} \times \mathcal{C})^J$.

Proof. Straightforward. □

The idea of using the module category language in the study of dynamical twists is due to Ostrik, see [O1]. In *loc. cit.* the author relates the classification of module categories over $\text{Rep}(G)$, G a finite group, with the results obtained by Etingof and Nikshych on dynamical twists over the group algebra of the group G [EN1]. This idea was used with profit in [M].

2.3. Dynamical twists and dynamical adjoint functors. Let \mathcal{C} be a tensor category. Cocycles in \mathcal{C} are in bijective correspondence with natural associative operations in the space $\text{Hom}_{\mathcal{C}}$. If J is a cocycle then

$$\otimes : \text{Hom}_{\mathcal{C}}(V, U) \otimes \text{Hom}_{\mathcal{C}}(V', U') \rightarrow \text{Hom}_{\mathcal{C}}(V \otimes V', U \otimes U')$$

defined by $\phi \otimes \psi = (\phi \otimes \psi) J_{VU}^{-1}$ is an associative operation for all $V, U, V', U' \in \mathcal{C}$.

The following result is analogous to [DM1, Lemma 6.1].

Lemma 2.6. *Assume that there is an associative operation*

$$\otimes : \text{Hom}_{\mathcal{C}}(V, U) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(V', U') \rightarrow \text{Hom}_{\mathcal{C}}(V \otimes V', U \otimes U')$$

for all $V, U, V', U' \in \mathcal{C}$ such that

$$(2.11) \quad (\phi \otimes \psi) = (\phi \otimes \psi)(\text{id} \otimes \text{id}),$$

$$(2.12) \quad \phi \otimes \xi = \phi \otimes \xi, \quad \xi \otimes \phi = \xi \otimes \phi$$

for all morphisms $\phi, \psi, \alpha, \beta$ in \mathcal{C} and $\xi \in \text{Hom}_{\mathcal{C}}(V, \mathbf{1})$. Assume also that for any $U, V \in \mathcal{C}$ $I_{UV} = \text{id}_U \otimes \text{id}_V$ is invertible. Then $J_{UV} = I_{UV}^{-1}$, $U, V \in \mathcal{C}$, is a cocycle in \mathcal{C} .

Proof. The proof is entirely similar to the proof of [DM1, Lemma 6.1]. \square

Let $\mathcal{C}, \mathcal{C}'$ be two tensor categories. Let (\mathcal{M}, m, l) be a module category over \mathcal{C} and (\mathcal{M}', m', l') be a module category over \mathcal{C}' . Let $(R, b) : \mathcal{C} \rightarrow \mathcal{C}'$ be a tensor functor. The following definition is [DM1, Def. 6.2] for right module categories.

Definition 2.7. A functor $T : \mathcal{M}' \rightarrow \mathcal{M}$ is said to be a *dynamical adjoint* to R if there exists a family of natural isomorphisms

$$\xi_{X, M, N} : \text{Hom}_{\mathcal{M}}(X \overline{\otimes} T(M), T(N)) \xrightarrow{\cong} \text{Hom}_{\mathcal{M}'}(R(X) \overline{\otimes} M, N),$$

for all $X \in \mathcal{C}$, $M, N \in \mathcal{M}'$. We further assume that for any $M \in \mathcal{M}'$

$$(2.13) \quad \xi_{\mathbf{1}, M, M}(l_{T(M)}) = l'_M.$$

Remark 2.8. For any $M, M', N, N' \in \mathcal{M}'$ and $X, Y \in \mathcal{C}$ and morphisms $f : N \rightarrow N'$, $g : M \rightarrow M'$, $\alpha : X \rightarrow Y$, the naturality of ξ implies that

$$(2.14) \quad f \circ \xi_{X, M, N}(\beta_1) = \xi_{X, M, N'}(T(f)\beta_1),$$

$$(2.15) \quad \xi_{X, M', N}(\beta_2) \circ (\text{id}_{R(X)} \otimes g) = \xi_{X, M, N}(\beta_2(\text{id}_X \otimes T(g))),$$

$$(2.16) \quad \xi_{Y, M, N}(\beta_3)(R(\alpha) \otimes \text{id}_M) = \xi_{X, M, N}(\beta_3(\alpha \otimes \text{id}_{T(M)})),$$

for any $\beta_1 \in \text{Hom}_{\mathcal{M}}(X \overline{\otimes} T(M), T(N))$, $\beta_2 \in \text{Hom}_{\mathcal{M}}(X \overline{\otimes} T(M'), T(N))$, $\beta_3 \in \text{Hom}_{\mathcal{M}}(Y \overline{\otimes} T(M), T(N))$.

The category \mathcal{M}' is a module category over \mathcal{C} via R . The action is given by $\tilde{\otimes} : \mathcal{C} \times \mathcal{M}' \rightarrow \mathcal{M}'$, $X \tilde{\otimes} M = R(X) \overline{\otimes} M$, for all $X \in \mathcal{C}$, $M \in \mathcal{M}'$. The associativity isomorphisms are

$$\tilde{m}_{X, Y, M} = m'_{R(X), R(Y), M}(b_{XY} \otimes \text{id}_M)$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}'$.

For any pair of dynamical adjoint functors (R, T) we will repeat the construction given in [DM1] of a dynamical twist for the extension $\mathcal{M}' \times \mathcal{C}$. For this we will define an associative operation in $\text{Hom}_{\mathcal{M}' \times \mathcal{C}}$.

In some sense, the dynamical twist constructed from the pair (R, T) *measures* how far is the functor T from being a module functor.

Let $X, Y, U, V \in \mathcal{C}$ and $\phi : F_X \rightarrow F_Y$, $\psi : F_U \rightarrow F_V$ be morphisms in $\mathcal{M}' \times \mathcal{C}$. So for each $M \in \mathcal{M}'$ we have that $\phi_M : R(X) \otimes M \rightarrow R(Y) \otimes M$, $\psi_M : R(U) \otimes M \rightarrow R(V) \otimes M$ are morphisms in \mathcal{M}' .

Set $\tilde{\phi}_M = \xi_{X, M, R(Y) \otimes M}^{-1}(\phi_M)$, and $\tilde{\psi}_M = \xi_{U, M, R(V) \otimes M}^{-1}(\psi_M)$. Thus we define $(\phi \otimes \psi)_M$ as the image by ξ of the composition

$$\begin{aligned} (X \otimes U) \otimes T(M) &\xrightarrow{m_{X, U, T(M)}} X \otimes (U \otimes T(M)) \xrightarrow{\text{id}_X \otimes \tilde{\psi}_M} X \otimes T(R(V) \otimes M) \rightarrow \\ &\xrightarrow{\tilde{\phi}_{R(V) \otimes M}} T(R(Y) \otimes (R(V) \otimes M)) \xrightarrow{T(\tilde{m}_{YVM}^{-1})} T(R(Y \otimes V) \otimes M). \end{aligned}$$

That is, for all $M \in \mathcal{M}'$, $(\phi \otimes \psi)_M$ equals to

$$\xi_{X \otimes U, M, R(Y \otimes V) \otimes M}(T(\tilde{m}_{YVM}^{-1}) \tilde{\phi}_{R(V) \otimes M}(\text{id}_X \otimes \tilde{\psi}_M) m_{X, U, T(M)}).$$

