MODULE CATEGORIES OVER FINITE POINTED TENSOR CATEGORIES

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Abstract. We study exact module categories over the representation categories of finite-dimensional quasi-Hopf algebras. As a consequence we classify exact module categories over some families of pointed tensor categories with cyclic group of invertible objets of order p, where p is a prime number.

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Introduction

For a given tensor category $\mathcal{C}$ a *module category* over $\mathcal{C}$, or a $\mathcal{C}$-*module*, is the categorification of the notion of module over a ring, it consist of an Abelian category $\mathcal{M}$ together with a biexact functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ satisfying natural associativity and unit axioms. A module category $\mathcal{M}$ is *exact* [EO1] if for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$ the object $P \otimes M$ is again projective.

The notion of module category has been used with profit in the theory of tensor categories, see [DGNO],[ENO1], [ENO2]. Interestingly, the notion of module categories is related with diverse areas of mathematics and mathematical physics such as subfactor theory [Oc], [BEK]; extensions of vertex algebras [KO], Calabi-Yau algebras [Gi], Hopf algebras [N], affine Hecke algebras [BO] and conformal field theory, see for example [BFS], [CS1], [CS2], [FS1], [FS2], [O1].

The classification of exact module categories over a given tensor category was undertaken by several authors:

1. When $\mathcal{C}$ is the semisimple quotient of $U_q(\mathfrak{sl}_2)$ [Oc], [KO], [EO2],
2. over the category of finite-dimensional $SL_q(2)$-comodules [O3],
3. over the tensor categories of representations of finite supergroups [EO1],
4. for any group-theoretical tensor category [O2],
5. over the Tambara-Yamagami categories [Ga2], [MM],

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6. over the Hageerup fusion categories [GS],
7. over $\text{Rep}(H)$, where H is a lifting of a quantum linear space [Mo2].

In this paper we are concerned with the classification of exact module categories over some families of finite non-semisimple pointed tensor categories that are not equivalent to the representation categories of Hopf algebras.

An object X in a tensor category is invertible if there is another object Y such that $X \otimes Y \simeq \mathbf{1} \simeq Y \otimes X$. A *pointed tensor category* is a tensor category such that every simple object is invertible. The invertible objects form a group. Pointed tensor categories with cyclic group of invertible objects were studied in [EG1], [EG2], [EG3] and later in [A].

Any finite pointed tensor category is equivalent to the representation category of a finite-dimensional quasi-Hopf algebra A. In the case when the group of invertible elements is a cyclic group G there exists an action of G on $\text{Rep}(A)$ such that the equivariantization $\text{Rep}(A)^G$ is equivalent to the representation category of a finite-dimensional pointed Hopf algebra H, see [A]. The purpose of this work is to relate module categories over $\text{Rep}(A)$ and module categories over $\text{Rep}(H)$ and whenever is possible obtain a classification of exact module categories over $\text{Rep}(A)$ assuming that we know the classification for $\text{Rep}(H)$. Module categories over any quasi-Hopf algebra are parameterized by Morita equivariant equivalence classes of comodule algebras. We would like to establish a correspondence as follows:

$$\left\{ \begin{array}{c} \text{Morita quivalence classes} \\ \text{of } H\text{-comodule algebras} \\ \text{such that } G \subseteq K_0 \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \begin{array}{c} \text{Morita equivalence classes} \\ \text{of } A\text{-comodule algebras} \\ (\mathcal{K}, \Phi_\lambda) \end{array} \right\}.$$

The contents of the paper are the following. In Section 2 we recall the notion of exact module category, the notion of tensor product of module categories over a tensor category. In Section 3 we recall the notion of G-graded tensor categories, G-actions of tensor categories and crossed products tensor categories. We also recall the G-equivariantization construction of tensor categories and module categories.

Section 4 is devoted to study comodule algebras over quasi-Hopf algebras and how they give rise to module categories. Next, in Section 5 we study the equivariantization of the representation category of a quasi-Hopf algebra and the equivariantization of comodule algebras. We describe the datum that gives rise to an action in a representation category of a comodule algebra, that we call a *crossed system* and we prove that the equivariantization of module categories are modules over a certain crossed product comodule algebra.

In Section 6.1 we recall the definition of a family of finite-dimensional basic quasi-Hopf algebras introduced by I. Angiono [A] that are denoted by $A(H, s)$, where H is a coradically graded Hopf algebra with cyclic group of group-like elements. A particular class of these quasi-Hopf algebras were introduced by S. Gelaki [Ge] and later used by Etingof and Gelaki to classify certain families of pointed tensor categories. There is an action of a group $G \subseteq G(H)$ on $\text{Rep}(A(H,s))$ such that $\text{Rep}(A(H,s))^G \simeq \text{Rep}(H)$ [A]. For any left H-comodule algebra K such that $\Bbbk G \subseteq K_0$ we construct a left $A(H,s)$-comodule algebra. We prove that in the case that $\mid G(H) \mid = p^2$, where p is a prime number, the representation category of this family of comodule algebras is big enough to contain all module categories over $\text{Rep}(A(H,s))$. We apply this result to classify module categories in the case when H is the bosonization of a quantum linear space.

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1. Preliminaries and notation

Hereafter $\Bbbk$ will denote an algebraically closed field of characteristic 0. All vector spaces and algebras will be considered over $\Bbbk$.

If H is a Hopf algebra and A is an H-comodule algebra via $\lambda : A \to H \otimes_{\Bbbk} A$, we shall say that a (right) ideal J is H-costable if $\lambda(J) \subseteq H \otimes_{\Bbbk} J$. We shall say that A is (right) H-simple, if there is no nontrivial (right) ideal H-costable in A.

If H is a finite-dimensional Hopf algebra then $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$ will denote the coradical filtration. When $H_0 \subseteq H$ is a Hopf subalgebra then the associated graded algebra $\text{gr}\, H$ is a coradically graded Hopf algebra. If (A, λ) is a left H-comodule algebra, the coradical filtration on H induces a filtration on A, given by $A_n = \lambda^{-1}(H_n \otimes_{\Bbbk} A)$ called the *Loewy filtration.*

1.1. Finite tensor categories and tensor functors.

A *tensor category over* $\Bbbk$ is a $\Bbbk$-linear Abelian rigid monoidal category. A *finite tensor category* [EO1] is a tensor category such that Hom spaces are finite-dimensional $\Bbbk$-vector spaces, all objects have finite lenght, every simple object has a projective cover and the unit object is simple.

Hereafter all tensor categories will be considered over $\Bbbk$ and every functor will be assumed to be $\Bbbk$-linear.

If $\mathcal{C}$, $\mathcal{D}$ are tensor categories, the collection $(F, \xi, \phi) : \mathcal{C} \to \mathcal{D}$ is a *tensor functor* if $F : \mathcal{C} \to \mathcal{D}$ is a functor, $\phi : F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ is an isomorphism and for any $X, Y \in \mathcal{C}$ the family of natural isomorphisms $\zeta_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ satisfies

(1.1)
$$\zeta_{X,Y\otimes Z}(\text{id}_{F(X)} \otimes \zeta_{Y,Z}) a_{F(X),F(Y),F(Z)} = F(a_{X,Y,Z}) \zeta_{X\otimes Y,Z}(\zeta_{X,Y} \otimes \text{id}_{F(Z)}),$$

$$l_{F(X)} = F(l_X) \zeta_{\mathbf{1},X}(\phi \otimes \text{id}_{F(X)}), \tag{1.2}$$

$$r_{F(X)} = F(r_X) \zeta_{X,\mathbf{1}}(\text{id}_{F(X)} \otimes \phi), \tag{1.3}$$

If $(F,\zeta),(G,\xi):\mathcal{C}\to\mathcal{D}$ are tensor functors, a *natural tensor transformation* $\gamma: F\to G$ is a natural transformation such that $\gamma_{X\otimes Y}\zeta_{X,Y} = \xi_{X,Y}(\gamma_X\otimes\gamma_Y)$ for all $X,Y\in\mathcal{C}$.

2. Module categories

A (left) *module category* over a tensor category $\mathcal{C}$ is an Abelian category $\mathcal{M}$ equipped with an exact bifunctor $\overline{\otimes}:\mathcal{C}\times\mathcal{M}\to\mathcal{M}$, that we will sometimes refer as the *action*, natural associativity and unit isomorphisms $m_{X,Y,M}:(X\otimes Y)\overline{\otimes}M\to X\otimes(Y\overline{\otimes}M)$, $\ell_M:\mathbf{1}\overline{\otimes}M\to M$ subject to natural associativity and unity axioms. See for example [EO1]. A module category $\mathcal{M}$ is *exact*, [EO1], if for any projective object $P\in\mathcal{C}$ the object $P\overline{\otimes}M$ is projective in $\mathcal{M}$ for all $M\in\mathcal{M}$. Sometimes we shall also say that $\mathcal{M}$ is a $\mathcal{C}$-module. Right module categories and bimodule categories are defined similarly.

If $\mathcal{M}$ is a left $\mathcal{C}$-module then $\mathcal{M}^{\rm op}$ is the right $\mathcal{C}$-module over the opposite Abelian category with action $\mathcal{M}^{\rm op}\times\mathcal{C}\to\mathcal{M}^{\rm op}$, $(M,X)\mapsto X^*\overline{\otimes}M$ and associativity isomorphisms $m^{\rm op}_{M,X,Y}=m_{Y^*,X^*,M}$ for all $X,Y\in\mathcal{C}, M\in\mathcal{M}$.

If $\mathcal{C},\mathcal{C}',\mathcal{E}$ are tensor categories, $\mathcal{M}$ is a $(\mathcal{C},\mathcal{E})$-bimodule category and $\mathcal{N}$ is an $(\mathcal{E},\mathcal{C}')$-bimodule category, we shall denote the tensor product over $\mathcal{E}$ by $\mathcal{M}\boxtimes_{\mathcal{E}}\mathcal{N}$. This category is a $(\mathcal{C},\mathcal{C}')$-bimodule category. For more details on the tensor product of module categories the reader is referred to [ENO3], [Gr].

A module functor between module categories $\mathcal{M}$ and $\mathcal{M}'$ over a tensor category $\mathcal{C}$ is a pair (T,c), where $T:\mathcal{M}\to\mathcal{M}'$ is a functor and $c_{X,M}: T(X\overline{\otimes}M)\to X\overline{\otimes}T(M)$ is a natural isomorphism such that for any $X,Y\in\mathcal{C}$, $M\in\mathcal{M}$:

$$(\mathrm{id}_X\otimes c_{Y,M})c_{X,Y\overline{\otimes}M}T(m_{X,Y,M}) = m_{X,Y,T(M)}\,c_{X\otimes Y,M} \tag{2.1}$$

$$\ell_{T(M)}\,c_{\mathbf{1},M}=T(\ell_M). \tag{2.2}$$

We shall use the notation $(T,c):\mathcal{M}\to\mathcal{M}'$. There is a composition of module functors: if $\mathcal{M}''$ is another module category and $(U,d):\mathcal{M}'\to\mathcal{M}''$ is another module functor then the composition

$$(U\circ T,e):\mathcal{M}\to\mathcal{M}'',\qquad \text{where } e_{X,M}=d_{X,U(M)}\circ U(c_{X,M}), \tag{2.3}$$

is also a module functor.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be module categories over $\mathcal{C}$. We denote by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)$ the category whose objects are module functors $(\mathcal{F},c)$ from $\mathcal{M}_1$ to $\mathcal{M}_2$. A morphism between $(\mathcal{F},c)$ and $(\mathcal{G},d)\in\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)$ is a natural transformation $\alpha:\mathcal{F}\to\mathcal{G}$ such that for any $X\in\mathcal{C}$, $M\in\mathcal{M}_1$:

$$d_{X,M}\alpha_{X\overline{\otimes}M}=(\mathrm{id}_X\overline{\otimes}\alpha_M)c_{X,M}. \tag{2.4}$$

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exist module functors $F:\mathcal{M}_1\to\mathcal{M}_2$ and $G:\mathcal{M}_2\to\mathcal{M}_1$ and natural isomorphisms $\mathrm{id}_{\mathcal{M}_1}\to F\circ G$, $\mathrm{id}_{\mathcal{M}_2}\to G\circ F$ that satisfy (2.4).

The direct sum of two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ is the $\Bbbk$-linear category $\mathcal{M}_1 \times \mathcal{M}_2$ with coordinate-wise module structure. A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories.

If $(F, \xi) : \mathcal{C} \to \mathcal{C}$ is a tensor functor and $(\mathcal{M}, \overline{\otimes}, m)$ is a module category over $\mathcal{C}$ we shall denote by $\mathcal{M}^F$ the module category $(\mathcal{M}, \overline{\otimes}^F, m^F)$ with the same underlying Abelian category with action and associativity isomorphisms defined by

$$X\overline{\otimes}^F M = F(X)\overline{\otimes} M, \quad m^F_{X,Y,M} = m_{F(X),F(Y),M}(\xi^{-1}_{X,Y}\overline{\otimes}\,\mathrm{id}_M),$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$.

3. Equivariantization of tensor categories

3.1. Group actions on tensor categories. We briefly recall the group actions on tensor categories and the equivariantization construction. For more details the reader is referred to [DGNO].

Let $\mathcal{C}$ be a tensor category and let $\underline{\mathrm{Aut}_\otimes(\mathcal{C})}$ be the monoidal category of tensor auto-equivalences of $\mathcal{C}$, arrows are tensor natural isomorphisms and tensor product the composition of monoidal functors. We shall denote by $\mathrm{Aut}_\otimes(\mathcal{C})$ the group of isomorphisms classes of tensor auto-equivalences of $\mathcal{C}$, with the multiplication induced by the composition, *i.e.* $[F][F'] = [F \circ F']$.

For any group G we shall denote by $\underline{G}$ the monoidal category where objects are elements of G and tensor product is given by the product of G. An action of the group G over a $\mathcal{C}$, is a monoidal functor $* : \underline{G} \to \underline{\mathrm{Aut}_\otimes(\mathcal{C})}$. In another words for any $\sigma \in G$ there is a tensor functor $(F_\sigma, \zeta_\sigma) : \mathcal{C} \to \mathcal{C}$, and for any $\sigma, \tau \in G$, there are natural tensor isomorphisms $\gamma_{\sigma,\tau} : F_\sigma \circ F_\tau \to F_{\sigma\tau}$.

3.2. *G*-graded tensor categories. Let G be a group and $\mathcal{C}$ be a tensor category. We shall say that $\mathcal{C}$ is G-graded, if there is a decomposition

$$\mathcal{C} = \oplus_{\sigma\in G}\mathcal{C}_\sigma$$

of $\mathcal{C}$ into a direct sum of full Abelian subcategories, such that for all $\sigma, \tau \in G$, the bifunctor $\otimes$ maps $\mathcal{C}_\sigma \times \mathcal{C}_\tau$ to $\mathcal{C}_{\sigma\tau}$. Given a G-graded tensor category $\mathcal{C}$, and a subgroup $H \subset G$, we shall denote by $\mathcal{C}_H$ the tensor subcategory $\oplus_{h\in H}\mathcal{C}_h$.

3.3. *G*-equivariantization of tensor categories. Let G be a group acting on a tensor category $\mathcal{C}$. An *equivariant* object in $\mathcal{C}$ is a pair (X, u) where $X \in \mathcal{C}$ is an object together with isomorphisms $u_\sigma : F_\sigma(X) \to X$ satisfying

$$u_{\sigma\tau} \circ (\gamma_{\sigma,\tau})_X = u_\sigma \circ F_\sigma(u_\tau),$$

for all $\sigma, \tau \in G$. A G-equivariant morphism $\phi : (V, u) \to (W, u')$ between G-equivariant objects (V, f) and (W, σ), is a morphism $\phi : V \to W$ in $\mathcal{M}$ such that $\phi \circ u_\sigma = u'_\sigma \circ F_\sigma(\phi)$ for all $\sigma \in G$.

The tensor category of equivariant objects is denoted by $\mathcal{C}^G$ and it is called the *equivariantization* of $\mathcal{C}$. The tensor product of $\mathcal{C}^G$ is defined by

$$(V, u) \otimes (W, u') := (V \otimes W, \tilde{u}),$$

where $\tilde{u}_\sigma = (u_\sigma \otimes u'_\sigma)\zeta_\sigma^{-1}$, for any $\sigma \in G$. The unit object is $(1, \mathrm{id}_1)$.