Lemma 2.9. *For any $X, Y, U, V \in \mathcal{C}$ and morphisms $f : X \rightarrow Y$, $g : U \rightarrow V$, $\phi : F_X \rightarrow F_Y$, $\psi : F_U \rightarrow F_V$*

$$(2.17) \quad (\eta_f \otimes \eta_g) = (\text{id}_Y \otimes \text{id}_V)(\eta_f \otimes \eta_g).$$

$$(2.18) \quad \phi \otimes \psi = (\phi \otimes \psi)(\text{id}_X \otimes \text{id}_U).$$

Proof. Using (2.16) we have that $(\eta_f \otimes \eta_g)$ equals to

$$\begin{aligned} &= \xi_{X \otimes U, M, R(Y \otimes V) \otimes M}(T(\tilde{m}_{YVM}^{-1}) \xi_{Y, M, R(Y) \otimes M}^{-1}(\text{id})(f \otimes \text{id}_{T(M)}) \\ &\quad (\text{id}_X \otimes \xi_{V, M, R(V) \otimes M}^{-1}(\text{id})(g \otimes \text{id}_{T(M)})) m_{X, U, T(M)}) \\ &= \xi_{X \otimes U, M, R(Y \otimes V) \otimes M}(T(\tilde{m}_{YVM}^{-1}) \xi_{Y, M, R(Y) \otimes M}^{-1}(\text{id})(\text{id}_Y \otimes \xi_{V, M, R(V) \otimes M}^{-1}(\text{id})) \\ &\quad (f \otimes (g \otimes \text{id}_{T(M)})) m_{X, U, T(M)}) \\ &= \xi_{X \otimes U, M, R(Y \otimes V) \otimes M}(T(\tilde{m}_{YVM}^{-1}) \xi_{Y, M, R(Y) \otimes M}^{-1}(\text{id})(\text{id}_Y \otimes \xi_{V, M, R(V) \otimes M}^{-1}(\text{id})) \\ &\quad m_{Y, V, T(M)}((f \otimes g) \otimes \text{id}_{T(M)})) \\ &= \xi_{X \otimes U, M, R(Y \otimes V) \otimes M}(T(\tilde{m}_{YVM}^{-1}) \xi_{Y, M, R(Y) \otimes M}^{-1}(\text{id})(\text{id}_Y \otimes \xi_{V, M, R(V) \otimes M}^{-1}(\text{id})) \\ &\quad m_{Y, V, T(M)}(R(f \otimes g) \otimes \text{id}_M) = (\text{id}_Y \otimes \text{id}_V)(\eta_f \otimes \eta_g). \end{aligned}$$

The third equality follows from the naturality of m and the fourth equality, again, follows from (2.16). Thus we have proved (2.17).

Note that for any $\phi : F_X \rightarrow F_Y$, $\psi : F_U \rightarrow F_V$, equations (2.14) and (2.15) implies that for any $M \in \mathcal{M}'$ $\tilde{\phi}_{R(V)\overline{\otimes}M}(\text{id}_X \otimes \tilde{\psi}_M)$ equals to

$$\xi_{X,R(U)\overline{\otimes}M,R(Y)\overline{\otimes}(R(V)\overline{\otimes}M)}^{-1}(\phi_{R(V)\overline{\otimes}M}(\text{id}_X \otimes \psi_M))(\text{id}_X \otimes \xi_{U,M,R(U)\overline{\otimes}M}^{-1}(\text{id})).$$

Also, using (2.14) we get that $\xi_{X,R(U)\overline{\otimes}M,R(Y)\overline{\otimes}(R(V)\overline{\otimes}M)}^{-1}(\phi_{R(V)\overline{\otimes}M}(\text{id}_X \otimes \psi_M))$ is equal to

$$T(\phi_{R(V)\overline{\otimes}M}(\text{id}_X \otimes \psi_M))\xi_{X,R(U)\overline{\otimes}M,R(X)\overline{\otimes}(R(U)\overline{\otimes}M)}^{-1}(\text{id}).$$

Thus, for any $M \in \mathcal{M}'$, $\xi_{X\otimes U,M,R(Y\otimes V)\overline{\otimes}M}^{-1}((\phi \otimes \psi)_M)$ is equal to

$$T(\tilde{m}_{Y,V,M}^{-1}\phi_{R(V)\overline{\otimes}M}(\text{id}_X \otimes \psi_M))\xi_{X,R(U)\overline{\otimes}M,R(X)\overline{\otimes}(R(U)\overline{\otimes}M)}^{-1}(\text{id})(\text{id}_X \otimes \xi_{U,M,R(U)\overline{\otimes}M}^{-1}(\text{id}))$$

Follows from (2.1) that

$$T(\tilde{m}_{Y,V,M}^{-1}\phi_{R(V)\overline{\otimes}M}(\text{id}_X \otimes \psi_M)) = T((\phi \otimes \psi)_M \tilde{m}_{X,U,M}^{-1}),$$

hence $(\phi \otimes \psi)_M$ equals to

$$\xi_{X\otimes U,M,R(Y\otimes V)\overline{\otimes}M}(T((\phi \otimes \psi)_M \tilde{m}_{X,U,M}^{-1})\xi_{X,R(U)\overline{\otimes}M,R(X)\overline{\otimes}(R(U)\overline{\otimes}M)}^{-1}(\text{id})(\text{id}_X \otimes \xi_{U,M,R(U)\overline{\otimes}M}^{-1}(\text{id})))$$

Using again (2.14) we get that $(\phi \otimes \psi)_M$ equals to

$$(\phi \otimes \psi)_M \xi_{X\otimes U,M,R(Y\otimes V)\overline{\otimes}M}(T(\tilde{m}_{X,U,M}^{-1})\xi_{X,R(U)\overline{\otimes}M,R(X)\overline{\otimes}(R(U)\overline{\otimes}M)}^{-1}(\text{id})(\text{id}_X \otimes \xi_{U,M,R(U)\overline{\otimes}M}^{-1}(\text{id}))),$$

and by definition this is equal to $(\phi \otimes \psi)_M(\text{id}_X \otimes \text{id}_U)_M$. \square

Definition 2.10. For any $X, Y \in \mathcal{C}$ set

$$(2.19) \quad I_{X,Y} = \text{id}_X \otimes \text{id}_Y.$$

Lemma 2.11. For any $X \in \mathcal{C}$, $I_{X,1} = \text{id}_X \otimes \text{id}_1$, $I_{1,X} = \text{id}_1 \otimes \text{id}_X$.

Proof. By definition, for all $M \in \mathcal{M}'$, $\xi_{X\otimes 1,M,R(X\otimes 1)\overline{\otimes}M}^{-1}((\text{id}_X \otimes \text{id}_1)_M)$ equals to

$$(2.20) \quad T(\tilde{m}_{X,1,M}^{-1})\xi_{X,R(1)\overline{\otimes}M,R(X)\overline{\otimes}(1\overline{\otimes}M)}^{-1}(\text{id})(\text{id}_X \otimes \xi_{1,M,1\overline{\otimes}M}^{-1}(\text{id}))m_{X,1,M}$$

Using (1.5) and (2.14) for the map $\text{id}_{R(X)} \otimes l_M$ we get that (2.20) is equal to

$$(2.21) \quad T(b_{X,1}r'_{R(X)}^{-1} \otimes \text{id}_M)\xi_{X,1\overline{\otimes}M,R(X)\overline{\otimes}M}^{-1}(\text{id}_{R(X)} \otimes l'_M)(\text{id}_X \otimes \xi_{1,M,1\overline{\otimes}M}^{-1}(\text{id}))m_{X,1,M}.$$

From (2.14) and (2.13) we get that

$$\xi_{\mathbf{1},M,\mathbf{1}\overline{\otimes}M}^{-1}(\text{id}) = T(l'_M)^{-1}\xi_{\mathbf{1},M,M}^{-1}(l'_M) = T(l'_M)^{-1}l_{T(M)}.$$

Now, using (1.5) follows that (2.21) equals to

(2.22)

$$T(b_{X,\mathbf{1}}r'_{R(X)} \otimes \text{id}_M) \xi_{X,\mathbf{1}\overline{\otimes}M,R(X)\overline{\otimes}M}^{-1}(\text{id}_{R(X)} \otimes l'_M)(\text{id}_X \otimes T(l'_M)^{-1})(r_X \otimes \text{id}_M).$$

From (2.15) and (2.15) we get that

$$\xi_{X,\mathbf{1}\overline{\otimes}M,R(X)\overline{\otimes}M}^{-1}(\text{id}_{R(X)} \otimes l'_M)(\text{id}_X \otimes T(l'_M)^{-1}) = \xi_{X,M,R(X)\overline{\otimes}M}^{-1}(\text{id}),$$

$$\xi_{X,M,R(X)\overline{\otimes}M}^{-1}(\text{id})(r_X \otimes \text{id}_M) = \xi_{X\otimes\mathbf{1},M,R(X)\overline{\otimes}M}^{-1}(R(r_X) \otimes \text{id}_M)$$

thus (2.22) is equal to

$$\begin{aligned} T(b_{X,\mathbf{1}}r'_{R(X)} \otimes \text{id}_M) \xi_{X,M,R(X)\overline{\otimes}M}^{-1}(\text{id})(r_X \otimes \text{id}_M) &= \\ &= T(b_{X,\mathbf{1}}r'_{R(X)} \otimes \text{id}_M) \xi_{X\otimes\mathbf{1},M,R(X)\overline{\otimes}M}^{-1}(R(r_X) \otimes \text{id}_M). \end{aligned}$$

Finally, $(\text{id}_X \otimes \text{id}_{\mathbf{1}})_M$ is equal to

$$\begin{aligned} &= \xi_{X\otimes\mathbf{1},M,R(X\otimes\mathbf{1})\overline{\otimes}M}(T(b_{X,\mathbf{1}}r'_{R(X)} \otimes \text{id}_M) \xi_{X\otimes\mathbf{1},M,R(X)\overline{\otimes}M}^{-1}(R(r_X) \otimes \text{id}_M)) \\ &= (b_{X,\mathbf{1}}r'_{R(X)} \otimes \text{id}_M) \xi_{X\otimes\mathbf{1},M,R(X)\overline{\otimes}M}(\xi_{X\otimes\mathbf{1},M,R(X)\overline{\otimes}M}^{-1}(R(r_X) \otimes \text{id}_M)) \\ &= (b_{X,\mathbf{1}}r'_{R(X)} \otimes \text{id}_M)(R(r_X) \otimes \text{id}_M) = \text{id}_{R(X)} \otimes \text{id}_{\mathbf{1}} \otimes \text{id}_M. \end{aligned}$$

The equality $I_{\mathbf{1},X} = \text{id}_{\mathbf{1}} \otimes \text{id}_X$ follows in a similar way. \square

Lemma 2.12. *For any $U, V, W, X, Y, Z \in \mathcal{C}$, $\phi : F_X \rightarrow F_Y$, $\psi : F_U \rightarrow F_V$, $\chi : F_Z \rightarrow F_W$*

$$(\phi \otimes \psi) \otimes \chi = \phi \otimes (\psi \otimes \chi).$$

Proof. The proof is done by a tedious but straightforward computation. \square

The following result is a re-statement of [DM1, Prop. 6.3].

Theorem 2.13. *Let us assume that for any $X \in \mathcal{C}, M \in \mathcal{M}'$ the map $\xi_{X,M,R(X)\overline{\otimes}M}^{-1}(\text{id})$ is an isomorphism. Then for any $X, Y \in \mathcal{C}$ the maps $I_{X,Y}$ are invertible and $J_{X,Y} = I_{X,Y}^{-1}$ is a dynamical twist for the extension $\mathcal{M}' \rtimes \mathcal{C}$.*

Proof. Once we prove that $I_{X,Y}$ are invertible for any $X, Y \in \mathcal{C}$, the proof that $I_{X,Y}^{-1}$ is a dynamical twist for the extension $\mathcal{M}' \rtimes \mathcal{C}$ follows immediately from Lemmas 2.6, 2.9, 2.11 and 2.12.

Let $X \in \mathcal{C}, M, N \in \mathcal{M}'$ and $\beta : X \otimes T(M) \rightarrow T(N)$ be an invertible map with inverse $\gamma : T(N) \rightarrow X \otimes T(M)$. There exists an $f : R(X)\overline{\otimes}M \rightarrow N$ such that $\beta = \xi_{X,M,N}^{-1}(f)$. Using (2.14) we obtain that $\beta = T(f) \xi_{X,M,R(X)\overline{\otimes}M}^{-1}(\text{id})$, hence f is invertible. Therefore $\xi_{X,M,N}(\beta)$ is invertible. In another words, ξ maps isomorphisms to isomorphisms.

Let $X, Y \in \mathcal{C}$, $M \in \mathcal{M}'$. By definition

$$(I_{X,Y})_M = \xi_{X \otimes Y, M, R(X \otimes Y) \overline{\otimes} M} (T(\tilde{m}_{X,Y,M}^{-1}) \theta m_{X,Y,T(M)}),$$

where $\theta = \xi_{X, R(Y) \overline{\otimes} M, R(X) \overline{\otimes} (R(Y) \overline{\otimes} M)} (\text{id}) (\text{id}_X \otimes \xi_{Y, M, R(Y) \overline{\otimes} M}^{-1} (\text{id}))$. By assumption θ is invertible, thus $T(\tilde{m}_{X,Y,M}^{-1}) \theta m_{X,Y,T(M)}$ is invertible, hence $I_{X,Y}$ is invertible. \square

Remark 2.14. For any $X, Y \in \mathcal{C}$ $I_{X,Y} = \text{id}_{X \otimes Y}$ if and only if the functor $(T, c) : \mathcal{M}' \rightarrow \mathcal{M}$ is a module functor, where $c_{X,M} : T(R(X) \overline{\otimes} M) \rightarrow X \overline{\otimes} T(M)$ is defined by $c_{X,M} = (\xi_{X, M, R(X) \overline{\otimes} M}^{-1} (\text{id}))^{-1}$ for any $X \in \mathcal{C}$, $M \in \mathcal{M}'$.

Lemma 2.15. *Let T be a dynamical adjoint to R and let J be the dynamical twist associated. Then the functor $(T, c) : (\mathcal{M}')^{(J)} \rightarrow \mathcal{M}$ is a module functor, where $c_{X,M} = (\xi_{X, M, R(X) \overline{\otimes} M}^{-1} (\text{id}))^{-1}$ for any $X \in \mathcal{C}$, $M \in \mathcal{M}'$. \square*

3. DYNAMICAL TWISTS OVER HOPF ALGEBRAS

In this section we shall focus our attention to the computation of dynamical twists for a dynamical extension of the category $\text{Rep}(H)$, where H is a finite-dimensional Hopf algebras, and we shall give explicit examples.