3.4. **Crossed product tensor categories and G-invariant module categories.** Given an action $* : \underline{G} \to \underline{\mathrm{Aut}_\otimes(\mathcal{C})}$ of G on $\mathcal{C}$, the G-crossed product tensor category, denoted by $\mathcal{C} \rtimes G$ is defined as follows. As an Abelian category $\mathcal{C} \rtimes G = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma$, where $\mathcal{C}_\sigma = \mathcal{C}$ as an Abelian category, the tensor product is

$$[X, \sigma] \otimes [Y, \tau] := [X \otimes F_\sigma(Y), \sigma\tau], \quad X, Y \in \mathcal{C}, \quad \sigma, \tau \in G,$$

and the unit object is $[1, e]$. See [Ta] for the associativity constraint and a proof of the pentagon identity.

If $\mathcal{C} = \mathrm{Rep}(A)$ is the representation category of a finite-dimensional quasi-Hopf algebra A then $\mathcal{C} \rtimes G$ is also a representation category of a finite-dimensional quasi-Hopf algebra B. This is an immediate consequence of [EO1, Prop. 2.6] since each simple object $W \in \mathcal{C} \rtimes G$ is isomorphic to $[V, e] \otimes [1, \sigma]$, where $\sigma \in G$ and $V \in \mathrm{Rep}(A)$ is simple. Let $d : K_0(\mathcal{C}) \to \mathbb{Z}$ the Perron-Frobenius dimension, then $d([V, e] \otimes [1, \sigma]) = d(V)d([1, \sigma]) = d(V) \in \mathbb{Z}$, where $d([1, \sigma]) = 1$ because $[1, \sigma]$ is multiplicatively invertible.

3.5. **Equivariantization of module categories.** We shall explain analogous procedures for equivariantization in module categories. Equivariant module categories appeared in [ENO2]. We shall use the approach given in [Ga1].

Let G be a group and $\mathcal{C}$ be a tensor category equipped with an action of G. Let $\mathcal{M}$ be a module category over $\mathcal{C}$. For any $g \in G$ we shall denote by $\mathcal{M}^\sigma$ the module category $\mathcal{M}^{F_\sigma}$. If $\sigma \in G$, we shall say that an endofunctor $T : \mathcal{M} \to \mathcal{M}$ is σ-*invariant* if it has a module structure $(T, c) : \mathcal{M} \to \mathcal{M}^\sigma$.

If $\sigma, \tau \in G$ and T is σ-invariant and U is τ-invariant then $T \circ U$ is $\sigma\tau$-invariant. Indeed, let us assume that the functors $(T, c) : \mathcal{M} \to \mathcal{M}^\sigma$, $(U, d) : \mathcal{M} \to \mathcal{M}^\tau$ are module functors then $(T \circ U, b) : \mathcal{M} \to \mathcal{M}^{\sigma\tau}$ is a module functor, where

$$b_{X,M} = ((\gamma_{\sigma,\tau})_X \otimes \mathrm{id}\,) c_{F_\tau(X),M} T(d_{X,M}), \tag{3.1}$$

for all $X \in \mathcal{C}$, $M \in \mathcal{M}$.

Definition 3.1. Let $F \subseteq G$ be a subgroup.

1. The monoidal category of σ-equivariant functors for some $\sigma \in F$ in $\mathcal{M}$ will be denoted by $\underline{\mathrm{Aut}\,^F_\mathcal{C}(\mathcal{M})}$.
3. An F-*equivariant* module category is a module category $\mathcal{M}$ equipped with a monoidal functor $(\Phi, \mu) : \underline{F} \to \underline{\mathrm{Aut}\,^F_\mathcal{C}(\mathcal{M})}$, such that $\Phi(\sigma)$ is a σ-invariant functor for any $\sigma \in F$.

In another words, an F-equivariant module category is a module category $\mathcal{M}$ endowed with a family of module functors $(U_\sigma, c^\sigma) : \mathcal{M} \to \mathcal{M}^\sigma$ for any $\sigma \in F$ and a family of natural isomorphisms $\mu_{\sigma,\tau} : (U_\sigma \circ U_\tau, b) \to (U_{\sigma\tau}, c^{\sigma\tau})$ $\sigma, \tau \in F$ such that

$$(\mu_{\sigma,\tau\nu})_M \circ U_\sigma(\mu_{\tau,\nu})_M = (\mu_{\sigma\tau,\nu})_M \circ (\mu_{\sigma,\tau})_{U_\nu(M)}, \tag{3.2}$$

$$c^{\sigma\tau}_{X,M} \circ (\mu_{\sigma,\tau})_{X\overline{\otimes}M} = ((\gamma_{\sigma,\tau})_X \overline{\otimes}(\mu_{\sigma,\tau})_M) \circ c^\sigma_{F_\tau(X),U_\tau(M)} \circ U_\sigma(c^\tau_{X,M}), \tag{3.3}$$

for all $\sigma, \tau, \nu \in F$, $X \in \mathcal{C}$, $M \in \mathcal{M}$. Equation (3.2) follows from (1.1) and (3.3) follows from (2.4).

Example 3.2. $\mathcal{C}$ is a G-equivariant module category over itself. For any $g \in G$ set $(U_\sigma, c^\sigma) = (F_\sigma, \theta_\sigma)$ and $\mu_{\sigma,\tau} = \gamma_{\sigma,\tau}$ for all $\sigma, \tau \in G$.

If $\mathcal{M}$ is an F-equivariant module category, an *equivariant object* (see [ENO2, Def. 5.3]) is an object $M \in \mathcal{M}$ together with isomorphisms $\{v_\sigma : U_\sigma(M) \to M : \sigma \in F\}$ such that for all $\sigma, \tau \in F$

$$v_{\sigma\tau} \circ (\mu_{\sigma,\tau})_M = v_\sigma \circ U_\sigma(v_\tau). \tag{3.4}$$

The category of F-equivariant objects is denoted by $\mathcal{M}^F$. A morphism between two F-equivariant objects (M, v), (M', v') is a morphism $f : M \to M'$ in $\mathcal{M}$ such that $f \circ v_\sigma = v'_\sigma \circ U_\sigma(f)$ for all $\sigma \in F$.

Lemma 3.3. *The category $\mathcal{M}^F$ is a $\mathcal{C}^G$-module category.*

Proof. If $(X, u) \in \mathcal{C}^G$ and $(M, v) \in \mathcal{M}^F$ the action is defined by

$$(X, u)\overline{\otimes}(M, v) = (X\overline{\otimes}M, \widetilde{v}),$$

where $\widetilde{v}_\sigma = (u_\sigma \otimes v_\sigma) c^\sigma_{X,M}$ for all $\sigma \in F$. The object $(X\overline{\otimes}M, \widetilde{v})$ is equivariant due to equation (3.3). The associativity isomorphisms are the same as in $\mathcal{M}$. □

The notion of F-equivariant module category is equivalent to the notion of $\mathcal{C} \rtimes F$-module cateory. If $\mathcal{M}$ is an F-equivariant $\mathcal{C}$-module category for some subgroup F of G, then $\mathcal{M}$ is a $\mathcal{C} \rtimes F$-module with action $\overline{\otimes} : \mathcal{C} \rtimes F \times \mathcal{M} \to \mathcal{M}$ given by $[X, g]\overline{\otimes}M = X\overline{\otimes}U_g(M)$, for all $X \in \mathcal{C}$, $g \in F$ and $M \in \mathcal{M}$. The associativity isomorphisms are given by

$$m_{[X,g],[Y,h],M} = \big(\mathrm{id}_X \otimes (c^g_{Y,U_h(M)})^{-1} (\mathrm{id}_{F_g(Y)} \otimes \mu^{-1}_{g,h}(M))\big)\, m_{X,F_g(Y),U_{gh}(M)},$$

for all $X, Y \in \mathcal{C}$, $g, h \in F$ and $M \in \mathcal{M}$.

In the next statement we collect several well-known results that are, by now, part of the folklore of the subject.

Proposition 3.4. *Let G be a finite group acting over a finite tensor category $\mathcal{C}$. If $F \subset G$ is a subgroup, and $\mathcal{M}$ is an F-equivariant $\mathcal{C}$-module category, then:*

1. *If $\mathcal{M}$ is an exact (indecomposable) $\mathcal{C}$-module category then $\mathcal{M}$ is an exact (respectively indecomposable) $\mathcal{C} \rtimes F$-module category.*

2. *$\mathcal{M}^F$ is an exact module category if and only if $\mathcal{M}$ is an exact module category.*
3. *There is an equivalence of $\mathcal{C}^G$-module categories*

$$\mathcal{M}^F \simeq \mathrm{Hom}_{\mathcal{C}\rtimes F}(\mathcal{C},\mathcal{M}) \simeq \mathcal{C}^{\mathrm{op}} \boxtimes_{\mathcal{C}\rtimes F} \mathcal{M} \simeq \big(\mathcal{C} \rtimes G \boxtimes_{\mathcal{C}\rtimes F} \mathcal{M}\big)^G. \tag{3.5}$$

4. *If $\mathcal{N}$ is an indecomposable (exact) module category over $\mathcal{C}^G$ there exists a subgroup F of G and an F-equivariant indecomposable (exact) module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathcal{N} \simeq \mathcal{M}^F$.*
5. *If $\mathcal{M}_1$, $\mathcal{M}_2$ are G-equivariant $\mathcal{C}$-module categories such that $\mathcal{M}_1^G \simeq \mathcal{M}_2^G$ as $\mathcal{C}^G$-module categories then $\mathcal{M}_1 \simeq \mathcal{M}_2$ as $\mathcal{C}$-module categories.*

Proof. 1. Let $P \in \mathcal{C} \rtimes G$ be a projective object. Thus, there exists a family of projective objects $P_\sigma \in \mathcal{C}$ such that $P = \oplus_{\sigma\in G}[P_\sigma,\sigma]$. Let $M \in \mathcal{M}$, then $P\overline{\otimes}M = \bigoplus_{\sigma\in G} P_\sigma\overline{\otimes}U_\sigma(M)$, and since $\mathcal{M}$ is an exact $\mathcal{C}$-module category $P_\sigma\overline{\otimes}U_\sigma(M)$ is projective for all σ, thus $P\overline{\otimes}M$ is projective.

2. Under the correspondence described in [Ta, Thm. 4.1] is enough to show that a $\mathcal{C} \rtimes F$-module category $\mathcal{M}$ is exact if and only if $\mathcal{M}$ is an exact $\mathcal{C}$-module category. The proof follows from part (1) of this proposition.

3. An object $(F, c) \in \mathrm{Hom}_{\mathcal{C}\rtimes F}(\mathcal{C},\mathcal{M})$ is determined uniquely by an object $M \in \mathcal{M}$ such that $F(X) = X\overline{\otimes}M$ together with an isomorphism $v_\sigma = c_{[\mathbf{1},\sigma],\mathbf{1}} : U_\sigma(M) \to M$. This correspondence establish an equivalence $\mathcal{M}^F \simeq \mathrm{Hom}_{\mathcal{C}\rtimes F}(\mathcal{C},\mathcal{M})$. The equivalence $\mathrm{Hom}_{\mathcal{C}\rtimes F}(\mathcal{C},\mathcal{M}) \simeq \mathcal{C}^{\mathrm{op}} \boxtimes_{\mathcal{C}\rtimes F} \mathcal{M}$ follows from [Gr, Thm. 3.20].

Since $\mathcal{C} \rtimes G \boxtimes_{\mathcal{C}\rtimes F} \mathcal{M}$ is a $\mathcal{C} \rtimes G$-module then it is a G-equivariant $\mathcal{C}$-module category, thus

$$\big(\mathcal{C} \rtimes G \boxtimes_{\mathcal{C}\rtimes F} \mathcal{M}\big)^G \simeq \mathcal{C}^{\mathrm{op}} \boxtimes_{\mathcal{C}\rtimes G} \big(\mathcal{C} \rtimes G \boxtimes_{\mathcal{C}\rtimes F} \mathcal{M}\big) \simeq \mathcal{C}^{\mathrm{op}} \boxtimes_{\mathcal{C}\rtimes F} \mathcal{M} \simeq \mathcal{M}^F.$$

The first equivalence is [Ta, Thm 4.1].

4. By [EO1, Proposition 3.9] every indecomposable exact tensor category over a finite tensor category is a simple module category in the sense of [Ga1], so the result follows by the main result of [Ga1], and the item (1) of this proposition.

5. Since $\mathcal{M}_1$, $\mathcal{M}_2$ are G-equivariant then they are $\mathcal{C} \rtimes G$-module categories. It follows from [Ta, Thm. 4.1] that this are equivalent $\mathcal{C} \rtimes G$-module categories. This equivalence induces an equivalence of $\mathcal{C}$-module categories (see [Ta, Ex. 2.5]). □

It follows from Proposition 3.4 (4) that the equivariantization construction of module categories by a fixed subgroup is injective. Moreover, if the equivariantization of a module category by two subgroups gives the same result then the groups must be conjugate. We shall give the precise statement in the following. First we need a definition and a result from the paper [Ga2].

Definition 3.5. [Ga2, Def. 4.3] Let $\mathcal{C}$ be a G-graded tensor category. If $(\mathcal{M}, \otimes)$ is a $\mathcal{C}_e$-module category, then a $\mathcal{C}$-extension of $\mathcal{M}$ is a $\mathcal{C}$-module category $(\mathcal{M}, \odot)$ such that $(\mathcal{M}, \otimes)$ is obtained by restriction to $\mathcal{C}_e$.

Proposition 3.6. [Ga2, Prop. 4.6] *Let $\mathcal{C}$ be a G-graded finite tensor category and let $F, F' \subset G$ be subgroups and $(\mathcal{N}, \odot)$, $(\mathcal{N}', \odot')$ be a $\mathcal{C}_F$-extension and a $\mathcal{C}_{F'}$-extension of the indecomposable $\mathcal{C}_e$-module categories $\mathcal{N}$ and $\mathcal{N}'$, respectively. Then $\mathcal{C} \boxtimes_{\mathcal{C}_{F'}} \mathcal{N}' \cong \mathcal{C} \boxtimes_{\mathcal{C}_F} \mathcal{N}$ as $\mathcal{C}$-modules if and only if there exists $\sigma \in G$ such that $F = \sigma F' \sigma^{-1}$ and $\mathcal{C}_{\sigma F'} \boxtimes_{\mathcal{C}_{F'}} \mathcal{N}' \cong \mathcal{N}$ as $\mathcal{C}_e$-module categories.*

Theorem 3.7. *Let G be a finite group action on a finite tensor category $\mathcal{C}$ and let $F, F' \subset G$ be subgroups. Let $\mathcal{N}$ and $\mathcal{N}'$ be an F-equivariant and an F'-equivariant module categories respectively, such that $\mathcal{N}$ and $\mathcal{N}'$ are indecomposable as $\mathcal{C}$-module categories and $\mathcal{N}^F \cong \mathcal{N}'^{F'}$ as $\mathcal{C}^G$-module categories. Then F and F' are conjugate subgroups in G.*

Proof. It follows from Proposition 3.4 (3) that there is an equivalence of $\mathcal{C}^G$-modules
$$\big(\mathcal{C} \rtimes G \boxtimes_{\mathcal{C} \rtimes F} \mathcal{N}\big)^G \simeq \big(\mathcal{C} \rtimes G \boxtimes_{\mathcal{C} \rtimes F'} \mathcal{N}'\big)^G.$$
Hence by Proposition 3.4 (5) there is an equivalence of $\mathcal{C} \rtimes G$-modules $\mathcal{C} \rtimes G \boxtimes_{\mathcal{C} \rtimes F} \mathcal{N} \simeq \mathcal{C} \rtimes G \boxtimes_{\mathcal{C} \rtimes F'} \mathcal{N}'$, thus the result follows from Proposition 3.6. ☐

4. Quasi-Hopf algebras

A quasi-bialgebra [D] is a four-tuple $(A, \Delta, \varepsilon, \Phi)$ where A is an associative algebra with unit, $\Phi \in (A \otimes A \otimes A)^\times$ is called the *associator*, and $\Delta : A \to A \otimes A$, $\varepsilon : A \to k$ are algebra homomorphisms satisfying the identities

$$\Phi(\Delta \otimes \mathrm{id})(\Delta(h)) = (\mathrm{id} \otimes \Delta)(\Delta(h))\Phi, \tag{4.1}$$
$$(\mathrm{id} \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \quad (\varepsilon \otimes \mathrm{id})(\Delta(h)) = 1 \otimes h, \tag{4.2}$$

for all $h \in A$. The associator Φ has to be a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(\Phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi), \tag{4.3}$$
$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1 \otimes 1. \tag{4.4}$$

A is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism S of the algebra A and elements $\alpha, \beta \in A$ such that, for all $h \in A$, we have:

$$S(h_{(1)})\alpha h_{(2)} = \varepsilon(h)\alpha \quad \text{and} \quad h_{(1)}\beta S(h_{(2)}) = \varepsilon(h)\beta, \tag{4.5}$$
$$\Phi^1 \beta S(\Phi^2)\alpha \Phi^3 = 1 \quad \text{and} \quad S(\Phi^{-1})\alpha \Phi^{-2} \beta S(\Phi^{-3}) = 1. \tag{4.6}$$

Here we use the notation $\Phi = \Phi^1 \otimes \Phi^2 \otimes \Phi^3$, $\Phi^{-1} = \Phi^{-1} \otimes \Phi^{-3} \otimes \Phi^{-3}$. If A is a quasi-Hopf algebra, we shall denote by $\mathrm{Rep}(A)$ the tensor category of finite-dimensional representations of A.