Hereafter we shall denote by H a finite-dimensional Hopf algebra. Let S be a left H -comodule algebra.

Definition 3.1. A *dynamical twist with base S* for H is an invertible element $J \in H \otimes_{\mathbb{k}} H \otimes_{\mathbb{k}} S$ such that

$$(3.1) \quad J^1 s_{(-2)} \otimes J^2 s_{(-1)} \otimes J^3 s_{(0)} = s_{(-2)} J^1 \otimes s_{(-1)} J^2 \otimes s_{(0)} J^3 \quad \text{for all } s \in S,$$

$$(3.2) \quad j^1_{(1)} J^1 \otimes j^1_{(2)} J^2 \otimes j^2 J^3 \otimes j^3_{(-1)} \otimes j^3 J^3_{(0)} = j^1 \otimes j^2_{(1)} J^1 \otimes j^2_{(2)} J^2 \otimes j^3 J^3,$$

$$(3.3) \quad \langle \varepsilon, J^1 \rangle J^2 \otimes J^3 = 1_H \otimes 1_K = \langle \varepsilon, J^2 \rangle J^1 \otimes J^3.$$

Here we use the notation $J = J^1 \otimes J^2 \otimes J^3 = j^1 \otimes j^2 \otimes j^3$ avoiding the summation symbol.

Remark 3.2. Definition 3.1 coincides with the definition of dynamical twist over an Abelian group given in [EN1], [EN2].

Definition 3.3. Two dynamical twists for H over S , J and J' , are said to be *gauge equivalent* if there exists an invertible element $t \in H \otimes S$ such that

$$\langle \varepsilon, t^1 \rangle t^2 = 1,$$

$$t^1_{(1)} J^1 \otimes t^1_{(2)} J^2 \otimes t^2 J^3 = j^1 t^1 \otimes j^2 T^1 t^2_{(-1)} \otimes j^3 T^2 t^2_{(0)}.$$

Here $J = J^1 \otimes J^2 \otimes J^3$, $J' = j^1 \otimes j^2 \otimes j^3$, $t = t^1 \otimes t^2 = T^1 \otimes T^2$.

The following Lemma is a straightforward consequence of the definitions.

Lemma 3.4. *Let J be a dynamical twist with base S for H . For any $X, Y \in \text{Rep}(H)$, $M \in {}_S\mathcal{M}$ define $\mathcal{J}_{X,Y,M} : X \otimes_{\mathbb{k}} Y \otimes_{\mathbb{k}} M \rightarrow X \otimes_{\mathbb{k}} Y \otimes_{\mathbb{k}} M$ by*

$$\mathcal{J}_{X,Y,M}(x \otimes y \otimes m) = J^1 \cdot x \otimes J^2 \cdot y \otimes J^3 \cdot m,$$

for all $x \in X, y \in Y, m \in M$. Then \mathcal{J} is a dynamical twist for the dynamical extension ${}_S\mathcal{M} \rtimes \text{Rep}(H)$.

Moreover, any dynamical twist for the extension ${}_S\mathcal{M} \rtimes \text{Rep}(H)$ comes from a dynamical twist with base S over H . □

Lemma 3.5. *If J and J' are gauge equivalent dynamical twists for H with base S then ${}_S\mathcal{M}^{(J)} \simeq {}_S\mathcal{M}^{(J')}$ as module categories over $\text{Rep}(H)$.*

Proof. Let $(F, c) : {}_S\mathcal{M}^{(J)} \rightarrow {}_S\mathcal{M}^{(J')}$ be the functor defined as follows. For any $M \in {}_S\mathcal{M}$ and $X \in \text{Rep}(H)$, $F(M) = M$ and $c_{X,M} : X \otimes_{\mathbb{k}} M \rightarrow X \otimes_{\mathbb{k}} M$, $c_{X,M}(x \otimes m) = t^1 \cdot x \otimes t^2 \cdot m$, for any $x \in X, m \in M$. It is easy to verify that (F, c) gives an equivalence of module categories. □

3.1. Dynamical twists and Hopf Galois extensions. Let S be a left H -comodule algebra. For any dynamical twist with base S for H there is associated a H -Galois extension with coinvariants S .

Set $B = H^* \otimes_{\mathbb{k}} S$. The coproduct of H^* endows B with a right $H^{*\text{cop}}$ -comodule structure, that is $\delta : B \rightarrow B \otimes_{\mathbb{k}} H^*$, $\delta(\alpha \otimes s) = \alpha_{(2)} \otimes s \otimes \alpha_{(1)}$, for all $\alpha \in H^*, s \in S$. Clearly $B^{\text{co } H^{*\text{cop}}} = S$.

If $J \in H \otimes_{\mathbb{k}} H \otimes_{\mathbb{k}} S$ we endowed B with the following product:

$$(3.4) \quad (\alpha \otimes k)(\beta \otimes s) = (J^1 \rightharpoonup \alpha)(J^2 k_{(-1)} \rightharpoonup \beta) \otimes J^3 k_{(0)} s,$$

for all $\alpha, \beta \in H^*, k, s \in S$.

Proposition 3.6. *Assume that $J \in H \otimes_{\mathbb{k}} H \otimes_{\mathbb{k}} S$ is a dynamical twist with base S , then $B \supset S$ is a $H^{*\text{cop}}$ -Hopf Galois extension.*

Proof. First we prove that B is an associative algebra with the product described in (3.4). Let $\alpha, \beta, \gamma \in H^*, k, s, r \in S$, then

$$\begin{aligned} ((\alpha \otimes k)(\beta \otimes s))(\gamma \otimes r) &= ((J^1 \rightharpoonup \alpha)(J^2 k_{(-1)} \rightharpoonup \beta) \otimes J^3 k_{(0)} s)(\gamma \otimes r) \\ &= (j^1_{(1)} J^1 \rightharpoonup \alpha)(j^1_{(2)} J^2 k_{(-2)} \rightharpoonup \beta)(j^2 J^3_{(-1)} k_{(-1)} s_{(-1)} \rightharpoonup \gamma) \otimes j^3 J^3_{(0)} k_{(0)} s_{(0)} r. \end{aligned}$$

On the other hand

$$\begin{aligned} (\alpha \otimes k)((\beta \otimes s)(\gamma \otimes r)) &= (\alpha \otimes k)((J^1 \rightharpoonup \beta)(J^2 s_{(-1)} \rightharpoonup \gamma)) \otimes J^3 s_{(0)} r \\ &= (j^1 \rightharpoonup \alpha)(j^2_{(1)} k_{(-1)} J^1 \rightharpoonup \beta)(j^2_{(2)} k_{(-2)} J^2 s_{(-1)} \rightharpoonup \gamma) \otimes j^3 k_{(0)} J^3 s_{(0)} r \\ &= (j^1 \rightharpoonup \alpha)(j^2_{(1)} J^1 k_{(-1)} \rightharpoonup \beta)(j^2_{(2)} J^2 k_{(-2)} s_{(-1)} \rightharpoonup \gamma) \otimes j^3 J^3 k_{(0)} s_{(0)} r. \end{aligned}$$

The last equality follows by (3.1). From (3.2) follows that

$$((\alpha \otimes k)(\beta \otimes s))(\gamma \otimes r) = (\alpha \otimes k)((\beta \otimes s)(\gamma \otimes r)).$$

The proof that B is a H^{cop} -comodule algebra is straightforward. Let $\text{can} : B \otimes_S B \rightarrow B \otimes_{\mathbb{k}} H^{\text{cop}}$ be the canonical map; that is $\text{can}(a \otimes b) = ab_{(0)} \otimes b_{(1)}$, for all $a, b \in B$. In this case,

$$\text{can}((\alpha \otimes k) \otimes (\beta \otimes s)) = (J^1 \rightharpoonup \alpha)(J^2 k_{(-1)} \rightharpoonup \beta_{(2)}) \otimes J^3 k_{(0)} s \otimes \beta_{(1)},$$

for all $\alpha, \beta \in H^*$, $k, s \in S$. It is easy to see that the inverse of can is given by $\text{can}^{-1} : B \otimes_{\mathbb{k}} H^{\text{cop}} \rightarrow B \otimes_S B$,

$$\text{can}^{-1}(\gamma \otimes r \otimes \beta) = J^{-1} \rightharpoonup (\gamma \mathcal{S}(\beta_{(2)})) \otimes 1 \otimes (J^{-2} \rightharpoonup \beta_{(1)}) \otimes J^{-3} r,$$

for all $\gamma, \beta \in H^*$, $r \in S$. Thus $B \supset S$ is a H^{cop} -Galois extension. \square

3.2. Dynamical twists coming from dynamical datum.

In this subsection we shall give a method for constructing dynamical twists with base A , where $A \subset H$ is a Hopf subalgebra. This construction is based on the same ideas contained in [EN1], [M] without assuming commutativity nor cocommutativity of the base of the dynamical twist.

The following definition generalizes [EN1, Def. 4.5], see also [M, Def. 3.8].

Definition 3.7. A *dynamical datum* for H over A is a pair (K, T) where K is a left H -comodule algebra H -simple, with trivial coinvariants, $T : \text{Rep}(A) \rightarrow {}_K \mathcal{M}$ is a functor such that there are natural H -module isomorphisms

$$(3.5) \quad \omega_{VW} : \text{Stab}_K(T(V), T(W)) \xrightarrow{\cong} (\text{Ind}_A^H (V \otimes_{\mathbb{k}} W^*))^*,$$

for any $V, W \in \text{Rep}(A)$. We shall further assume that

$$(3.6) \quad \omega_{VV}(1)(\overline{h \otimes v \otimes f}) = \langle \varepsilon, h \rangle \langle f, v \rangle,$$

for all $h \in H$, $v \in V$, $f \in V^*$.

Two dynamical data (K, T) and (S, U) are *equivalent* if and only if there exists an element $P \in {}^H \mathcal{M}_K$ such that $S \simeq \text{End}_K(P_K)$ as H -comodule algebras, and there exists a family of natural K -module isomorphisms

$$\phi_V : P \otimes_K T(V) \xrightarrow{\cong} U(V),$$

for all $V \in \text{Rep}(A)$.

Remark 3.8. If (K, T) is a dynamical datum, then for any $V \in \text{Rep}(A)$

$$(3.7) \quad \dim A (\dim T(V))^2 = \dim K (\dim V)^2.$$

This formula follows straightforward from the definition of dynamical datum and formula (1.2).

Denote by $R : \text{Rep}(H) \rightarrow \text{Rep}(A)$ the restriction functor.

Proposition 3.9. *If (K, T) is a dynamical datum then T is a dynamical adjoint to R .*

Proof. Category ${}_K\mathcal{M}$ is a module category over $\text{Rep}(H)$ as explained in section 1.3. The category $\text{Rep}(A)$ is a module category over itself. Let $V, W \in {}_K\mathcal{M}$ and $X \in \text{Rep}(H)$. Then

$$\begin{aligned} \text{Hom}_K(X \otimes_{\mathbb{k}} T(V), T(W)) &\simeq \text{Hom}_H(X, \text{Stab}_K(T(V), T(W))) \simeq \\ &\text{Hom}_H(X, (\text{Ind}_A^H(V \otimes_{\mathbb{k}} W^*))) \simeq \text{Hom}_H(\text{Ind}_A^H(V \otimes_{\mathbb{k}} W^*), X^*) \simeq \\ &\simeq \text{Hom}_A(V \otimes_{\mathbb{k}} W^*, R(X^*)) \simeq \text{Hom}_A(R(X), W \otimes_{\mathbb{k}} V^*) \simeq \\ &\simeq \text{Hom}_A(R(X) \otimes_{\mathbb{k}} V, W). \end{aligned}$$

The first isomorphism follows from Proposition 1.2 (3), the second because (K, T) is a dynamical datum and the fourth isomorphism is Frobenius reciprocity.

Let us denote by $\xi : \text{Hom}_K(X \otimes_{\mathbb{k}} T(V), T(W)) \rightarrow \text{Hom}_A(R(X) \otimes_{\mathbb{k}} V, W)$ the composition of the above isomorphisms. Is clear that ξ satisfies (2.13) since we requested that the isomorphisms ω_{VW} satisfy (3.6). \square

Definition 3.10. For any dynamical datum (K, T) we shall denote by J_T the associated dynamical twists for the Hopf algebra H with base A according to Theorem 2.13.

In the following we shall prove that the construction of the dynamical twist does not depend on the equivalence class of the dynamical datum.

Proposition 3.11. *Let (K, T) and (S, U) be two equivalent dynamical data over A . Then J_T is gauge equivalent to J_S .*

For the proof we will need first some technical results. From the hypothesis we know that there exists $P \in {}^H\mathcal{M}_K$ such that $S \simeq \text{End}_K(P_K)$ as H -module algebras and natural isomorphisms

$$\phi_V : P \otimes_K T(V) \xrightarrow{\simeq} U(V),$$

for all $V \in \text{Rep}(A)$. For any $X \in \text{Rep}(H)$, $V \in \text{Rep}(A)$ let us denote by

$$\xi : \text{Hom}_K(X \otimes_{\mathbb{k}} T(V), T(W)) \rightarrow \text{Hom}_A(R(X) \otimes_{\mathbb{k}} V, W),$$

$$\zeta : \text{Hom}_S(X \otimes_{\mathbb{k}} U(V), U(W)) \rightarrow \text{Hom}_A(R(X) \otimes_{\mathbb{k}} V, W),$$

the family of natural isomorphisms constructed in the proof of Proposition 3.9.

For any $X \in \text{Rep}(H)$, $V \in \text{Rep}(A)$, $M \in {}_K\mathcal{M}$ let us define

$$\theta_{X,M} : X \otimes_{\mathbb{k}} (P \otimes_K M) \rightarrow P \otimes_K (X \otimes_{\mathbb{k}} M)$$

as follows. For any $x \in X, p \in P, m \in M$

$$\theta_{X,M}(x \otimes (p \otimes m)) = p_{(0)} \otimes \mathcal{S}^{-1}(p_{(-1)}) \cdot x \otimes m.$$

Let us also define

$$\sigma_{X,V} : X \otimes_{\mathbb{k}} U(V) \rightarrow U(X \otimes_{\mathbb{k}} V)$$

as the composition

$$\sigma_{X,V} = \phi_{X \otimes_{\mathbb{k}} V}(\text{id}_P \otimes \xi_{X,V,X \otimes_{\mathbb{k}} V}^{-1}(\text{id})) \theta_{X,T(V)}(\text{id}_X \otimes \phi_V^{-1}).$$

Clearly, $\sigma_{X,V}$ and $\theta_{X,M}$ are isomorphisms.