An invertible element $J \in A \otimes A$ is called a *twist* if $(\varepsilon \otimes \mathrm{id})(J) = 1 = (\mathrm{id} \otimes \varepsilon)(J)$. If A is a quasi-Hopf algebra and $J = J^1 \otimes J^2 \in A \otimes A$ is a twist with inverse $J^{-1} = J^{-1} \otimes J^{-2}$, then we can define a quasi-Hopf algebra on the same algebra A keeping the counit and antipode and replacing the comultiplication, associator and the elements α and β by

$$\Delta_J(h) = J\Delta(h)J^{-1}, \tag{4.7}$$

$$\Phi_J = (1 \otimes J)(\mathrm{id} \otimes \Delta)(J)\Phi(\Delta \otimes \mathrm{id})(J^{-1})(J^{-1} \otimes 1), \tag{4.8}$$

$$\alpha_J = S(J^{-1})\alpha J^{-2}, \quad \beta_J = J^1 \beta S(J^2). \tag{4.9}$$

We shall denote this new quasi-Hopf algebra by (A_J, Φ_J). If $\Phi = 1$ then, in this case, we shall denote $\Phi_J = dJ$.

4.1. Comodule algebras over quasi-Hopf algebras.

Let $(A, \Phi, \alpha, \beta, 1)$ be a finite dimensional quasi-Hopf algebra.

Definition 4.1. A left A-comodule algebra is a family $(\mathcal{K}, \lambda, \Phi_\lambda)$ such that $\mathcal{K}$ is an algebra, $\lambda : \mathcal{K} \to A \otimes \mathcal{K}$ is an algebra map, $\Phi_\lambda \in A \otimes A \otimes \mathcal{K}$ is an invertible element such that

$$(1 \otimes \Phi_\lambda)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi_\lambda)(\Phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\Phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_\lambda), \tag{4.10}$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi_\lambda) = 1, \tag{4.11}$$

$$\Phi_\lambda(\Delta \otimes \mathrm{id})\lambda(x) = \big((\mathrm{id} \otimes \lambda)\lambda(x)\big)\Phi_\lambda, \quad x \in \mathcal{K} \tag{4.12}$$

We shall say that a comodule algebra $(\mathcal{K}, \lambda, \Phi_\lambda)$ is right A-simple if it has no non-trivial right ideals $J \subseteq \mathcal{K}$ such that J is costable, that is $\lambda(J) \subseteq A \otimes \mathcal{K}$.

Remark 4.2. The notion of comodule algebra for quasi-Hopf algebras does not coincide with the notion of comodule algebra for (usual) Hopf algebras. For quasi-Hopf algebras the coaction may not be coassociative.

If $(\mathcal{K}, \lambda, \Phi_\lambda)$ is a left A-comodule algebra, the category ${}^A_{\mathcal{K}}\mathcal{M}_A$ consists of $(\mathcal{K}, A)$-bimodules M equipped with a $(\mathcal{K}, A)$-bimodule map $\delta : M \to A \otimes M$ such that for all $m \in M$

$$\Phi_\lambda(\Delta \otimes \mathrm{id})\delta(m) = (\mathrm{id} \otimes \delta)\delta(m)\Phi, \tag{4.13}$$

$$(\varepsilon \otimes \mathrm{id})\delta = \mathrm{id}. \tag{4.14}$$

The following result will be useful to present examples of exact module categories, it is a consequence of some freeness results on comodule algebras over quasi-Hopf algebras proven by H. Henker.

Lemma 4.3. *Let $(\mathcal{K}, \lambda, \Phi_\lambda)$ be a right A-simple left A-comodule algebra. If $M \in {}_{\mathcal{K}}\mathcal{M}$ then $A \otimes M \in {}_{\mathcal{K}}\mathcal{M}$ is projective.*

Proof. The object $A\otimes M$ is in the category ${}^{A}_{\mathcal{K}}\mathcal{M}_A$ as follows. The left $\mathcal{K}$-action and the right A-action on $A\otimes M$ are determined by

$$x\cdot(a\otimes m) = x_{(-1)}a\otimes x_{(0)}\cdot m, \quad (a\otimes m)\cdot b = ab\otimes m,$$

for all $x\in\mathcal{K}$, $a,b\in A$ and $m\in M$. The coaction is determined by $\delta : A\otimes M\to A\otimes A\otimes M$, $\delta = \Phi_\lambda(\Delta\otimes \mathrm{id}_M)$. It follows from [He, Lemma 3.6] that $A\otimes M$ is a projective $\mathcal{K}$-module. □

4.2. **Comodule algebras over radically graded quasi-Hopf algebras.** Let A be a quasi-Hopf algebra *radically graded*, that is there is an algebra grading $A=\oplus_{i=0}^m A[i]$, where $I := \mathrm{Rad}\, A = \oplus_{i\geq 1}A[i]$ and $I^k = \oplus_{i\geq k}A[i]$ for any $k=0\dots m$. Here $I^0 = A$. Since $\Delta(I)\subseteq I\otimes A + A\otimes I$ then $\Delta(I)\subseteq \sum_{j=0}^k I^j\otimes I^{k-j}$ for any $k=0\dots m$. In this case $A[0]$ is semisimple, A is generated by $A[0]$ and $A[1]$, and the associator Φ is an element in $A[0]^{\otimes 3}$, see [EG1, Lemma 2.1].

If $(\mathcal{K},\lambda,\Phi_\lambda)$ is a left A-comodule algebra, define

$$\mathcal{K}_i = \lambda^{-1}(I^i\otimes\mathcal{K}), \quad i=0\dots m.$$

This is an algebra filtration, thus we can consider the associated graded algebra $\mathrm{gr}\,\mathcal{K} = \oplus_{i=0}^m \mathcal{K}[i]$, $\mathcal{K}[i] = \mathcal{K}_i/\mathcal{K}_{i+1}$.

Lemma 4.4. 1. *The above filtration satisfies*

$$\lambda(\mathcal{K}_i)\subseteq \sum_{j=0}^{i} I^j\otimes\mathcal{K}_{i-j}. \tag{4.15}$$

2. *There is a left A-comodule algebra structure $(\mathrm{gr}\,\mathcal{K},\overline{\lambda},\overline{\Phi}_\lambda)$ satisfying*

$$\overline{\lambda}(\mathrm{gr}\,\mathcal{K}(n))\subseteq \oplus_{k=0}^n A[k]\otimes\mathcal{K}[n-k]. \tag{4.16}$$

3. *$(\mathcal{K}[0],\overline{\lambda},\overline{\Phi}_\lambda)$ is a left $A[0]$-comodule algebra.*

Proof. Item (1) follows from the definition of $\mathcal{K}_i$ and equation (4.12). For each $n=0\dots m$ there is a linear map $\overline{\lambda}:\mathrm{gr}\,\mathcal{K}\to A\otimes\mathrm{gr}\,\mathcal{K}$ such that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{K}_n & \xrightarrow{\lambda} & \sum_{j=0}^n I^j\otimes\mathcal{K}_{n-j} \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow \\
\mathcal{K}_n/\mathcal{K}_{n+1} & \longrightarrow & \left(\sum_{j=0}^n I^j\otimes\mathcal{K}_{n-j}\right)/\sum_{j=0}^{n+1} I^j\otimes\mathcal{K}_{n+1-j} \\
\big\downarrow & & \big\downarrow{\scriptstyle\simeq} \\
\mathcal{K}[n] & \xrightarrow{\overline{\lambda}} & \oplus_{k=0}^n A[k]\otimes_{\Bbbk}\mathcal{K}[n-k].
\end{array}$$

Defining $\overline{\Phi}_\lambda$ as the projection of Φ_λ to $A[0]\otimes A[0]\otimes\mathcal{K}[0]$ follows immediately that $(\mathrm{gr}\,\mathcal{K},\overline{\lambda},\overline{\Phi}_\lambda)$ is a left A-comodule algebra. □

Lemma 4.5. *The following statements are equivalent:*

1. *$\mathcal{K}$ is a right A-simple left A-comodule algebra.*
2. *$\mathcal{K}[0]$ is a right $A[0]$-simple left $A[0]$-comodule algebra.*
3. *$\operatorname{gr}\mathcal{K}$ is a right A-simple left A-comodule algebra.*

Proof. Assume $\mathcal{K}[0]$ is a right $A[0]$-simple. Let $J \subseteq A$ be a right ideal A-costable. Consider the filtration $J = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_m$ given by $J_k = \lambda^{-1}(I^k \otimes J)$ for all $k = 0 \dots m$. Set $\overline{J}(k) = J_k/J_{k+1}$ for any k and $\overline{J} = \oplus_k \overline{J}(k)$. It follows that for any $n = 0 \dots m$

$$\overline{\lambda}(\overline{J}(n)) \subseteq \oplus_{k=0}^{n} A[k] \otimes \overline{J}(k). \tag{4.17}$$

In particular $\overline{J}(0) \subseteq \mathcal{K}[0]$ is a right ideal $A[0]$-costable thus $\overline{J} = \mathcal{K}[0]$ or $\overline{J} = 0$. In the first case $J = A$ and in the second case $J = J_1$. It follows from (4.17) that $\overline{J}(1) \subseteq \mathcal{K}[0]$ is a a right ideal $A[0]$-costable. Hence $J = J_2$. Continuing this reasoning we obtain that $J = 0$.

Assume now that $\mathcal{K}$ is a right A-simple. Let $\overline{J} \subseteq \mathcal{K}[0]$ be a right $A[0]$-costable ideal. Denote $\pi : \mathcal{K} \to \mathcal{K}[0]$ the canonical projection and $J = \pi^{-1}(\overline{J})$. Clearly J is a right A-costable ideal thus $J = 0$ or $J = \mathcal{K}$, thus $\overline{J} = 0$ or $\overline{J} = \mathcal{K}[0]$ respectively. □

As a consequence we have the following result.

Corollary 4.6. *Let (A, Φ) be a radically graded quasi-Hopf algebra and $(\mathcal{K}, \lambda, \Phi_\lambda)$ be a left A-comodule algebra such that $\mathcal{K}[0] = \Bbbk 1$ then A is twist equivalent to a Hopf algebra.*

Proof. Since $(\mathcal{K}[0], \overline{\lambda}, \overline{\Phi}_\lambda)$ is a left $A[0]$-comodule algebra then there exists an invertible element $J \in A \otimes A$ such that $J \otimes 1 = \overline{\Phi}_\lambda$. Equation (4.10) implies that $\Phi = dJ$. □

4.3. **Module categories over quasi-Hopf algebras.** For any comodule algebra over a quasi-Hopf algebra A there is associated a module category over $\operatorname{Rep}(A)$.

Lemma 4.7. *Let A be a finite-dimensional quasi-Hopf algebra.*

1. *If $(\mathcal{K}, \lambda, \Phi_\lambda)$ is a left A-comodule algebra then the category ${}_{\mathcal{K}}\mathcal{M}$ is a module category over $\operatorname{Rep}(A)$. It is exact if $\mathcal{K}$ is right A-simple.*
2. *If $\mathcal{M}$ is an exact module category over $\operatorname{Rep}(A)$ there exists a left A-comodule algebra $(\mathcal{K}, \lambda, \Phi_\lambda)$ such that $\mathcal{M} \simeq {}_{\mathcal{K}}\mathcal{M}$ as module categories over $\operatorname{Rep}(A)$.*

Proof. 1. The action $\overline{\otimes} : \operatorname{Rep}(A) \times {}_{\mathcal{K}}\mathcal{M} \to {}_{\mathcal{K}}\mathcal{M}$ is given by the tensor product over the field $\Bbbk$ where the action on the tensor product is given by λ. The associativity isomorphisms $m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$ are given by

$$m_{X,Y,M}(x \otimes y \otimes m) = \Phi^1_\lambda \cdot x \otimes \Phi^2_\lambda \cdot y \otimes \Phi^3_\lambda \cdot m,$$

for all $x \in X$, $y \in Y$, $M \in M$, $X, Y \in \operatorname{Rep}(A)$, $M \in {}_{\mathcal{K}}\mathcal{M}$. To prove that ${}_{\mathcal{K}}\mathcal{M}$ is exact, it is enough to verify that $A \otimes M$ is projective for any $M \in {}_{\mathcal{K}}\mathcal{M}$ but this is Lemma 4.3.

2. This is a straightforward consequence of [EO1, Thm. 3.17], the proof of [AM, Prop. 1.19] extends *mutatis mutandis* to the quasi-Hopf setting. □

Definition 4.8. Two left A-comodule algebras $(\mathcal{K}, \lambda, \Phi_\lambda)$, $(\mathcal{K}', \lambda', \Phi'_\lambda)$ are *equivariantly Morita equivalent* if the corresponding module categories are equivalent.

4.4. **Comodule algebras coming from twisting.** Let (A, Φ) be a quasi-Hopf algebra and $J \in A{\otimes}A$ be a twist. Let $(K, \lambda, \Phi_\lambda)$ be a left A-comodule algebra. Let us denote by $(K_J, \lambda_J, \widetilde{\Phi}_\lambda)$ the following left A_J-comodule algebra. As algebras $K_J = K$, the coaction $\lambda_J = \lambda$ and $\widetilde{\Phi}_\lambda = \Phi_\lambda(J^{-1}{\otimes}1)$.

The following results are straightforward.

Lemma 4.9. *$(K_J, \lambda_J, \widetilde{\Phi}_\lambda)$ is a left A_J-comodule algebra. It is right A-simple if and only if $(K, \lambda, \Phi_\lambda)$ is right A-simple.* □

Lemma 4.10. *Let $J \in A{\otimes}A$ be a twist. If $(K, \lambda, \Phi_\lambda)$ and $(K', \lambda', \Phi'_\lambda)$ are equivariantly Morita equivalent A-comodule algebras then $(K_J, \lambda_J, \Phi_\lambda(J^{-1}{\otimes}1))$ and $(K'_J, \lambda'_J, \Phi'_\lambda(J^{-1}{\otimes}1))$ are equivariant Morita equivalent A_J-comodule algebras.* □

5. Equivariantization of quasi-Hopf algebras

For a quasi-Hopf algebra A we shall explain the notion of a *crossed system over* A and discuss its relation with the equivariantization of the category $\mathrm{Rep}(A)$.