Lemma 3.12. *For any $X, Y \in \text{Rep}(H)$, $V, W \in \text{Rep}(A)$, $M, N \in {}_K\mathcal{M}$ and any morphisms $f : X \rightarrow Y$, $\beta : V \rightarrow W$, $g : M \rightarrow N$*

$$(3.8) \quad \theta_{X \otimes_{\mathbb{k}} Y, M} = \theta_{X, Y \otimes_{\mathbb{k}} M}(\text{id}_X \otimes \theta_{X, M}),$$

$$(3.9) \quad (\text{id}_P \otimes \text{id}_X \otimes g) \theta_{X, M} = \theta_{X, N}(\text{id}_X \otimes \text{id}_P \otimes g),$$

$$(3.10) \quad (\text{id}_P \otimes f \otimes \text{id}_M) \theta_{X, M} = \theta_{Y, M}(f \otimes \text{id}_P \otimes \text{id}_M),$$

$$(3.11) \quad \sigma_{Y, V}(f \otimes \text{id}_V) = U(f \otimes \text{id}_V) \sigma_{X, V},$$

$$(3.12) \quad \sigma_{X, W}(\text{id}_X \otimes \beta) = U(\text{id}_X \otimes \beta) \sigma_{X, V}.$$

Proof. Equations (3.8), (3.9) and (3.10) are straightforward. By definition

$$\begin{aligned} \sigma_{Y, V}(f \otimes \text{id}_{U(V)}) &= \phi_{Y \otimes_{\mathbb{k}} V}(\text{id}_P \otimes \xi_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})) \theta_{Y, T(V)}(\text{id}_Y \otimes \phi_V^{-1})(f \otimes \text{id}_V) \\ &= \phi_{Y \otimes_{\mathbb{k}} V}(\text{id}_P \otimes \xi_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})) (\text{id}_P \otimes f \otimes \text{id}_{T(V)}) \theta_{X, T(V)}(\text{id}_Y \otimes \phi_V^{-1}) \\ &= \phi_{Y \otimes_{\mathbb{k}} V}(\text{id}_P \otimes \xi_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(f \otimes \text{id}_V)) \theta_{X, T(V)}(\text{id}_Y \otimes \phi_V^{-1}) \\ &= \phi_{Y \otimes_{\mathbb{k}} V}(\text{id}_P \otimes T(f \otimes \text{id}_V) \xi_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})) \theta_{X, T(V)}(\text{id}_Y \otimes \phi_V^{-1}) \\ &= U(\text{id}_X \otimes \beta) \sigma_{X, V}. \end{aligned}$$

The second equality follows by the naturality of ϕ and (3.10), the third equality follows by (2.16), the fourth by (2.14) and the fifth again by the naturality of ϕ . Equation 3.12 follows in a similar way. \square

For any $X \in \text{Rep}(H)$, $V \in \text{Rep}(A)$ set $t_{X, V} : X \otimes_{\mathbb{k}} V \rightarrow X \otimes_{\mathbb{k}} V$ the isomorphism of A -modules defined as

$$t_{X, V} = \zeta_{X, V, X \otimes_{\mathbb{k}} V}(\sigma_{X, V}).$$

Lemma 3.13. *The maps $t_{X, V}$ are natural isomorphisms. In particular there exists an invertible element $t = t^1 \otimes t^2 \in H \otimes_{\mathbb{k}} A$ such that for any $x \in X$, $v \in V$, $t_{X, V}(x \otimes v) = t^1 \cdot x \otimes t^2 \cdot v$.*

Proof. Let $X, Y \in \text{Rep}(H)$ and let $f : X \rightarrow Y$ be any morphism.

$$\begin{aligned} t_{Y, V}(f \otimes \text{id}_V) &= \zeta_{Y, V, Y \otimes_{\mathbb{k}} V}(\sigma_{Y, V})(f \otimes \text{id}_V) = \zeta_{X, V, Y \otimes_{\mathbb{k}} V}(\sigma_{Y, V}(f \otimes \text{id}_{U(V)})) \\ &= \zeta_{X, V, Y \otimes_{\mathbb{k}} V}(U(f \otimes \text{id}_V) \sigma_{X, V}) = (f \otimes \text{id}_V) \zeta_{X, V, X \otimes_{\mathbb{k}} V}(\sigma_{X, V}) \\ &= (f \otimes \text{id}_V) t_{X, V}. \end{aligned}$$

The second equality follows from (2.16), the third by (3.11) and the fourth one by (2.14). The naturality of t in the second variable follows in an analogous way using (3.12). \square

Proof of Proposition 3.11. Let $X, Y \in \text{Rep}(H)$ and $V \in \text{Rep}(A)$. Let

$$I_{X,Y,V}, \tilde{I}_{X,Y,V} : (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V \rightarrow (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V$$

be the isomorphisms defined as

$$I_{X,Y,V}(x \otimes y \otimes v) = J_K^{-1} \cdot x \otimes J_K^{-2} \cdot y \otimes J_K^{-3} \cdot v,$$

$$\tilde{I}_{X,Y,V}(x \otimes y \otimes v) = J_S^{-1} \cdot x \otimes J_S^{-2} \cdot y \otimes J_S^{-3} \cdot v,$$

for any $x \in X, y \in Y, v \in V$. In another words, the family of natural isomorphisms I and \tilde{I} are given by

$$I_{X,Y,V} = \xi_{X \otimes_{\mathbb{k}} Y, V, (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}(\xi_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(\text{id}) (\text{id}_X \otimes \xi_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})))$$

and

$$\tilde{I}_{X,Y,V} = \zeta_{X \otimes_{\mathbb{k}} Y, V, (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}(\zeta_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(\text{id}) (\text{id}_X \otimes \zeta_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id}))).$$

We shall prove that

$$(3.13) \quad I_{X,Y,V} t_{X \otimes_{\mathbb{k}} Y, V} = t_{X, Y \otimes_{\mathbb{k}} V}(\text{id}_X \otimes t_{Y, V}) \tilde{I}_{X,Y,V}.$$

Using 2.14 we obtain that

$$\begin{aligned} \zeta_{X \otimes_{\mathbb{k}} Y, V, (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}^{-1}(I_{X,Y,V} t_{X \otimes_{\mathbb{k}} Y, V}) &= U(I_{X,Y,V}) \zeta_{X \otimes_{\mathbb{k}} Y, V, (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}^{-1}(t_{X \otimes_{\mathbb{k}} Y, V}) \\ &= U(I_{X,Y,V}) \sigma_{X \otimes_{\mathbb{k}} Y, V}. \end{aligned}$$

Now, using the naturality of ϕ and the naturality of ξ (2.14) we obtain that $U(I_{X,Y,V}) \sigma_{X \otimes_{\mathbb{k}} Y, V}$ is equal to

$$\phi_{(X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}(\text{id}_P \otimes \xi_{X \otimes_{\mathbb{k}} Y, V, (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}^{-1}(T(I_{X,Y,V}))) \theta_{X \otimes_{\mathbb{k}} Y, T(V)}(\text{id}_{X \otimes_{\mathbb{k}} Y} \otimes \phi_V^{-1}).$$

By definition of the isomorphism $I_{X,Y,V}$ and (3.8) this last expression equals to

$$\begin{aligned} \phi_{(X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}(\text{id}_P \otimes \xi_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(\text{id})(\text{id}_X \otimes \xi_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id}))) \theta_{X, Y \otimes_{\mathbb{k}} T(V)} \\ (\text{id}_X \otimes \theta_{X, T(V)}) (\text{id}_{X \otimes_{\mathbb{k}} Y} \otimes \phi_V^{-1}), \end{aligned}$$

and using (3.9) equals to

$$\begin{aligned} \phi_{(X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}(\text{id}_P \otimes \xi_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(\text{id})) \theta_{X, T(Y \otimes_{\mathbb{k}} V)} \\ (\text{id}_{X \otimes_{\mathbb{k}} P} \otimes \xi_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})) (\text{id}_X \otimes \theta_{X, T(V)}) (\text{id}_{X \otimes_{\mathbb{k}} Y} \otimes \phi_V^{-1}), \end{aligned}$$

which is equal to

$$\sigma_{X, Y \otimes_{\mathbb{k}} V}(\text{id}_X \otimes \sigma_{Y, V}).$$

From (2.14) follows that $\zeta_{X \otimes_{\mathbb{k}} Y, V, (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} V}^{-1}(t_{X, Y \otimes_{\mathbb{k}} V}(\text{id}_X \otimes t_{Y, V}) \tilde{I}_{X, Y, V})$ equals to

$$\begin{aligned}
&= T(t_{X, Y \otimes_{\mathbb{k}} V}) T(\text{id}_X \otimes t_{Y, V}) \zeta_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(\text{id})(\text{id}_X \otimes \zeta_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})) \\
&= T(t_{X, Y \otimes_{\mathbb{k}} V}) \zeta_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(\text{id}_X \otimes t_{Y, V})(\text{id}_X \otimes \zeta_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})) \\
&= T(t_{X, Y \otimes_{\mathbb{k}} V}) \zeta_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(\text{id})(\text{id}_X \otimes T(t_{Y, V}) \zeta_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(\text{id})) \\
&= \zeta_{X, Y \otimes_{\mathbb{k}} V, X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} V)}^{-1}(t_{X, Y \otimes_{\mathbb{k}} V})(\text{id}_X \otimes \zeta_{Y, V, Y \otimes_{\mathbb{k}} V}^{-1}(t_{Y, V})) \\
&= \sigma_{X, Y \otimes_{\mathbb{k}} V}(\text{id}_X \otimes \sigma_{Y, V}).
\end{aligned}$$

The second equality follows from (2.14), the third by (2.15) and the fourth again from (2.14). Thus, we have proven equation (3.13). Follows immediately from (3.13) that the element t^{-1} is a gauge equivalence for J_K and J_S . \square

There is a reciprocate construction, that is, for any dynamical twist over A we can associate a dynamical datum as follows.

Let J be a dynamical twist for the dynamical extension ${}_A \mathcal{M} \rtimes \text{Rep}(H)$. By Proposition 2.5 the category ${}_A \mathcal{M}^{(J)}$ is an exact indecomposable module category over $\text{Rep}(H)$, therefore by Theorem 1.3 there exists an H -simple H -comodule algebra K such that $K^{\text{co}H}$ and ${}_A \mathcal{M}^{(J)} \simeq {}_K \mathcal{M}$ as module categories over $\text{Rep}(H)$. Let us denote by $T : {}_A \mathcal{M}^{(J)} \rightarrow {}_K \mathcal{M}$ such equivalence.

Proposition 3.14. *The pair (K, T) as above is a dynamical datum.*

Proof. The proof is entirely analogous to the proof of [M, Prop. 3.18]. For completeness we write the proof. Let $V, W \in \text{Rep}(A)$, $X \in \text{Rep}(H)$ then

$$\begin{aligned}
&\text{Hom}_H(X, \text{Stab}_K(T(V), T(W))) \simeq \text{Hom}_K(X \otimes_{\mathbb{k}} T(V), T(W)) \simeq \\
&\simeq \text{Hom}_K(T(X \otimes_{\mathbb{k}} V), T(W)) \simeq \text{Hom}_A(X \otimes_{\mathbb{k}} V, W) \simeq \text{Hom}_A(R(X), W \otimes_{\mathbb{k}} V^*) \\
&\simeq \text{Hom}_A(V \otimes_{\mathbb{k}} W^*, R(X^*)) \simeq \text{Hom}_H(\text{Ind}_A^H(V \otimes_{\mathbb{k}} W^*), X^*) \simeq \\
&\simeq \text{Hom}_H(X, \text{Ind}_A^H(V \otimes_{\mathbb{k}} W^*)^*).
\end{aligned}$$

The first isomorphism is a consequence of Proposition 1.2 (3), the sixth isomorphism is Frobenius reciprocity. Thus, the statement follows from Yoneda's Lemma. \square

The construction of the dynamical datum from a dynamical twist is not canonical but it does not depend on the gauge equivalence class of the dynamical twist.

Proposition 3.15. *Let J, J' two gauge equivalent dynamical twists and let (K, T) and (S, U) the dynamical data associated as in Proposition 3.14. Then (K, T) is equivalent to (S, U) .*

Proof. By construction the functors $T : {}_A\mathcal{M}^{(J)} \rightarrow {}_K\mathcal{M}$ and $U : {}_A\mathcal{M}^{(J')} \rightarrow {}_S\mathcal{M}$ are equivalences of module categories over $\text{Rep}(H)$. By Lemma 3.5 the categories ${}_A\mathcal{M}^{(J)}$ and ${}_A\mathcal{M}^{(J')}$ are equivalent. Let $G : {}_K\mathcal{M} \rightarrow {}_A\mathcal{M}^{(J)}$ be the inverse of T . The functor $U \circ G : {}_K\mathcal{M} \rightarrow {}_S\mathcal{M}$ is an equivalence of module categories, thus Proposition 1.4 implies that there exists an object $P \in {}^H\mathcal{M}_K$ such that $S \simeq \text{End}_K(P_K)$ as H -module algebras and natural isomorphisms $U(G(M)) \simeq P \otimes_K M$ for all $M \in {}_K\mathcal{M}$. In particular there are natural isomorphisms

$$U(V) \simeq U(G(T(V))) \simeq P \otimes_K T(V),$$

for all $V \in \text{Rep}(A)$. □

Remark 3.16. It would be interesting to know, for a fixed Hopf algebra H , which module categories are equivalent to ${}_A\mathcal{M}^{(J)}$ for some Hopf subalgebra A and a dynamical twist J with base A .

3.3. Some examples. We shall give concrete examples of dynamical datum and explicit computations of the corresponding dynamical twist.

3.3.1. K is an A -Galois extension.

Let us assume that K is an A -Galois extension with trivial coinvariants. Let us denote by $\gamma : A \rightarrow K \otimes_{\mathbb{k}} K$ the map

$$\gamma(a) = \text{can}^{-1}(a \otimes 1) = a^{[1]} \otimes a^{[2]},$$

for all $a \in A$.

Let us assume that $T : \text{Rep}(A) \rightarrow {}_K\mathcal{M}$ is a functor such that for any $V, W \in \text{Rep}(A)$ there are A -module isomorphisms

$$T(W) \otimes_{\mathbb{k}} T(V)^* \xrightarrow{\simeq} W \otimes_{\mathbb{k}} V^*.$$

The A -module structure on $T(W) \otimes_{\mathbb{k}} T(V)^*$ is given as follows. If $w \in T(W)$, $f \in T(V)^*$ and $a \in A$, then

$$a \cdot (w \otimes f) = a^{[1]} \cdot w \otimes f \cdot a^{[2]},$$

where $(f \cdot k)(v) = f(k \cdot v)$, for any $k \in K, v \in V$. This is a well defined action, see [AM, Lemma 2.21].