Let A_1, A_2 be quasi-Hopf algebras. A *twisted homomorphism* between A_1 and A_2 is pair (f, J) consisting of a homomorphism of algebras $f : A_1 \to A_2$ and an invertible element $J \in A_2^{\otimes 2}$ such that

$$\Phi_2(\Delta \otimes \mathrm{id}\,)(J)(J \otimes 1) = (\mathrm{id}\, \otimes \Delta)(J)(1 \otimes J)(f^{\otimes 3})(\Phi_1), \tag{5.1}$$

$$(\varepsilon{\otimes}\mathrm{id}\,)(J) = (\mathrm{id}\,{\otimes}\varepsilon)(J) = 1, \tag{5.2}$$

$$\varepsilon(f(a)) = \varepsilon(a), \tag{5.3}$$

$$\Delta(f(a))J = J(f^{2\otimes}(\Delta(a))), \quad \text{for all } a \in A. \tag{5.4}$$

Remark 5.1. If $(f, J) : A_1 \to A_2$ is a twisted homomorphism, then $J^{-1} \in A_2 \otimes A_2$ is a twist and $f : A_1 \to (A_2)_{J^{-1}}$ is a homomorphism of quasi-bialgebras.

We define the category $\underline{\mathrm{End}}^{\mathrm{Tw}}(A_1, A_2)$ whose objects are twisted homomorphism from A_1 to A_2. A *morphism* between two twisted homomorphisms $(f, J), (f', J') : A_1 \to A_2$ is an element $c \in A_2$ such that $cf(a) = f'(a)c$ for any $a \in A_1$ and $\Delta(c)J = J'(c{\otimes}c)$. The composition of $a : f \to g$, $b : g \to h$, is $ba : f \to h$. If $(f, J_f) : A_1 \to A_2$ and $(g, J_g) : A_2 \to A_3$ are twisted homomorphism, we define the composition as the twisted homomorpshism $(g \circ f, J_g(g \otimes g)(J_f)) : A_1 \to A_3$.

To any twisted homomorphism $(f, J) : A_1 \to A_2$ there is associated a tensor functor

$$(f^*, \xi^J) : \mathrm{Rep}(A_2) \to \mathrm{Rep}(A_1),$$

where $f^*(V) = V$ for all $V \in \mathrm{Rep}(A_2)$, and f^* is the identity over arrows. The A_1-action on $f^*(V)$ is given through the morphism f. The monoidal structure is given by applying the element $J \in A_2^{\otimes 2}$:

$$\xi^J{}_{M,N} : f^*(M) \otimes f^*(N) \to f^*(M \otimes N), \quad \xi^J{}_{M,N}(m \otimes n) = J(m \otimes n),$$

for any $M, N \in \mathrm{Rep}(A_2)$, $m \in M$, $n \in N$. Morphisms between twisted homomorphisms $f, f' : A_1 \to A_2$ of quasi-Hopf algebras correspond to tensor natural transformations between the associated tensor functors.

5.1. **Crossed system over a quasi-Hopf algebra.** Given a quasi-Hopf algebra A we shall denote by $\underline{\mathrm{Aut}}^{\mathrm{Tw}}(A)$ the (monodial) subcategory of $\underline{\mathrm{End}}^{\mathrm{Tw}}(A)$ where objects are twisted automorphisms of A, and arrows are isomorphisms of twisted automorphisms.

Let G be a group, and let A be a quasi-Hopf algebra. A *G-crossed system* over A is a monoidal functor $* : \underline{G} \to \underline{\mathrm{Aut}}^{\mathrm{Tw}}(A)$ such that $e_* = (\mathrm{id}_A, 1 \otimes 1)$.

More explicitly a G-crossed system consists of the following data:

- A twisted automorphism (σ_*, J_σ) for each $\sigma \in G$,
- an element $\theta_{(\sigma,\tau)} \in A^\times$ for each $\sigma, \tau \in G$,

such that for all $a \in A$, $\sigma, \tau, \rho \in G$,

$$\varepsilon(\theta_{(\sigma,\tau)}) = 1, \tag{5.5}$$

$$(1_*, J_1) = (\mathrm{id}, 1 \otimes 1), \tag{5.6}$$

$$\theta_{(\sigma,\tau)}(\sigma\tau)_*(a) = \sigma_*(\tau_*(a))\theta_{(\sigma,\tau)}, \tag{5.7}$$

$$\theta_{(\sigma,\tau)}\theta_{(\sigma\tau,\rho)} = \sigma_*(\theta_{(\tau,\rho)})\theta_{\sigma,\tau\rho}, \tag{5.8}$$

$$\theta_{(1,\sigma)} = \theta_{(\sigma,1)} = 1, \tag{5.9}$$

$$\Delta(\theta_{(\sigma,\tau)})J_{\sigma\tau} = J_\sigma(\sigma_* \otimes \sigma_*)(J_\tau)(\theta_{(\sigma,\tau)} \otimes \theta_{(\sigma,\tau)}). \tag{5.10}$$

Let $A\#G$ be the vector space $A \otimes_{\Bbbk} \Bbbk G$ with product and coproduct

$$(x\#\sigma)(y\#\tau) = x\sigma_*(y)\theta_{(\sigma,\tau)}\#\sigma\tau, \quad \Delta(x\#\sigma) = x_{(1)}J_\sigma^1\#\sigma \otimes x_{(2)}J_\sigma^2\#\sigma,$$

for all $x, y \in A$, $\sigma, \tau \in G$.

Proposition 5.2. *The foregoing operations makes the vector space $A\#G$ into a quasi-bialgebra with associator $\Phi^1\#e \otimes \Phi^2\#e \otimes \Phi^3\#e$, and counit $\varepsilon(x\#\sigma) = \varepsilon(x)$ for all $x \in A, \sigma \in G$.*

Proof. It is straightforward to see that $A\#G$ is an associative algebra with unit $1\#e$. Equation (4.1) follows from (5.1). The map ε is an algebra morphism by (5.3) and (5.5). Equations (4.2) follow from (5.2), equations (4.3) and (4.4) follow by the definition of the associator. Finally Δ is an algebra morphism by (5.4) and (5.10). □

5.2. **Antipodes of crossed systems.** Let G be a group, $(A, \Phi, S, \alpha, \beta)$ be a quasi-Hopf algebra and $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ a G-crossed system over A. An antipode for $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ is a function $\upsilon : G \to A^\times$ such that

$$\upsilon_{\sigma\tau}(\sigma\tau)_*(S(\theta_{\sigma,\tau})) = \upsilon_\tau(\tau^{-1})_*(\upsilon_\sigma)\theta_{\tau^{-1},\sigma^{-1}}, \tag{5.11}$$

$$\upsilon_\sigma^{-1}S(x)\upsilon_\sigma = (\sigma^{-1})_*(S(\sigma_*(x))), \tag{5.12}$$

$$\upsilon_\sigma(\sigma^{-1})_*((S(J_\sigma^1)\alpha J_\sigma^2))\theta_{\sigma^{-1},\sigma} = \alpha, \tag{5.13}$$

$$J_\sigma^1\sigma_*(\beta\upsilon_\sigma(\sigma^{-1})_*(S(J_\sigma^2)))\theta_{\sigma,\sigma^{-1}} = \beta, \tag{5.14}$$

for all $\sigma \in G$, where $J_\sigma = J_\sigma^1 \otimes J_\sigma^2$. The next proposition follows by a straightforward verification.

Proposition 5.3. *Let $\upsilon : G \to A^\times$ be an antipode for $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$. Then $(S, \alpha\#e, \beta\#e)$ is an antipode for $A\#G$, where*

$$S(x\#\sigma) = \upsilon_\sigma(\sigma^{-1})_*(S(x))\#\sigma^{-1},$$

for all $\sigma \in G, x \in A$. □

5.3. **Equivariantization and crossed systems.** Let us assume that G is an Abelian group. In this case a G-crossed system over A gives rise to a G-action on the category Rep(A). Indeed, for any $\sigma \in G$ we can define the tensor functors $(F_\sigma, \zeta_\sigma) : \text{Rep}(A) \to \text{Rep}(A)$ described as follows. For any $V \in \text{Rep}(A)$, $F_\sigma(V) = V$ as vector spaces and the action on $F_\sigma(V)$ is given by $a \cdot v = \sigma_*(a)v$ for all $a \in A$, $v \in V$. For any $V, W \in \text{Rep}(A)$ the isomorphisms $(\zeta_\sigma)_{V,W} : V\otimes W \to V\otimes W$ are given by $(\zeta_\sigma)_{V,W}(v\otimes w) = J_\sigma \cdot (v\otimes w)$ for all $v \in V$, $w \in W$. For any $\sigma, \tau \in G$ the natural tensor transformation $\gamma_{\sigma,\tau} : F_\sigma \circ F_\tau \to F_{\sigma\tau}$, $(\gamma_{\sigma,\tau})_V(v) = \theta^{-1}_{(\sigma,\tau)}v$ for all $V \in \text{Rep}(A)$, $v \in V$.

Lemma 5.4. *If $\theta_{(\sigma,\tau)} = \theta_{(\tau,\sigma)}$ for all $\sigma, \tau \in G$ then the tensor functors (F_σ, ζ_σ) described above define a G-action on* Rep(A).

Proof. The conmutativity of G and equation $\theta_{(\sigma,\tau)} = \theta_{(\tau,\sigma)}$ for all $\sigma, \tau \in G$ imply that the maps $\gamma_{\sigma,\tau}$ are morphisms of A-modules. The proof that the tensor functors (F_σ, ζ_σ) define a G-action is straightforward. □

Given a G-crossed system $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ over A we consider the category Rep$(A)^G$ of G-equivariant A-modules.

Proposition 5.5. *Let G be an Abelian group, A be a quasi-Hopf algebra and $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ a G-crossed system over A such that $\theta_{(\sigma,\tau)} = \theta_{(\tau,\sigma)}$ for all $\sigma, \tau \in G$. Then there is a tensor equivalence between* Rep$(A)^G$ *and* Rep$(A\#G)$.

Proof. Let (V, u) be a G-equivariant object. The linear isomorphisms $u_\sigma : F_\sigma(V) \to V$ satisfy

$$u_\sigma(\sigma_*(a) \cdot v) = a \cdot u_\sigma(v), \quad u_\sigma(u_\tau(v)) = u_{\sigma\tau}(\theta_{(\sigma,\tau)} \cdot v) \tag{5.15}$$

for all $v \in V$, $a \in A$, $\sigma, \tau \in G$. Equation (5.15) together with the fact that $\theta_{(\sigma,\tau)} = \theta_{(\tau,\sigma)}$ for all $\sigma, \tau \in G$ imply that there is a well-defined action of the crossed product $A\#G$ on V determined by

$$(a\#\sigma) \cdot v = au_\sigma^{-1}(v), \tag{5.16}$$

for all $a \in A, v \in V, \sigma \in G$. Morphisms of G-equivariant representations are exactly morphisms of $A\#G$-modules. Hence we have defined a functor

$$\mathcal{F} : \mathrm{Rep}(A)^G \to \mathrm{Rep}(A\#G),$$

which clearly is a tensor functor. Assume that $W \in \mathrm{Rep}(A\#G)$. Then, by restriction, W is a representation of A. Moreover (W, u) is a G-equivariant object in $\mathrm{Rep}(A)$, letting

$$u_\sigma : W \to W, \quad u_\sigma(w) = (\theta^{-1}_{(\sigma,\sigma^{-1})}\#\sigma^{-1}) \cdot w,$$

for every $\sigma \in G$. We have thus a functor $\mathcal{G} : \mathrm{Rep}(A\#G) \to \mathrm{Rep}(A)^G$. It is clear that $\mathcal{F}$ and $\mathcal{G}$ are inverse equivalences of categories. □

Remark 5.6. A version of the above result appears in [Na, Prop. 3.2].

5.4. **Crossed product of quasi-bialgebras.**

Definition 5.7. Let $(A, \Phi, S, \alpha, \beta)$ be a quasi-Hopf algebra, and let G be a group. We shall say that A is a *G-crossed product* if there is a decomposition $A = \bigoplus_{\sigma\in G} A_\sigma$, where:

- $\Phi \in A_e \otimes A_e \otimes A_e$,
- $A_\sigma A_\tau \subseteq A_{\sigma\tau}$ for all $\sigma, \tau \in G$,
- A_σ has an invertible element for each $\sigma \in G$,
- $\Delta(A_\sigma) \subseteq A_\sigma \otimes A_\sigma$ for each $\sigma \in G$.
- $S(A_\sigma) \subseteq A_{\sigma^{-1}}$, for each $\sigma \in G$.
- $\alpha, \beta \in A_e$

Proposition 5.8. *Every G-crossed product A is of the form $B\#G$ for some quasi-Hopf algebra B. Moreover, there exists an antipode $\upsilon : G \to B^\times$ such that $B\#G$ is isomorphic to A as quasi-Hopf algebras.*

Proof. Let A be a G-crossed product. Set $B = A_e$. Since every A_σ has an invertible element, we may choose for each $\sigma \in G$ some invertible element $t_\sigma \in A_\sigma$, with $t_e = 1$. Then it is clear that $A_\sigma = t_\sigma A_e = A_e t_\sigma$, and the set $\{t_\sigma : \sigma \in G\}$ is a basis for A as a left (and right) A_e-module. Note that $\varepsilon(t_\sigma) \neq 0$, because ε is an algebra map and t_σ is invertible. Thus, we may and shall assume that $\varepsilon(t_\sigma) = 1$ for each $\sigma \in G$. Let us define the maps

$$\sigma_*(a) = t_\sigma a t_\sigma^{-1}, \text{ for each } \sigma\ G \text{ and } a \in A_e,$$

and

$$\theta : G \times G \to A \quad \text{by} \quad \theta_{(\sigma,\tau)} = t_\sigma t_\tau t_{\sigma\tau}^{-1} \quad \text{for } \sigma, \tau \in G.$$

We have that $\Delta(t_\sigma) \in A_\sigma \otimes A_\sigma$ can be uniquely expressed as $\Delta(t_\sigma) = J_\sigma(t_\sigma \otimes t_\sigma)$, with $J_\sigma \in A_e \otimes A_e$. Since Δ is an algebra morphism, J_σ is invertible, and for the normalization $\varepsilon(t_\sigma)=1$, $(\varepsilon\otimes \mathrm{id})(J_\sigma)=(\mathrm{id}\otimes\varepsilon)(J_\sigma)=1$.

Then, it is straightforward to see that the data $(\sigma_*,\theta_{(\sigma,\tau)},J_\sigma)_{\sigma,\tau\in G}$, defines a G-crossed system over the sub-quasi-bialgebra $A_e\subseteq A$, and $A_e\#G$ is isomorphic to A as quasi-bialgebras.

The antipode $S:A\to A$ is anti-isomorphism of algebras, and the condition $S(A_\sigma)\subset A_{\sigma^{-1}}$ implies that there is a unique function $\upsilon:G\to A_e^\times$ such that $S(t_\sigma)=\theta_\sigma t_{\sigma^{-1}}$ for all $\sigma\in G$. Hence, it is straightforward to see that υ is antipode for the crossed system $(\sigma_*,\theta_{(\sigma,\tau)},J_\sigma)_{\sigma,\tau\in G}$, and $A_e\#G$ is isomorphic to A as quasi-Hopf algebras. □

5.5. **Twisted homomorphisms of comodule algebras.** Let A be a quasi-Hopf algebra. A *twisted homomorphism* of left A-comodule algebras $(\mathcal{K},\lambda,\Phi_\lambda)$ and $(\mathcal{K}',\lambda',\Phi'_\lambda)$ is pair $(\mathfrak{f},\mathfrak{J})$ consisting of a homomorphism of algebras $\mathfrak{f}:\mathcal{K}\to\mathcal{K}'$ and an invertible element $\mathfrak{J}\in A\otimes\mathcal{K}'$ such that

$$\Phi_{\lambda'}(\Delta\otimes \mathrm{id})(\mathfrak{J})=(\mathrm{id}\otimes\lambda')(\mathfrak{J})(1\otimes J)(\mathrm{id}\otimes\mathrm{id}\otimes\mathfrak{f})(\Phi_\lambda), \tag{5.17}$$

$$(\varepsilon\otimes \mathrm{id})(\mathfrak{J})=1, \tag{5.18}$$

$$\lambda'(\mathfrak{f}(a))\mathfrak{J}=\mathfrak{J}(\mathrm{id}\otimes\mathfrak{f})(\lambda(a)),\qquad \text{for all } a\in\mathcal{K}. \tag{5.19}$$

A *morphism* between two twisted homomorphisms $(\mathfrak{f}_1,\mathfrak{J}_1),(\mathfrak{f}_2,\mathfrak{J}_2):\mathcal{K}\to\mathcal{K}'$ is an element $c\in\mathcal{K}'$ such that $c\mathfrak{f}_1(a)=\mathfrak{f}_2(a)c$ for any $a\in\mathcal{K}$ and $\lambda'(c)\mathfrak{J}_1=\mathfrak{J}_2(1\otimes c)$.