Lemma 3.17. *Under the above conditions (K, T) is a dynamical datum.*

Proof. For any $V, W \in \text{Rep}(A)$ we have that

$$\begin{aligned} \text{Stab}_K(T(V), T(W)) &\simeq \text{Hom}_A(H, \text{Hom}_{\mathbb{k}}(T(V), T(W))) \\ &\simeq \text{Hom}_A(H, T(W) \otimes_{\mathbb{k}} T(V)^*) \simeq \text{Hom}_A(H, W \otimes_{\mathbb{k}} V^*) \\ &\simeq (\text{Ind}_A^H V \otimes_{\mathbb{k}} W^*)^*. \end{aligned}$$

The first isomorphism follows from (1.3), and the last is Lemma 1.1. □

3.3.2. Dynamical twists over $\mathbb{A}(G, \chi, g)$.

Let G be a finite group, $g \in Z(G)$ and $\chi : G \rightarrow \mathbb{C}^\times$ a character such that $n = |g| = |\chi(g)|$ and $\chi^n = 1$.

Let us denote by $H = \mathbb{A}(G, \chi, g)$ the algebra generated by x, g subject to the relations: $x^n = 0$, $xh = \chi(h)hx$ for all $h \in G$. The algebra $\mathbb{A}(G, \chi, g)$ has a Hopf algebra structure as follows:

$$\Delta(x) = 1 \otimes x + x \otimes g, \quad \Delta(f) = f \otimes f, \quad \varepsilon(x) = 0, \quad \varepsilon(f) = 1,$$

for all $f \in G$.

These Hopf algebras are a special class of monomial Hopf algebras. See [CYZ].

Let $\lambda \in \mathbb{C}^\times$ and let $F \subseteq G$ be a subgroup such that $g \in F$. In this case $\mathbb{A}(F, \chi, g) \subseteq \mathbb{A}(G, \chi, g)$ is a Hopf subalgebra. Let us denote by $\mathcal{A}(F, \lambda)$ the algebra generated by elements $y, e_h : h \in F$ subject to the relations

$$y^n = \lambda 1, \quad e_h e_f = e_{hf}, \quad y e_h = \chi(h) e_h y,$$

for all $h, f \in F$.

Lemma 3.18. *Let us denote $\delta : \mathcal{A}(F, \lambda) \rightarrow \mathbb{A}(G, \chi, g) \otimes \mathcal{A}(F, \lambda)$, the map given by*

$$\delta(y) = g^{-1} \otimes y - x g^{-1} \otimes 1, \quad \delta(e_h) = h \otimes e_h,$$

for all $h \in F$. Then $\mathcal{A}(F, \lambda)$ is a left $\mathbb{A}(G, \chi, g)$ -comodule algebra with trivial coinvariants. Moreover, $\mathcal{A}(F, \lambda)$ is a $\mathbb{A}(F, \chi, g)$ -Galois extension.

Proof. Straightforward. \square

Let $B \subseteq F$ be a subgroup such that $B \cap \langle g \rangle = \{1\}$. Let A be the group algebra of the group B .

Let $T : \text{Rep}(A) \rightarrow \mathcal{A}(F, \lambda)\text{-}\mathcal{M}$ be the functor defined as follows. For any left A -module V , $T(V) = \bigoplus_{i=0}^{n-1} \mathbb{C} v_i \otimes V$. The action of $\mathcal{A}(F, \lambda)$ on $T(V)$ is the following. For any $v \in V$, $i = 0 \dots n-1$, $h \in F$

$$\begin{aligned} y \cdot (v_i \otimes v) &= \mu v_{i-1} \otimes v, & y \cdot (v_0 \otimes v) &= \mu v_{n-1} \otimes v \\ g \cdot (v_i \otimes v) &= \chi^i(g) v_i \otimes v, & e_h \cdot (v_i \otimes v) &= \chi^i(h) v_i \otimes h \cdot v. \end{aligned}$$

Here $\mu^n = \lambda$ is a fixed n -th root of λ .

Proposition 3.19. *The pair $(\mathcal{A}(F, \lambda), T)$ is a dynamical datum for H over A .*

Proof. Let $V, W \in \text{Rep}(A)$. Since $\mathcal{A}(F, \lambda)$ is $\mathbb{A}(F, \chi, g)$ -Galois, using (1.3), we obtain that

$$\text{Stab}_{\mathcal{A}(F, \lambda)}(T(V), T(W)) \simeq \text{Hom}_{\mathbb{A}(F, \chi, g)}(\mathbb{A}(G, \chi, g), \text{Hom}(T(V), T(W))).$$

Also, by Lemma 1.1, we have that

$$(\text{Ind}_{\mathbb{C}B}^{\mathbb{A}(G, \chi, g)} V \otimes W^*)^* \simeq \text{Hom}_{\mathbb{C}B}(\mathbb{A}(G, \chi, g), \text{Hom}(V, W)).$$

Thus, the proof will end if we prove that $\text{Hom}_{\mathbb{C}B}(\mathbb{A}(G, \chi, g), \text{Hom}(V, W))$ is isomorphic to $\text{Hom}_{\mathbb{A}(F, \chi, g)}(\mathbb{A}(G, \chi, g), \text{Hom}(T(V), T(W)))$.

Let $G = \bigcup_l F c_l$, $F = \bigcup_{j=0}^{n-1} B g^j$ be right coset decompositions of G and F . Then the algebra $\mathbb{A}(G, \chi, g)$ has a basis consisting of elements $\{B g^j x^i c_l\}_{j,i,l}$.

Let us define the maps

$$\phi : \text{Hom}_{\mathbb{C}B}(H, \text{Hom}(V, W)) \longrightarrow \text{Hom}_{\mathbb{A}(F, \chi, g)}(H, \text{Hom}(T(V), T(W)))$$

and

$$\psi : \text{Hom}_{\mathbb{A}(F, \chi, g)}(H, \text{Hom}(T(V), T(W))) \longrightarrow \text{Hom}_{\mathbb{C}B}(H, \text{Hom}(V, W))$$

defined as follows. If $\xi \in \text{Hom}_{\mathbb{C}B}(H, \text{Hom}(V, W))$ then

$$\phi(\xi)(c_l)(v_k \otimes v) = \sum_{s=0}^{n-1} w_s \otimes \xi(b_s x^k c_l)(v),$$

$$\psi(\alpha)(g^j x^i c_l)(v) = (p_j \otimes \text{id})(\alpha(c_l)(v_i \otimes v))$$

for any $v \in V$, $k = 0 \dots n-1$. Here $p_j : \bigoplus_{i=0}^{n-1} \mathbb{C} w_i \rightarrow \mathbb{C}$, $p_j(w_i) = \delta_{ij}$.

It is immediate to verify that these two maps are well defined and they are one the inverse of each other.

□

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MATHEMATISCHES INSTITUT, UNIVERSITÄT MÜNCHEN, THERESIENSTRASSE 39,
D-80333 MÜNCHEN, GERMANY.

E-mail address: mombelli@mate.uncor.edu

URL: <http://www.mate.uncor.edu/mombelli>