To any twisted homomorphism of comodule algebras $(\mathfrak{f},\mathfrak{J}):\mathcal{K}\to\mathcal{K}'$ there is associated a $\mathrm{Rep}(A)$-module functor

$$(\mathfrak{f}^*,\xi^{\mathfrak{J}}):\mathrm{Rep}(\mathcal{K}')\to\mathrm{Rep}(\mathcal{K}),$$

where, for all $V\in\mathrm{Rep}(\mathcal{K}_2)$, $\mathfrak{f}^*(V)=V$ with action given by $x\cdot v=\mathfrak{f}(x)v$, $x\in\mathcal{K}$, $v\in V$. The natural transformation $\xi^{\mathfrak{J}}$ is given by

$$\xi^{\mathfrak{J}}{}_{X,M}:\mathfrak{f}^*(X\otimes M)\to X\otimes\mathfrak{f}^*(M),\quad \xi^{\mathfrak{J}}{}_{X,M}(x\otimes m)=\mathfrak{J}^{-1}\cdot(x\otimes m),$$

for any $X\in\mathrm{Rep}(A),M\in\mathrm{Rep}(\mathcal{K}_2)$, $x\in X$, $m\in M$. Morphisms between twisted homomorphisms $\mathfrak{f},\mathfrak{f}':\mathcal{K}\to\mathcal{K}'$ of A-comodule algebras correspond to module natural transformations between the module functors.

Let A be a quasi-Hopf algebra and $(\mathcal{K},\lambda,\Phi_\lambda)$ be a left A-comodule algebra. For each twisted endomorphism $(f,J):A\to A$, we define a new left A-comodule algebra $(\mathcal{K}^f,\lambda^f,\Phi^f_\lambda)$, where $\mathcal{K}^f=\mathcal{K}$ as algebras and

$$\lambda^f(x)=(f\otimes \mathrm{id})\lambda(x),\quad \Phi^f_\lambda=(f\otimes f\otimes \mathrm{id})(\Phi_\lambda)(J^{-1}\otimes 1),$$

for all $x\in\mathcal{K}$.

Definition 5.9. Let A be a quasi-Hopf algebra and $(\mathcal{K},\lambda,\Phi_\lambda)$ be a left A-comodule algebra. Given a twisted endomorphism (f,J) of A, a (f,J)-*twisted endomorphism of* $\mathcal{K}$ is a twisted homomorphism from $(\mathcal{K}^f,\lambda^f,\Phi^f_\lambda)$

to $(\mathcal{K}, \lambda, \Phi_\lambda)$. Explicitly a (f, J)-twisted endomorphism is a pair $(\mathfrak{f}, \mathfrak{J})$ consisting of an algebra endomorphism $\mathfrak{f} : \mathcal{K} \to \mathcal{K}$ and an invertible element $\mathfrak{J} \in A \otimes \mathcal{K}$, such that:

$$(\varepsilon \otimes \mathrm{id})(\mathfrak{J}) = 1, \tag{5.20}$$

$$\Phi_\lambda(\Delta \otimes \mathrm{id})(\mathfrak{J})(J \otimes 1) = (\mathrm{id} \otimes \lambda)(\mathfrak{J})(1 \otimes \mathfrak{J})(f \otimes f \otimes \mathfrak{f})(\Phi_\lambda) \tag{5.21}$$

$$\lambda(\mathfrak{f}(x))\mathfrak{J} = \mathfrak{J}(f \otimes \mathfrak{f})(\lambda(x)), \quad \text{for all } x \in \mathcal{K}. \tag{5.22}$$

Lemma 5.10. *Let $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ be a crossed system over a quasi-Hopf algebra A, and $(\mathcal{K}, \lambda, \Phi_\lambda)$ a left A-comodule algebra. If $(\mathfrak{f}_\sigma, \mathfrak{J}_\sigma), (\mathfrak{f}_\tau, \mathfrak{J}_\tau) : \mathcal{K} \to \mathcal{K}$ are (σ_*, J_σ)-twisted and (τ_*, J_τ)-twisted endomorphism, then*

$$(\mathfrak{f}_\sigma, \mathfrak{J}_\sigma)\bar{\circ}(\mathfrak{f}_\tau, \mathfrak{J}_\tau) = (\mathfrak{f}_\sigma \circ \mathfrak{f}_\tau, \mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau} \otimes 1))$$

is a $((\sigma\tau)_*, J_{\sigma\tau})$-twisted endomorphism. Moreover, this composition is associative, i.e., if $(\mathfrak{f}_\sigma, \mathfrak{J}_\sigma), (\mathfrak{f}_\tau, \mathfrak{J}_\tau), (\mathfrak{f}_\rho, \mathfrak{J}_\rho) : \mathcal{K} \to \mathcal{K}$ are (σ_*, J_σ)-twisted, (τ_*, J_τ)-twisted, and (ρ_*, J_ρ)-twisted endomorphism, then

$$[(\mathfrak{f}_\sigma, \mathfrak{J}_\sigma)\bar{\circ}(\mathfrak{f}_\tau, \mathfrak{J}_\tau)]\bar{\circ}(\mathfrak{f}_\rho, \mathfrak{J}_\rho) = (\mathfrak{f}_\sigma, \mathfrak{J}_\sigma)\bar{\circ}[(\mathfrak{f}_\tau, \mathfrak{J}_\tau)\bar{\circ}(\mathfrak{f}_\rho, \mathfrak{J}_\rho)]$$

Proof. If we use the following notation

$$\mathfrak{J}_\sigma \bar{\circ}\, \mathfrak{J}_\tau = \mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau} \otimes 1),$$

thus we need to prove:

(1) $(\varepsilon \otimes \mathrm{id})(\mathfrak{J}_\sigma \bar{\circ}\, \mathfrak{J}_\tau) = 1$,
(2) $\Phi_\lambda(\Delta \otimes \mathrm{id})(\mathfrak{J}_\sigma \bar{\circ}\, \mathfrak{J}_\tau)(J_{\sigma\tau} \otimes 1) = (\mathrm{id} \otimes \lambda)(\mathfrak{J}_\sigma \bar{\circ}\, \mathfrak{J}_\tau)(1 \otimes \mathfrak{J}_\sigma \bar{\circ}\, \mathfrak{J}_\tau)((\sigma\tau)_* \otimes (\sigma\tau)_* \otimes \mathfrak{f}_\sigma \circ \mathfrak{f}_\tau)(\Phi_\lambda)$,
(3) $\lambda(\mathfrak{f}_\sigma \circ \mathfrak{f}_\tau(x))\mathfrak{J}_\sigma \bar{\circ}\, \mathfrak{J}_\tau = \mathfrak{J}_\sigma \bar{\circ}\, \mathfrak{J}_\tau(f \otimes \mathfrak{f})(\lambda(x))$ for all $x \in \mathcal{K}$.

(1) The first equation follows immediately using $\varepsilon \otimes \mathrm{id}(\mathfrak{J}_\sigma) = 1$, and ε is an algebra morphism that commutes with σ_* for all $\sigma \in G$.

(2) For the second equation, first we shall see some equalities:

$$(\Delta \otimes \mathrm{id})(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(J_\sigma \otimes 1) = (J_\sigma \otimes 1)(\sigma_* \otimes \sigma_* \otimes \mathfrak{f}_\sigma)(\Delta \otimes \mathrm{id})(\mathfrak{J}_\tau) \tag{5.23}$$

$$(1 \otimes \mathfrak{J}_\sigma)(\sigma_* \otimes \sigma_* \otimes \mathfrak{f}_\sigma)[(\mathrm{id} \otimes \lambda)(\mathfrak{J}_\tau)] = (\mathrm{id} \otimes \lambda)(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(1 \otimes \mathfrak{J}_\sigma) \tag{5.24}$$

The equation (5.23) follows by axiom (5.4) of $J_{\sigma\tau}$, and the equation (5.24) follows by axiom (5.22) of $\mathfrak{J}_\sigma$.

$$\begin{aligned} &\Phi_\lambda(\Delta \otimes \mathrm{id})(\mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau))(J_\sigma \otimes 1) \\ &\overset{(5.23)}{=} \Phi_\lambda(\Delta \otimes \mathrm{id})(\mathfrak{J}_\sigma)(\Delta \otimes \mathrm{id})(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(J_\sigma \otimes 1) \\ &\overset{(5.21)}{=} \Phi_\lambda(\Delta \otimes \mathrm{id})(\mathfrak{J}_\sigma)(J_\sigma \otimes 1)(\sigma_* \otimes \sigma_* \otimes \mathfrak{f}_\sigma)(\Delta \otimes \mathrm{id})(\mathfrak{J}_\tau) \\ &= (\mathrm{id} \otimes \lambda)(\mathfrak{J}_\sigma)(1 \otimes \mathfrak{J}_\sigma)(\sigma_* \otimes \sigma_* \otimes \mathfrak{f}_\sigma)(\Phi_\lambda)(\sigma_* \otimes \sigma_* \otimes \mathfrak{f}_\sigma)(\Delta \otimes \mathrm{id})(\mathfrak{J}_\tau) \\ &= (\mathrm{id} \otimes \lambda)(\mathfrak{J}_\sigma)(1 \otimes \mathfrak{J}_\sigma)(\sigma_* \otimes \sigma_* \otimes \mathfrak{f}_\sigma)[\Phi_\lambda(\Delta \otimes \mathrm{id})(\mathfrak{J}_\tau)] \end{aligned} \tag{5.25}$$

$$
\begin{aligned}
(5.26)\quad & \Phi_\lambda(\Delta\otimes \mathrm{id}\,)(\mathfrak{J}_\sigma(\sigma_*\otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau))[J_\sigma(\sigma_*\otimes\sigma_*)(J_\tau)\otimes 1]\\
\overset{(5.25)}{=} & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma)(1\otimes\mathfrak{J}_\sigma)(\sigma_*\otimes\sigma_*\otimes\mathfrak{f}_\sigma)[\Phi_\lambda(\Delta\otimes\mathrm{id}\,)(\mathfrak{J}_\tau)]((\sigma_*\otimes\sigma_*)(J_\tau))\otimes 1\\
= & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma)(1\otimes\mathfrak{J}_\sigma)(\sigma_*\otimes\sigma_*\otimes\mathfrak{f}_\sigma)[\Phi_\lambda(\Delta\otimes\mathrm{id}\,)(\mathfrak{J}_\tau)(J_\tau\otimes 1)]\\
\overset{(5.21)}{=} & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma)(1\otimes\mathfrak{J}_\sigma)(\sigma_*\otimes\sigma_*\otimes\mathfrak{f}_\sigma)[(\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\tau)(1\otimes\mathfrak{J}_\tau)(\tau_*\otimes\tau_*\otimes\mathfrak{f}_\tau)(\Phi_\lambda)]\\
= & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma)(1\otimes\mathfrak{J}_\sigma)(\sigma_*\otimes\sigma_*\otimes\mathfrak{f}_\sigma)[(\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\tau)(1\otimes\mathfrak{J}_\tau)](\sigma_*\tau_*\otimes\sigma_*\tau_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)\\
= & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma)(1\otimes\mathfrak{J}_\sigma)(\sigma_*\otimes\sigma_*\otimes\mathfrak{f}_\sigma)[(\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\tau)]\\
& \times[1\otimes(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)](\sigma_*\tau_*\otimes\sigma_*\tau_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)\\
\overset{(5.24)}{=} & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma)(\mathrm{id}\,\otimes\lambda)(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(1\otimes\mathfrak{J}_\sigma)\\
& \times[1\otimes(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)](\sigma_*\tau_*\otimes\sigma_*\tau_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)\\
= & (\mathrm{id}\,\otimes\lambda)[(\mathfrak{J}_\sigma)(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)](1\otimes\mathfrak{J}_\sigma)\\
& \times[1\otimes(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)](\sigma_*\tau_*\otimes\sigma_*\tau_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)\\
= & (\mathrm{id}\,\otimes\lambda)(\mathfrak{f}_\sigma\bar{\circ}\,\mathfrak{f}_\tau)(1\otimes\mathfrak{f}_\sigma\bar{\circ}\,\mathfrak{f}_\tau)(\theta^{-1}_{(\sigma,\tau)}\otimes\theta^{-1}_{(\sigma,\tau)}\otimes 1)(\sigma_*\tau_*\otimes\sigma_*\tau_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)\\
\overset{(5.7)}{=} & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma\bar{\circ}\,\mathfrak{J}_\tau)(1\otimes\mathfrak{J}_\sigma\bar{\circ}\,\mathfrak{J}_\tau)((\sigma\tau)_*\otimes(\sigma\tau)_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)(\theta^{-1}_{(\sigma,\tau)}\otimes\theta^{-1}_{(\sigma,\tau)}\otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
& \Phi_\lambda(\Delta\otimes\mathrm{id}\,)(\mathfrak{J}_\sigma\bar{\circ}\,\mathfrak{J}_\tau)(J_{\sigma\tau}\otimes 1)\\
= & \Phi_\lambda(\Delta\otimes\mathrm{id}\,)(\mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau}\otimes 1))(J_{\sigma\tau}\otimes 1)\\
= & \Phi_\lambda(\Delta\otimes\mathrm{id}\,)(\mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau))(\Delta(\theta_{\sigma,\tau})J_{\sigma\tau}\otimes 1)\\
\overset{(5.10)}{=} & \Phi_\lambda(\Delta\otimes\mathrm{id}\,)(\mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau))[(J_\sigma(\sigma_*\otimes\sigma_*)(J_\tau)(\theta_{(\sigma,\tau)}\otimes\theta_{(\sigma,\tau)}))\otimes 1]\\
= & \Phi_\lambda(\Delta\otimes\mathrm{id}\,)(\mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau))[J_\sigma(\sigma_*\otimes\sigma_*)(J_\tau)\otimes 1]\\
& \times[\theta_{(\sigma,\tau)}\otimes\theta_{(\sigma,\tau)}\otimes 1]\\
\overset{(5.26)}{=} & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma\bar{\circ}\,\mathfrak{J}_\tau)(1\otimes\mathfrak{f}_\sigma\bar{\circ}\,\mathfrak{f}_\tau)((\sigma\tau)_*\otimes(\sigma\tau)_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)(\theta^{-1}\otimes\theta^{-1}\otimes 1)\\
& \times[\theta_{(\sigma,\tau)}\otimes\theta_{(\sigma,\tau)}\otimes 1]\\
= & (\mathrm{id}\,\otimes\lambda)(\mathfrak{J}_\sigma\bar{\circ}\,\mathfrak{J}_\tau)(1\otimes\mathfrak{f}_\sigma\bar{\circ}\,\mathfrak{f}_\tau)((\sigma\tau)_*\otimes(\sigma\tau)_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\Phi_\lambda)
\end{aligned}
$$

The proof of the second equation is over.

(3) Now we shall prove the third equation:

$$
\begin{aligned}
& \lambda(\mathfrak{f}_\sigma\circ\mathfrak{f}_\tau(x))\mathfrak{J}_\sigma\bar{\circ}\,\mathfrak{J}_\tau\\
= & \lambda(\mathfrak{f}_\sigma\circ\mathfrak{f}_\tau(x))\mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau}\otimes 1)\\
\overset{(5.22)}{=} & \mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)\lambda(\mathfrak{f}_\tau(x))(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau}\otimes 1)\\
= & \mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)[\lambda(\mathfrak{f}_\tau(x))\mathfrak{J}_\tau](\theta_{\sigma,\tau}\otimes 1)\\
\overset{(5.22)}{=} & \mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)[\mathfrak{J}_\tau(\tau_*\otimes\mathfrak{f}_\tau)\lambda(x)](\theta_{\sigma,\tau}\otimes 1)\\
= & \mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\sigma_*\tau_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\lambda(x))(\theta_{\sigma,\tau}\otimes 1)\\
\overset{(5.7)}{=} & \mathfrak{J}_\sigma(\sigma_*\otimes\mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau}\otimes 1)((\sigma\tau)_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)(\lambda(x))\\
= & (\mathfrak{J}_\sigma\bar{\circ}\,\mathfrak{J}_\tau)((\sigma\tau)_*\otimes\mathfrak{f}_\sigma\mathfrak{f}_\tau)\lambda(x)]
\end{aligned}
$$

Finally, we shall prove the associativity of $\overline{\circ}$,

$$\begin{aligned}[\mathfrak{J}_\sigma \overline{\circ}\, \mathfrak{J}_\tau]\overline{\circ}\, \mathfrak{J}_\rho &= [\mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau} \otimes 1)]\overline{\circ}\, \mathfrak{J}_\rho \\ &= \mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\theta_{\sigma,\tau} \otimes 1)((\sigma\tau)_* \otimes (\mathfrak{f}_\sigma \circ \mathfrak{f}_\tau))(\mathfrak{J}_\rho)(\theta_{\sigma\tau,\rho} \otimes 1) \\ \overset{(5.7)}{=}& \mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\sigma_*\tau_* \otimes (\mathfrak{f}_\sigma \circ \mathfrak{f}_\tau))(\mathfrak{J}_\rho)(\theta_{\sigma,\tau}\theta_{\sigma\tau,\rho} \otimes 1) \\ \overset{(5.8)}{=}& \mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)(\mathfrak{J}_\tau)(\sigma_*\tau_* \otimes (\mathfrak{f}_\sigma \circ \mathfrak{f}_\tau))(\mathfrak{J}_\rho)(\sigma_*(\theta_{\tau,\rho})\theta_{\sigma,\tau\rho} \otimes 1) \\ &= \mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)[\mathfrak{J}_\tau(\tau_* \otimes \mathfrak{f}_\tau)(\mathfrak{J}_\rho)(\theta_{\tau,\rho} \otimes 1)](\theta_{\sigma,\tau\rho} \otimes 1) \\ &= \mathfrak{J}_\sigma(\sigma_* \otimes \mathfrak{f}_\sigma)[\mathfrak{J}_\tau \overline{\circ}\, \mathfrak{J}_\rho](\theta_{\sigma,\tau\rho} \otimes 1) = \mathfrak{J}_\sigma \overline{\circ}\, [\mathfrak{J}_\tau \overline{\circ}\, \mathfrak{J}_\rho].\end{aligned}$$

□

5.6. **Crossed system of comodule algebras.** Let A be a quasi-Hopf algebra $(\mathcal{K}, \lambda, \Phi_\lambda)$ be a left A-comodule algebra and $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ a G-crossed system over A.

We define the monoidal category $\underline{\mathrm{Aut}}_G^{\mathrm{Tw}}(\mathcal{K})$ of twisted automorphisms as follows. Objects in $\underline{\mathrm{Aut}}_G^{\mathrm{Tw}}(\mathcal{K})$ are (σ_*, J_σ)-twisted automorphisms of $\mathcal{K}$ for $\sigma \in G$, the set of arrows are the isomorphisms of twisted homomorphisms of A-comodule algebras, the tensor product of object is defined by the composition explained in Lemma 5.10. The unity object is the $\mathrm{id}_{\mathcal{K}}$, and tensor product of arrows is as in $\underline{\mathrm{Aut}}^{\mathrm{Tw}}(A)$.

Let $F \subset G$ be a subgroup. An F-*crossed system for* a left A-comodule algebra $\mathcal{K}$, compatible with the G-crossed system $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ is a monoidal functor $\overline{(\ )} : \underline{F} \to \underline{\mathrm{Aut}}_G^{\mathrm{Tw}}(\mathcal{K})$, that is, an F-crossed system consists of the following data:

- A (σ_*, J_σ)-twisted automorphism $(\overline{\sigma}, \overline{J_\sigma})$ for each $\sigma \in F$,
- an element $\overline{\theta_{(\sigma,\tau)}} \in \mathcal{K}^\times$ for each $\sigma, \tau \in F$,

such that

$$(\overline{1}, \overline{J_1}) = (\mathrm{id}\,, 1\otimes 1), \tag{5.27}$$

$$\overline{\theta}_{(\sigma,\tau)}\overline{(\sigma\tau)}(k) = \overline{\sigma}(\overline{\tau}(k))\overline{\theta}_{(\sigma,\tau)}, \tag{5.28}$$

$$\overline{\theta}_{(\sigma,\tau)}\overline{\theta}_{(\sigma\tau,\rho)} = \overline{\sigma}(\overline{\theta}_{(\tau,\rho)})\overline{\theta}_{\sigma,\tau\rho}, \tag{5.29}$$

$$\overline{\theta}_{(1,\sigma)} = \overline{\theta}_{(\sigma,1)} = 1, \tag{5.30}$$

$$\lambda(\overline{\theta}_{(\sigma,\tau)})\overline{J}_{\sigma\tau} = \overline{J}_\sigma\big((\sigma_* \otimes \overline{\sigma})(\overline{J}_\tau)\big)\theta_{(\sigma,\tau)} \otimes \overline{\theta}_{(\sigma,\tau)}, \tag{5.31}$$

for all $k \in \mathcal{K}$, $\sigma, \tau, \rho \in F$. Let $\mathcal{K}\#F$ be the vector space $\mathcal{K}\otimes_{\Bbbk}\Bbbk F$ with product and coaction given by

(5.32)

$$(x\#\sigma)(y\#\tau) = x\overline{\sigma}(y)\overline{\theta}_{(\sigma,\tau)}\#\sigma\tau, \quad \delta(x\#\sigma) = x_{(-1)}\overline{J}^1_\sigma\#\sigma \otimes x_{(0)}\overline{J}^2_\sigma\#\sigma,$$

for all $x, y \in \mathcal{K}$, $\sigma, \tau \in F$.

Proposition 5.11. *The foregoing operations make the space* $\mathcal{K}\#F$ *into a left* $A\#G$*-comodule algebra with associator* $\Phi_\delta = \Phi^1_\lambda\#1\otimes\Phi^2_\lambda\#1\otimes\Phi^3_\lambda\#1$. □

Definition 5.12. Let G be a group, and $F \subseteq G$ be a subgroup. Let A be a G-crossed product quasi-bialgebra, and let $(\mathcal{K}, \lambda, \Phi_\lambda)$ be a left A-comodule algebra. We shall say that $\mathcal{K}$ is an *F-crossed product*, if there is a decomposition $\mathcal{K} = \bigoplus_{\sigma\in F} \mathcal{K}_\sigma$, such that

- $\Phi_\lambda \in A_e \otimes A_e \otimes \mathcal{K}_e$,
- $\mathcal{K}_\sigma \mathcal{K}_\tau \subseteq \mathcal{K}_{\sigma\tau}$ for all $\sigma, \tau \in F$,
- $\mathcal{K}_\sigma$ has an invertible element for each $\sigma \in F$,
- $\lambda(\mathcal{K}_\sigma) \subseteq A_\sigma \otimes \mathcal{K}_\sigma$ for each $\sigma \in F$.

Let A be a quasi-Hopf algebra and $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ be a crossed system for the group G. We have similar results as for quasi-Hopf algebras. The proof is analogous to the proof of Proposition 5.8.

Proposition 5.13. *Let $(\mathcal{L}, \delta)$ be a F-crossed $A\#G$-comodule algebra, for a subgroup $F \subseteq G$. Then there is an A-comodule algebra $\mathcal{K}$, and an F-crossed system over $\mathcal{K}$ compatible with the crossed system $(\sigma_*, \theta_{(\sigma,\tau)}, F_\sigma)_{\sigma,\tau\in G}$, such that $\mathcal{K}\#F$ and $\mathcal{L}$ are isomorphic $A\#G$-comodule algebras.*

Proof. Let $\mathcal{L}$ be a F-crossed $A\#G$-comodule algebra, for a subgroup $F \subseteq G$. Set $\mathcal{K} = \mathcal{L}_e$. Since every $\mathcal{L}_\sigma$ has an invertible element, we may choose for each $\sigma \in F$ some invertible element $u_\sigma \in \mathcal{L}_\sigma$, with $u_e = 1$. Then it is clear that $\mathcal{L}_\sigma = u_\sigma \mathcal{L}_e = \mathcal{L}_e u_\sigma$, and the set $\{u_\sigma : \sigma \in F\}$ is a basis for $\mathcal{L}$ as a left (and right) $\mathcal{L}_e$-module. Let us define the maps

$$\overline{\sigma}(a) = u_\sigma a u_\sigma^{-1}, \text{ for each } \sigma\ F \text{ and } a \in \mathcal{L}_e,$$

and

$$\overline{\theta} : G \times G \to \mathcal{L}_e \quad \text{by} \quad \overline{\theta}_{(\sigma,\tau)} = u_\sigma u_\tau u_{\sigma\tau}^{-1} \quad \text{for } \sigma, \tau \in F.$$

Note that $\{(1\#\sigma) \otimes u_\tau\}_{\sigma\in G,\tau\in F}$ is a basis for $A\#F \otimes \mathcal{L}$ as a left (and right) $A \otimes \mathcal{L}_e$-module. We have that $\delta(u_\sigma) \in A\#\sigma \otimes \mathcal{L}_\sigma$ can be uniquely expressed as $\delta(u_\sigma) = \overline{J}_\sigma((1\#\sigma) \otimes u_\sigma)$, with $\overline{J}_\sigma \in A \otimes \mathcal{L}_e$, for all $\sigma \in F$.

Then, it is straightforward to see that the data $(\overline{\sigma}, \overline{\theta}_{(\sigma,\tau)}, \overline{J}_\sigma)_{\sigma,\tau\in F}$, define an F-crossed system over the A-comodule algebra $\mathcal{L}_e$, and $\mathcal{L}_e\#F$ is isomorphic to $\mathcal{L}$ as $A\#G$ comodule algebras. □

Let G be an Abelian group, $F \subseteq G$ a subgroup, $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ be a crossed system over a quasi-Hopf algebra A, and $(\overline{\sigma}, \overline{\theta}_{(\sigma,\tau)}, \overline{J}_\sigma)_{\sigma,\tau\in F}$ be an F-crossed system for a A-comodule algebra $\mathcal{K}$. We shall further assume that

$$\theta_{(\sigma,\tau)} = \theta_{(\tau,\sigma)}, \quad \overline{\theta}_{(\rho,\nu)} = \overline{\theta}_{(\nu,\rho)}, \tag{5.33}$$

for all $\sigma, \tau \in G$, $\rho, \nu \in F$. We can consider the action of G on the category $\mathrm{Rep}(A)$ described in Lemma 5.4.

Proposition 5.14. *Under the above assumptions the following assertions hold.*

1. *The* $\mathrm{Rep}(A)$*-module category* ${}_{\mathcal{K}}\mathcal{M}$ *is F-equivariant.*
2. *There is an equivalence between* $({}_{\mathcal{K}}\mathcal{M})^F$ *and* ${}_{\mathcal{K}\#F}\mathcal{M}$ *as* $\mathrm{Rep}(A)^G$*-module categories.*

Proof. 1. For any $\rho \in F$ define $(U_\rho, c^\rho) : {}_{\mathcal{K}}\mathcal{M} \to ({}_{\mathcal{K}}\mathcal{M})^\rho$ the Rep(A)-module functor given as follows. For any $M \in {}_{\mathcal{K}}\mathcal{M}$, $U_\rho(M) = M$ as vector spaces and the action of $\mathcal{K}$ is given by: $x \cdot v = \overline{\rho}(x) \cdot v$, for all $x \in \mathcal{K}, v \in M$. For any $X \in \text{Rep}(A)$, $M \in {}_{\mathcal{K}}\mathcal{M}$ the maps $c^\rho_{X,M} : U_\rho(X \otimes_{\Bbbk} M) \to F_\rho(X) \otimes_{\Bbbk} U_\rho(M)$ are defined by $c^\rho_{X,M}(x \otimes v) = \overline{J}_\rho^{-1} \cdot (x \otimes v)$, for any $x \in X, v \in M$. Equation (2.1) for the pair (U_ρ, c^ρ) follows from (5.21).

For any $\sigma, \tau \in F$ define $\mu_{\sigma,\tau} : U_\sigma \circ U_\tau, \to U_{\sigma\tau}$ as follows. For any $M \in {}_{\mathcal{K}}\mathcal{M}$, $m \in M$

$$\mu_{\sigma,\tau}(m) = \overline{\theta}^{-1}_{(\sigma,\tau)} \cdot m.$$

It follows from equation (5.7) that $\mu_{\sigma,\tau}$ is a morphism of $\mathcal{K}$-modules. Equation (3.2) follows from (5.7) and (3.3) follows from (5.10).

2. Let $\mathcal{T} : ({}_{\mathcal{K}}\mathcal{M})^F \to {}_{\mathcal{K}\#F}\mathcal{M}$ be the module functor defined as follows. If (M, v) is an F-equivariant object then for any $\sigma \in F$ we have isomorphisms $v_\sigma : U_\sigma(M) \to M$ satisfying

$$v_{\sigma\tau}(\theta^{-1}_{(\sigma,\tau)} \cdot m) = v_\sigma(v_\tau(m), \quad v_\sigma(\overline{\sigma}(x) \cdot m) = x \cdot v_\sigma(m),$$

for all $\sigma, \tau \in F$, $x \in \mathcal{K}$, $m \in M$. In this case there is a well-defined action of $\mathcal{K}\#F$ on M determined by

$$(x\#\sigma) \cdot m = x \cdot v_\sigma^{-1}(m), \tag{5.34}$$

for all $\sigma \in F$, $x \in \mathcal{K}$, $m \in M$. We define $\mathcal{T}(M) = M$ with the above described action. If $(X, u) \in \text{Rep}(A)^G$, $(M, v) \in ({}_{\mathcal{K}}\mathcal{M})^F$ the action of $\mathcal{K}\#F$ on $X \otimes M$ using the coaction given in (5.32) coincides with the action (5.34) using the isomorphism $\widetilde{v}$ described in Lemma 3.3. The proof that $\mathcal{T}$ is an equivalence is analogous to the proof of Proposition 5.5. ☐

The category of F-equivariant objects in a module category is always of the form ${}_{\mathcal{K}\#F}\mathcal{M}$ for some left A-comodule algebra $\mathcal{K}$ and some group F.

Proposition 5.15. *Let A be a finite dimensional quasi-Hopf algebra and G be a finite Abelian group and $F \subset G$ a subgroup. Let $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ be a G-crossed system over A, and $\mathcal{M}$ be an exact F-equivariant* Rep(A)-*module category. Then there is a left A-comodule algebra $(\mathcal{K}, \lambda, \Phi^\lambda)$ such that ${}_{\mathcal{K}}\mathcal{M} \cong \mathcal{M}$ as* Rep(A)*-module categories and there is an F-crossed system compatible with $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau\in G}$ such that ${}_{\mathcal{K}\#F}\mathcal{M} \simeq \mathcal{M}^F$ as* Rep$(A)^G$-*module categories.*

Proof. Let B be a finite-dimensional quasi-Hopf algebra such that there is a quasi-Hopf algebra projection $\pi : B \to A$ and an equivalence Rep$(B) \simeq$ Rep$(A) \rtimes F$ of tensor categories, see section 3.4. Since $\mathcal{M}$ is F-equivariant follows from Proposition 3.4 that $\mathcal{M}$ is an exact Rep(B) module category.

Hence there exists a left B-comodule algebra $(\mathcal{K}, \lambda, \Phi^\lambda)$ such that $\mathcal{M} \simeq {}_{\mathcal{K}}\mathcal{M}$ as Rep(B)-modules. Let us recall that the equivariant structure is given by

$$(U_\sigma, c^\sigma) : \mathcal{M} \to \mathcal{M}^\sigma, \quad U_\sigma(M) = [\mathbf{1}, \sigma]\overline{\otimes} M,$$

for all $\sigma \in F$, $M \in \mathcal{M}$ together with a family of natural isomorphisms $\mu_{\sigma,\tau} : U_\sigma \circ U_\tau \to U_{\sigma\tau}$ for any $\sigma, \tau \in F$. Under the equivalence $\text{Rep}(B) \simeq \text{Rep}(A) \rtimes F$ the object $[\mathbf{1}, \sigma]$ correspond to a 1-dimensional representation of B. For any $\sigma \in F$ let us denote by $\chi_\sigma : B \to \Bbbk$ the corresponding character and the algebra map $\overline{\sigma} : \mathcal{K} \to \mathcal{K}$, $\overline{\sigma}(k) = \chi_\sigma(k_{(-1)})\, k_{(0)}$, for all $k \in \mathcal{K}$.

Define $\lambda^\pi = (\pi \otimes \text{id}\,)\lambda$, then $(\mathcal{K}, \lambda^\pi, (\pi \otimes \pi \otimes \text{id}\,)(\Phi^\lambda))$ is a left A-comodule algebra that we will denote by $\mathcal{K}^\pi$. The equivalence $\mathcal{M} \simeq {}_{\mathcal{K}}\mathcal{M}$ of $\text{Rep}(B)$-module categories induces an equivalence $\mathcal{M} \simeq {}_{\mathcal{K}^\pi}\mathcal{M}$ of $\text{Rep}(A)$-modules. Under this equivalence the functors $U_\sigma : {}_{\mathcal{K}^\pi}\mathcal{M} \to ({}_{\mathcal{K}^\pi}\mathcal{M})^\sigma$ are given as follows. For any $M \in {}_{\mathcal{K}^\pi}\mathcal{M}$, $U_\sigma(M) = M$ and the action of $\mathcal{K}$ on M is given by

$$k \cdot m = \overline{\sigma}(k) \cdot m, \qquad \text{for all } k \in \mathcal{K}, m \in M.$$

For any $\sigma, \tau \in F$ denote

$$\overline{J}_\sigma = c^\sigma_{A,\mathcal{K}}(1 \otimes 1)^{-1}, \quad \overline{\theta}_{\sigma,\tau} = (\mu_{\tau,\sigma})_{\mathcal{K}}(1)^{-1}.$$

Turns out that the collection $(\overline{\sigma}, \overline{\theta}_{(\sigma,\tau)}, \overline{J}_\sigma)_{\sigma,\tau \in F}$ is an F-crossed system compatible with $(\sigma_*, \theta_{(\sigma,\tau)}, J_\sigma)_{\sigma,\tau \in G}$ for the A-comodule algebra $\mathcal{K}^\pi$. Indeed for any $\sigma \in F$ the pair $(\overline{\sigma}, \overline{J_\sigma})$ is a (σ_*, J_σ)-twisted automorphism since equation (5.21) follows from the fact that c^σ satisfies (2.1) and equation (5.22) follows since c^σ is a $\mathcal{K}$-module morphism. Equation (5.28) follows since $\mu_{\sigma,\tau}$ is a morphism of $\mathcal{K}$-modules, equation (5.29) follows from (3.2) and equation (5.31) follows from (3.3). The equivalence ${}_{\mathcal{K}^\pi \# F}\mathcal{M} \simeq \mathcal{M}^F$ as $\text{Rep}(A)^G$-module categories follows from Proposition 5.14.

□

6. Module categories over the quasi-Hopf algebras $A(H, s)$

6.1. **Basic Quasi-Hopf algebras** $A(H, s)$**.** We recall the definition of a the family of basic quasi-Hopf algebras $A(H, s)$ introduced by I. Angiono [A] and used to give a classification of pointed tensor categories with cyclic group of invertible objects of order m such that $210 \nmid m$.

Let $m \in \mathbb{N}$ and $H = \oplus_{n \geq 0} H(n)$ be a finite-dimensional radically graded pointed Hopf algebra generated by a group like element χ of order m^2 and skew primitive elements $x_1, ..., x_\theta$ satisfying

$$\chi x_i \chi^{-1} = q^{d_i} x_i, \quad \Delta(x_i) = x_i \otimes 1 + \chi^{-b_i} \otimes x_i, \tag{6.1}$$

for any $i = 1, \ldots, \theta$, where q is a primitive root of 1 of order m^2, $H = \mathfrak{B}(V) \# \Bbbk C_{m^2}$, where $\mathfrak{B}(V)$ is the associated Nichols algebra of the Yetter-Drinfeld module $V \in {}^{\Bbbk C_{m^2}}_{\Bbbk C_{m^2}}\mathcal{YD}$.

We shall further assume that $\mathfrak{B}(V)$ has a basis $\{x_1^{s_1} \ldots x_\theta^{s_\theta} : 0 \leq s_i \leq N_i\}$.

Remark 6.1. The above condition does not hold for any Nichols algebra. If V has diagonal braiding with Cartan matrix of type A_3 then $\mathfrak{B}(V)$ is not generated by elements of degree 1. This conditions is satisfied for example for any quantum linear space.

Set $\sigma := \chi^m$ and denote by $\{1_i : i \in C_{m^2}\}$, $\{\mathbf{1}_j : j \in C_m\}$ the families of primitive idempotents in $\Bbbk C_{m^2}$ and $\Bbbk C_m$ respectively. That is

$$1_i = \frac{1}{m^2}\sum_{k=0}^{m^2-1} q^{-ki}\,\chi^k, \quad \mathbf{1}_j = \frac{1}{m}\sum_{l=0}^{m-1} q^{-mlj}\,\sigma^l.$$

For any $0 \le s \le m-1$ set $J_s = \sum_{i,j=0}^{m^2-1} c(i,j)^s 1_i \otimes 1_j$, where $c(i,j) := q^{j(i-i')}$. Here j' denotes the remainder in the division by m. The associator $\Phi_s = dJ_s$ is written explicitly as

$$\Phi_s := \sum_{i,j,k=0}^{m-1} \omega_s(i,j,k)\mathbf{1}_i \otimes \mathbf{1}_j \otimes \mathbf{1}_k, \tag{6.2}$$

where $\omega_s : (C_m)^3 \to \Bbbk^\times$ is the 3-cocycle defined by $\omega_s(i,j,k) = q^{sk(j+i-(j+i)')}$. Consider the quasi-Hopf algebra (H_{J_s}, Φ_s) obtained by twisting H. Denote $\Upsilon(H) = \{1 \le s \le m-1 : b_i \equiv sd_i \operatorname{mod}(m),\, 1 \le i \le \theta\}$. For any $s \in \Upsilon(H)$ the quasi-Hopf algebra $A(H,s)$ is defined as the subalgebra of H generated by σ and $x_1, ..., x_\theta$. The algebra $A(H,s)$ is a quasi-Hopf subalgebra of H_{J_s} with associator Φ_s such that $A(H,s)/\operatorname{Rad} A(H,s) \cong \Bbbk[C_m]$. See [A, Prop. 3.1.1].

For any $1 \le i \le \theta$ we have that

$$\Delta_{J_s}(x_i) = \sum_{y=0}^{m-1} q^{b_i y}\mathbf{1}_y \otimes x_i + \sum_{z=0}^{m-1}\left(\sum_{y=0}^{m-d_i'-1} q^{(d_i'-d_i)sz} x_i \mathbf{1}_y \otimes \mathbf{1}_z + \sum_{j=m-d_i'}^{m-1} q^{(d_i'+m-d_i)sz} x_i \mathbf{1}_y \otimes \mathbf{1}_z\right).$$

Remark 6.2. Our definition of $A(H,s)$ is slightly different that the one given in [A, § 3]. This is not a problem since our quasi-Hopf algebras are isomorphic to the ones defined in *loc. cit.* except that the s may change. The difference comes from the fact that we are using $(\chi^{-b_i}, 1)$ skew-primitive elements instead of $(1, \chi^{b_i})$ skew-primitive elements.

6.2. **C_m-crossed system over $A(H,s)$.** The cyclic group with m elements will be denoted by $C_m = \{1, h, h^2, \dots, h^{m-1}\}$. For any $0 \le i < m$ set $((h^i)_*, J_{h^i})$ the twisted endomorphism of $A(H,s)$ given by

$$J_{h^i} = 1 \otimes 1, \quad (h^i)_*(a) = \chi^{i'} a \chi^{-i'} \quad \text{for all } a \in A(H,s).$$

For any $0 \le i,j < m$ define $\theta_{(i,j)} = \theta_{(h^i,h^j)} = \sigma^{\frac{(i+j)-(i+j)'}{m}}$.

Remark 6.3. If $i+j \le m$ then $\theta_{(i,j)} = 1$ and if $i+j > m$ then $\theta_{(i,j)} = \sigma$. In principle the algebra maps $(h^i)_*$ are defined in H but when restricted to $A(H,s)$ they are well-defined.

These data is a C_m-crossed system over $A(H,s)$ such that the equivariantization $\mathrm{Rep}(A)^{C_m}$ is tensor equivalent to $\mathrm{Rep}(H)$. This is contained in the next result which gives an alternative proof for [A, Thm. 4.2.1].

Proposition 6.4. 1. $((h^i)_*, \theta_{(i,j)}, J_{h^i})_{h^i,h^j\in C_m}$ *is a C_m-crossed system over $A(H,s)$.*
2. *There is an isomorphism of quasi-Hopf algebras $A(H,s)\#C_m \simeq H_{J_s}$.*
3. *There is a tensor equivalence* $\mathrm{Rep}(A)^{C_m} \simeq \mathrm{Rep}(H)$.

Proof. 1. it follows by a straightforward computation.

2. Define $\varphi: A(H,s)\#C_m \to H_{J_s}$ the linear map given by

$$\varphi(a\#h^i) = a\chi^{i'},$$

for all $0 \le i < m$, $a \in A$. Let $0 \le i,j < m$, $a,b \in A$ then

$$\varphi((a\#h^i)(b\#h^j)) = \varphi(a(h^i)_*(b)\theta_{(i,j)}\#h^{i+j}) = a\chi^{i'} b\chi^{-i'}\theta_{(i,j)}\chi^{(i+j)'}.$$

On the other hand

$$\varphi(a\#h^i)\varphi(b\#h^j) = a\chi^{i'} b\chi^{j'}.$$

It is enough to prove that $\chi^{i'} b\chi^{j'} = \chi^{i'} b\chi^{-i'}\theta_{(i,j)}\chi^{(i+j)'}$ for $b = x_l$, $1 \le l \le \theta$. If $i + j \le m$ then

$$\chi^{i'} x_l\chi^{-i'}\theta_{(i,j)}\chi^{(i+j)'} = q^{d_l i}\, x_l\chi^{i+j} = \chi^i x_l\chi^j.$$

If $i + j = m + k$, $k > 0$ then

$$\chi^{i'} x_l\chi^{-i'}\theta_{(i,j)}\chi^{(i+j)'} = q^{d_l i}\, x_l\sigma\chi^k = q^{d_l i}\, x_l\chi^{i+j} = \chi^i x_l\chi^j.$$

It follows immediately that φ is a coalgebra map and it is injective and by a dimension argument is bijective.

3. It follows from Proposition 5.5. □

Remark 6.5. There is a grading on H compatible with the isomorphism of Proposition 6.4 (2). Namely, if $\sigma \in G$ then the vector space H_σ has basis $\{x_1^{s_1}\dots x_\theta^{s_\theta}\sigma\}$. Define $H^{(i)} = \oplus_{j=0}^{m-1} H_{\chi^{mj+i}}$, thus $H = \oplus_{j=0}^{m-1} H^{(j)}$. It is not difficult to prove that with this grading H is a C_m-crossed product (see definition 5.7) and this crossed product is compatible with the isomorphism of Proposition 6.4 (2).

6.3. **Right simple $A(H,s)$-comodule algebras.** We shall present some families of right $A(H,s)$-simple left $A(H,s)$-comodule algebras. This class will be big enough to classify module categories over $\mathrm{Rep}(A(H,s))$ in some cases.

Let (K,λ) be a finite-dimensional left H-comodule algebra. We say that (K,λ) is *of type 1* if the following assumptions are satisfied:

- There exists a subgroup $F \subseteq C_{m^2}$ and $t \in \mathbb{N}$ such that K has a basis $\{y_1^{r_1}\dots y_t^{r_t} e_f : 0 \le r_j < N_j, f \in F, t \le \theta\}$ such that

$$e_{\chi^a} y_l = q^{a d_l} y_l e_{\chi^a}, \quad \text{if } \chi^a \in F.$$

- there is an inclusion $\iota : K \hookrightarrow H$ of H-comodules such that

$$\iota(e_f) = f, \quad \iota(y_l) = x_l,$$

for all $f \in F$, $l = 1 \dots t$.

Observe that in this case we have that

$$\lambda(e_f) = f \otimes e_f, \quad \lambda(y_l) = x_l \otimes 1 + \chi^{-b_l} \otimes y_l.$$

Definition 6.6. We shall say that a Hopf algebra $H = \mathfrak{B}(V)\#\Bbbk G$ is of type 1 if

(1) $\mathfrak{B}(V)$ has a basis $\{x_1^{s_1} \dots x_\theta^{s_\theta} : 0 \leq s_i \leq N_i\}$, where V is the vector space generated by $\{x_1, \dots, x_\theta\}$,
(2) any right H-simple left H-comodule algebra (K, λ) is equivariantly Morita equivalent to a comodule algebra of type 1.

Remark 6.7. If $H = \mathfrak{B}(V)\#\Bbbk\Gamma$ is the bosonization of a Nichols algebra and a group algebra a finite group Γ then H is of type 1 when V is a quantum linear space and Γ is an Abelian group [Mo2] or when V is constructed from a rack and $\Gamma = \mathbb{S}_3, \mathbb{S}_4$ [GM].

Let (K, λ) be a type 1 left H-comodule algebra such that $K_0 = \Bbbk F$ where $F \subseteq C_{m^2}$ is a subgroup such that $< \sigma > \subseteq F$ we shall denote by $\lambda^{J_s} : K \to H \otimes K$ the map given by

$$\lambda^{J_s}(x) = J_s \lambda(x) J_s^{-1}, \quad \text{for all } x \in K.$$

Here J_s is identified with an element in $H \otimes K$ via the inclusion $\mathrm{id}_H \otimes \iota$. The same calculation as in [A, Prop. 3.1.1] proves that $\lambda^{J_s}(K) \subseteq H \otimes K$. Define $(K^{J_s}, \lambda^{J_s}, \Phi_s(J_s \otimes 1))$ the left H-comodule algebra with underlying algebra K^{J_s}, coaction λ^{J_s} and associator $\Phi_s(J_s \otimes 1)$. It follows from Lema 4.9 that $(K^{J_s}, \lambda^{J_s}, \Phi_s)$ is a left H_{J_s}-comodule algebra.

Lemma 6.8. *The left H-comodule algebras (K, λ) and $(K^{J_s}, \lambda^{J_s}, \Phi_s(J_s \otimes 1))$ are equivariantly Morita equivalent, that is ${}_K\mathcal{M}$, ${}_{K^{J_s}}\mathcal{M}$ are equivalent as* Rep(H)*-modules.* □

Proof. For any $X \in \mathrm{Rep}(H)$, $M \in {}_K\mathcal{M}$ and any $x \in X$, $m \in M$ define

$$c_{X,M} : X \otimes_{\Bbbk} M \to X \otimes_{\Bbbk} M, \quad c_{X,M}(x \otimes m) = J_s \cdot (x \otimes m).$$

It is immediate to prove that the identity functor $(\mathrm{Id}, c) : {}_K\mathcal{M} \to {}_{K^{J_s}}\mathcal{M}$ is an equivalence of module categories. □

Definition 6.9. Let $(\mathcal{K}, \lambda, \Phi^\lambda)$ be a left H_{J_s}-comodule algebra such that the associator $\Phi^\lambda \in A(H, s) \otimes_{\Bbbk} A(H, s) \otimes_{\Bbbk} \mathcal{K}$. Define $\widehat{\mathcal{K}} = \lambda^{-1}(A(H, s) \otimes_{\Bbbk} \mathcal{K})$ and denote $\widehat{\lambda}$ the restriction of λ to $\widehat{\mathcal{K}}$. Then $(\widehat{\mathcal{K}}, \widehat{\lambda}, \Phi_s)$ is a left $A(H, s)$-comodule algebra. Turns out that this procedure is the inverse of the crossed product.

6.4. **Actions on module categories $_{(\widehat{K},\widehat{\lambda},\Phi_s)}\mathcal{M}$.** For the rest of this section we shall assume now that $m = p$ is a prime number.

Let (K, λ) be a type 1 left H-comodule algebra such that $K_0 = \Bbbk F$ where $F = C_d$ is a cyclic group.

There are two possible cases; when $<\sigma>\subseteq F$ or $F = \{1\}$. Let us treat the first case. So we assume that $p|d$. Let $s, l \in \mathbb{N}$ be such that $d = ps$ and $sl = p$. Let us denote $\widehat{F} = C_s =<\chi^{lp}>$.

By hypothesis the vector space K has a decomposition $K = \oplus_{f\in F} K_f$ where K_f is the vector space with basis $\{y_1^{r_1}\dots y_t^{r_t}e_f : 0 \le r_j \le N_j\}$. For any $i = 0\dots s-1$ define

$$K^{(i)} = \bigoplus_{j:\,\chi^{mj+i}\in C_d} K_{\chi^{mj+i}}.$$

Observe that $\widehat{K} = K^{(0)}$. With this grading K is an $\widehat{F}$-crossed product.

Lemma 6.10. *Under the above assumptions $_{(\widehat{K},\widehat{\lambda},\Phi_s)}\mathcal{M}$ is an $\widehat{F}$-equivariant* $\mathrm{Rep}(A(H,s))$*-module category and $\left(_{(\widehat{K},\widehat{\lambda},\Phi_s)}\mathcal{M}\right)^{\widehat{F}} \simeq {}_K\mathcal{M}$ as module categories over* $\mathrm{Rep}(H)$.

Proof. It follows from Proposition 5.13 and Proposition 5.14. □

Now, let us assume that $F = \{1\}$. Let us endow the space $K\otimes_\Bbbk \Bbbk C_p$ with the product determined by

$$(y_l\otimes\sigma^a)(y_s\otimes\sigma^b) = q^{pad_s}\, y_l y_s\otimes\sigma^{a+b}.$$

The space $K\otimes_\Bbbk \Bbbk C_p$ is a left H-comodule algebra with coproduct determined by

$$\lambda(y_l\otimes\sigma^a) = x_l\sigma^a\otimes 1\otimes\sigma^a + \sigma^a\chi^{-b_l}\otimes y_l\otimes\sigma^a.$$

It is clear that $(K\otimes_\Bbbk \Bbbk C_p)_0 = \Bbbk C_p$. Thus we can consider the left $A(H,s)$-comodule algebra $(K\otimes_\Bbbk \Bbbk C_p, \widehat{\lambda}, \Phi_s)$.

Lemma 6.11. *Under the above conventions the following holds.*

1. *The module category $_{(\Bbbk C_p,\lambda,\Phi_s)}\mathcal{M}$ has a C_p-action such that there is an equivalence $\left(_{(\Bbbk C_p,\lambda,\Phi_s)}\mathcal{M}\right)^{C_p} \simeq \mathrm{Vect}_\Bbbk$ as* $\mathrm{Rep}(H)$*-modules.*
2. *The module category $_{(K\otimes_\Bbbk \Bbbk C_p,\widehat{\lambda},\Phi_s)}\mathcal{M}$ has a C_p-action such that there is an equivalence $\left(_{(K\otimes_\Bbbk \Bbbk C_p,\widehat{\lambda},\Phi_s)}\mathcal{M}\right)^{C_p} \simeq {}_K\mathcal{M}$ as* $\mathrm{Rep}(H)$*-modules.*

Proof. 1. It follows from (2) taking $K = \Bbbk$.

2. Set $\mathcal{M} = {}_{(K\otimes_\Bbbk \Bbbk C_p,\widehat{\lambda},\Phi_s)}\mathcal{M}$. For any $i = 0,\dots,p-1$ define the functors $(U_i, c^i) : \mathcal{M} \to \mathcal{M}^{\sigma^i}$ as follows. For any $M \in \mathcal{M}$ $U_i(M) = M$ with a new action $\triangleright : (K\otimes_\Bbbk \Bbbk C_p)\otimes_\Bbbk M \to M$ of $K\otimes_\Bbbk \Bbbk C_p$ given by

$$y_l \triangleright m = q^{id_l}\, y_l\cdot m, \quad \sigma\triangleright m = q^{ip}\,\sigma\cdot m,$$

for all $l = 1, \dots, t$, $m \in M$. For any $X \in \mathrm{Rep}(A), M \in \mathcal{M}$ the map $c^i_{X,M} : U_i(X \otimes_{\Bbbk} M) \to F_i(X) \otimes_{\Bbbk} U_i(M)$ is the identity.

The isomorphism $\mu_{i,j} : U_i \circ U_j \to U_{i+j}$ is given by the action of $\sigma^{-\frac{(i+j)-(i+j)'}{p}}$. Altogether makes the category ${}_{(K\otimes_{\Bbbk}\Bbbk C_p, \widehat{\lambda}, \Phi_s)}\mathcal{M}$ a C_p-equivariant $\mathrm{Rep}(A)$-module category.

Let $N \in {}_K\mathrm{Mod}$. Define $\mathcal{F}(N) = \oplus_{i=0}^{p-1} N_i$ where $N_i = N$ as vector spaces. Let us define a new action of $\rightharpoonup: K\otimes_{\Bbbk}\Bbbk C_p \otimes_{\Bbbk} \mathcal{F}(N) \to \mathcal{F}(N)$ as follows. If $n \in N_i$ then

$$\sigma \rightharpoonup n = q^{pi}\, n \in N_i, \quad y_l \rightharpoonup n = q^{d_l i}\, y_l \cdot n \in N_{(d_l+i)'}.$$

Recall that a' denotes the remainder of a in the division by p. Note also that for any $i, j = 0, \dots, p-1$ $U_i(N_j) = N_{i+j}$. The module $\mathcal{F}(N)$ is a C_p-equivariant object in ${}_{(K\otimes_{\Bbbk}\Bbbk C_p, \widehat{\lambda}, \Phi_s)}\mathcal{M}$, indeed for any $i = 0, \dots, p-1$ define the isomorphisms $v_i : U_i(\mathcal{F}(N)) \to \mathcal{F}(N)$ as follows: $v_i(n) = q^{-i}\, n \in N_{i+j}$ for any $n \in N_j$. This maps are $K\otimes_{\Bbbk}\Bbbk C_p$-module isomorphisms and they satisfy equation (3.4). This defines a functor $\mathcal{F} : {}_K\mathrm{Mod} \to {}_{(K\otimes_{\Bbbk}\Bbbk C_p, \widehat{\lambda}, \Phi_s)}\mathcal{M}$ that together with the identity isomorphisms $c_{X,N} : \mathcal{F}(X\otimes_{\Bbbk} N) \to X \otimes_{\Bbbk} \mathcal{F}(N)$ becomes a module functor.

If $M \in {}_{(K\otimes_{\Bbbk}\Bbbk C_p, \widehat{\lambda}, \Phi_s)}\mathcal{M}$ then $M = \oplus_{i=0}^{p-1} M_i$ where M_i is the eigenspace of the eigenvalue q^{pi} of the action of σ. The space M_0 has a K-action as follows. Since M is C_p-equivariant there are isomorphisms $v_i : U_i(M) \to M$ such that the restrictions $v_i \mid_{M_0} : M_0 \to M_i$ are isomorphisms. If $m \in M_0$, $y_l \in K$ then $y_l \cdot m \in M_{d_l}$, thus we can define $\rightharpoonup: K \otimes_{\Bbbk} M_0 \to M_0$

$$y_l \rightharpoonup m = v_{d_l}^{-1}(y_l \cdot m),$$

for all $m \in M_0$. The map $M \mapsto M_0$ is functorial and defines an inverse functor for $\mathcal{F}$. □

6.5. **Exact module categories over** $\mathrm{Rep}(A(H, s))$**.** Now we can formulate the main result of this section.

Theorem 6.12. *Let H be a Hopf algebra of type 1 (see definition 6.6) and let $\mathcal{M}$ be an exact indecomposable module category over* $\mathrm{Rep}(A(H, s))$*. Then the following statements hold.*

1. *there exists a right H-simple left H-comodule algebra (K, λ) with trivial coinvariants such that $K_0 \supseteq C_p$ and there is an equivalence of module categories $\mathcal{M} \simeq {}_{(\widehat{K}, \widehat{\lambda}, \Phi_s)}\mathcal{M}$.*
2. *If there is an equivalence ${}_{(\widehat{K'}, \widehat{\lambda'}, \Phi'_s)}\mathcal{M} \simeq {}_{(\widehat{K}, \widehat{\lambda}, \Phi_s)}\mathcal{M}$ as* $\mathrm{Rep}(A(H, s))$*-modules then (K, λ) and (K', λ') are equivariantly Morita equivalent H-comodule algebras.*

Proof. 1. By Lemma 4.7 there exists a left $A(H, s)$-comodule algebra $(\mathcal{K}, \lambda, \Phi)$ such that $\mathcal{M} \simeq {}_{\mathcal{K}}\mathcal{M}$. The category ${}_{\mathcal{K}}\mathcal{M}$ is F-equivariant for some subgroup $F \subseteq C_p$. Thus it follows from [AM, Thm 3.3] that there is a right

H-simple left H-comodule algebra (S, δ) with trivial coinvariants such that $\big({}_K\mathcal{M}\big)^F \simeq {}_S\mathcal{M}$ as $\text{Rep}(H)$-modules. Hence $S_0 = \Bbbk 1$, $S_0 = \Bbbk C_p$ or $S_0 = \Bbbk C_{p^2}$. In any case, it follows from Lemmas 6.10, 6.11 that there is a right H-simple left H-comodule algebra (K, λ) with trivial coinvariants such that $K_0 \supseteq C_p$ and there is an equivalence ${}_S\mathcal{M} \simeq \big({}_{(\widehat{K'}, \widehat{\lambda'}, \Phi'_s)}\mathcal{M}\big)^F$. Whence $\big({}_K\mathcal{M}\big)^F \simeq \big({}_{(\widehat{K'}, \widehat{\lambda'}, \Phi'_s)}\mathcal{M}\big)^F$, thus using Proposition 3.4 (5) we get the result.

2. There exists a subgroup $F \subseteq C_p$ such that both module categories ${}_{(\widehat{K'}, \widehat{\lambda'}, \Phi'_s)}\mathcal{M}$, ${}_{(\widehat{K}, \widehat{\lambda}, \Phi_s)}\mathcal{M}$ are F-equivariant and there are equivalences of module categories over $\text{Rep}(H_{J_s})$

$$ {}_{(K,\lambda,\Phi_s)}\mathcal{M} \simeq \big({}_{(\widehat{K}, \widehat{\lambda}, \Phi_s)}\mathcal{M}\big)^F \simeq \big({}_{(\widehat{K'}, \widehat{\lambda'}, \Phi'_s)}\mathcal{M}\big)^F \simeq {}_{(K', \lambda', \Phi'_s)}\mathcal{M}. $$

Thus by Lemma 6.8 follows that ${}_K\mathcal{M} \simeq {}_{K'}\mathcal{M}$. □

6.6. **Some classification results.** We apply Theorem 6.12 to obtain the classification of module categories over $\text{Rep}(A(H, s))$ where H is the bosonization of a quantum linear space.

Let $g_1, \dots, g_\theta \in C_{p^2}$, $\chi_1, \dots, \chi_\theta \in \widehat{C_{p^2}}$ be a datum for a quantum linear space and let $V = V(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ the associated Yetter-Drinfeld module over $\Bbbk C_{p^2}$ generated as a vector space by $x_1, \dots, x_\theta$. For more details see [AS].

The Hopf algebra $H = \mathfrak{B}(V)\#\Bbbk C_{p^2}$ is a type 1 Hopf algebra, see [Mo2].

Let us define now a family of right H-simple left H-comodule algebras. Let $F \subseteq C_{p^2}$ be a subgroup and $\xi = (\xi_i)_{i=1\dots\theta}$, $\alpha = (\alpha_{ij})_{1\le i<j\le\theta}$ be two families of elements in $\Bbbk$ satisfying

$$ \xi_i = 0 \ \text{ if } \ g_i^{N_i} \notin F \ \text{ or } \ \chi_i^{N_i}(f) \neq 1, \tag{6.3} $$

$$ \alpha_{ij} = 0 \ \text{ if } \ g_i g_j \notin F \ \text{ or } \ \chi_i\chi_j(f) \neq 1, \tag{6.4} $$

for all $f \in F$. In this case we shall say that the pair (ξ, α) is a *compatible comodule algebra datum* with respect to the quantum linear space V and the group F.

The algebra $\mathcal{A}(V, F, \xi, \alpha)$ is the algebra generated by elements in $\{v_i : i = 1 \dots \theta\}$, $\{e_f : f \in F\}$ subject to relations

$$ e_f e_g = e_{fg}, \quad e_f v_i = \chi_i(f)\, v_i e_f, \tag{6.5} $$

$$ v_i v_j - q_{ij}\, v_j v_i = \begin{cases} \alpha_{ij}\, e_{g_i g_j} & \text{if } g_i g_j \in F \\ 0 & \text{otherwise,} \end{cases} \tag{6.6} $$

$$ v_i^{N_i} = \begin{cases} \xi_i\, e_{g_i^{N_i}} & \text{if } g_i^{N_i} \in F \\ 0 & \text{otherwise,} \end{cases} \tag{6.7} $$

for any $1 \leq i < j \leq \theta$. If $W \subseteq V$ is a $\Bbbk C_{p^2}$-subcomodule invariant under the action of F, we define $\mathcal{A}(W, F, \xi, \alpha)$ as the subalgebra of $\mathcal{A}(V, F, \xi, \alpha)$ generated by W and $\{e_f : f \in F\}$.

The algebras $\mathcal{A}(V, F, \xi, \alpha)$ are right H-simple left H-comodule algebras with coaction determined by

$$\lambda(v_i) = x_i \otimes 1 + g_i \otimes v_i, \quad \lambda(e_f) = f \otimes e_f,$$

for all $i = 1, \ldots, \theta$, $f \in F$. The subalgebras $\mathcal{A}(W, F, \xi, \alpha)$ are also right H-simple left H-subcomodule algebras.

Theorem 6.13. [Mo2, Thm 4.6, Thm. 4.9] *Let $\mathcal{M}$ be an exact indecomposable module category over* $\text{Rep}(H)$.

1. *There exists a subgroup $F \subseteq C_{p^2}$, a compatible datum (ξ, α) and $W \subseteq V$ a subcomodule invariant under the action of F such that $\mathcal{M} \simeq {}_{\mathcal{A}(W,F,\xi,\alpha)}\mathcal{M}$ as module categories.*
2. *The left H-comodule algebras $\mathcal{A}(W, F, \xi, \alpha)$, $\mathcal{A}(W', F', \xi', \alpha')$ are equivariantly Morita equivalent if and only if $(W, F, \xi, \alpha) = (W', F', \xi', \alpha')$.*

□

Given a compatible datum (ξ, α) with respect to V and C_p define the left $A(H, s)$-comodule algebra $\widehat{\mathcal{A}}(V, \xi, \alpha)$ with underlying algebra equal to $\mathcal{A}(V, C_p, \xi, \alpha)$ and coaction $\widehat{\lambda} : \widehat{\mathcal{A}}(V, \xi, \alpha) \to A(H, s) \otimes_{\Bbbk} \widehat{\mathcal{A}}(V, \xi, \alpha)$ given by $\widehat{\lambda}(a) = J_s \lambda(a) J_s^{-1}$ for all $a \in \widehat{\mathcal{A}}(V, \xi, \alpha)$. If $W \subseteq V$ is a $\Bbbk C_{p^2}$-subcomodule invariant under the action of C_p define $\widehat{\mathcal{A}}(W, \xi, \alpha)$ as the subalgebra of $\widehat{\mathcal{A}}(V, \xi, \alpha)$ generated by W and C_p.

As a consequence of Theorem 6.12 we have the following result.

Theorem 6.14. *Let $\mathcal{M}$ be an exact indecomposable module category over* $\text{Rep}(A(H, s))$.

1. *There exists a compatible datum (ξ, α) and $W \subseteq V$ a subcomodule invariant under the action of C_p such that there is an equivalence $\mathcal{M} \simeq {}_{\widehat{\mathcal{A}}(W,\xi,\alpha)}\mathcal{M}$ as* $\text{Rep}(A(H, s))$*-module categories.*
2. *The comodule algebras $\widehat{\mathcal{A}}(W, \xi, \alpha)$, $\widehat{\mathcal{A}}(W', \xi', \alpha')$ are equivariantly Morita equivalent if and only if $(W, \xi, \alpha) = (W', \xi', \alpha')$.*

□

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