MODULE CATEGORIES OVER POINTED HOPF ALGEBRAS

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ABSTRACT. We develop some techniques for studying exact module categories over some families of pointed finite-dimensional Hopf algebras. As an application we classify exact module categories over the tensor category of representations of the small quantum groups $u_q(\mathfrak{sl}_2)$.

1. INTRODUCTION

Given a tensor category $\mathcal{C}$, a module category over $\mathcal{C}$ is an abelian category $\mathcal{M}$ together with an exact functor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ satisfying natural associativity and unity axioms. This rather general notion appears, and is implicitly present, in many different areas of mathematics and mathematical physics, such as the theory of weak Hopf algebras [O03a], subfactor theory [BEK00], extensions of vertex algebras [KO02], affine Hecke algebras [BO01] and conformal field theory, see for example [FS01], [CS07], [CS08]. In this case one is interested in the tensor category of representations of a certain vertex algebra, and irreducible objects of a module category are interpreted as boundary conditions for the conformal field theory.

The language of module categories was used with profit in diverse papers. It is present the study of fusion categories, see [ENO05], [ENO08], and in relation with dynamical twists [M07]. In [N07] module categories were used to describe some properties of semisimple Hopf algebras.

Etingof and Ostrik proposed [EO04], as an important problem, the classification of a certain class of module categories, called *exact*, and not only of semisimple module categories. This was done in the framework of the study of finite tensor categories not necessarily semisimple, motivated by its relation, among others, with logarithmic conformal field theories. Such classification is available only for few examples, see [EO04], [O03a], [O03b], [O08].

This paper arises mainly motivated by the problem of classifying module categories over the tensor category of representations of the small quantum groups $u_q(\mathfrak{g})$, for a simple Lie algebra $\mathfrak{g}$ and q a primitive root of unity.

One of the goals of this paper is the classification in the case $\mathfrak{g} = \mathfrak{sl}_2$. This particular problem was proposed by V. Ostrik at the meeting *Groupes quantiques dynamiques et categories de fusion*, at CIRM, Marseille 2008.

The paper is organized as follows. In section 2 we give a brief account of the basic definitions of graded and coradically graded Hopf algebras, twisting in Hopf algebras and Hopf-Galois extensions. In section 3 we recall some basic facts on exact module categories as introduced by Etingof and Ostrik [EO04], and we also recall some results on module categories over the tensor category of representations of a finite-dimensional Hopf algebra. In particular we recall the main result from [AM07] that any exact indecomposable module category over $\mathrm{Rep}(H)$, H a finite-dimensional Hopf algebra, is equivalent to the category of left K-modules, for some right H-simple left H-comodule K.

Section 4 is devoted to the study of filtered (graded) comodule algebras over a filtered (graded) Hopf algebra. In section 5 we study the relation between *liftings* of graded comodule algebras, that is comodule algebras A such that $\mathrm{gr}\, A \simeq G$, where G is a given graded comodule algebra, and formal deformations of G. This is completely analogous to the study of liftings of Hopf algebras in [DCY05].

The different liftings of a given comodule algebra are controlled by a certain double complex which we present in section 6 generalizing previous results of D. Yau [Y07], [Y08].

In section 7 we show that if A is a certain graded comodule algebra over a Hopf algebra H constructed by a bosonization as $H = \mathfrak{B}(V)\#\Bbbk G$, where $\mathfrak{B}(V)$ is the Nichols algebra of a Yetter-Drinfeld module V over the group algebra $\Bbbk G$ over a cyclic group G, then A is determined by a certain homogeneous coideal subalgebra in $\mathfrak{B}(V)$ and a subgroup of G.

Using the techniques developed previously, in section 8 we present the classification of exact module categories over the Taft Hopf algebras, the Radford Hopf algebras, the book Hopf algebras and over the Lusztig's small quantum groups $u_q(\mathfrak{sl}_2)$.

2. Preliminaries and notation

Throughout this work $\Bbbk$ will denote an algebraically closed field of characteristic zero. All vector spaces, algebras and linear categories will be considered over $\Bbbk$. By $\Bbbk^{\times}$ we shall denote the non-zero elements of $\Bbbk$.

If A is an algebra, we shall denote by $_A\mathcal{M}$ the category of finite-dimensional left A-modules. If H is a Hopf algebra $\mathrm{Rep}(H)$ will denote the tensor category of finite-dimensional representations of H and $_H^H\mathcal{Y}D$ will denote the category of Yetter-Drinfeld modules of H.

Given a Hopf algebra H, a left H-comodule algebra is a collection (A, μ, λ), where (A, μ) is an algebra, $\lambda : A \rightarrow H \otimes_{\Bbbk} A$ is the H-comodule structure

on A such that λ is an algebra map. We shall sometimes denote this co-module algebra by (A, λ). We will denote it by (A, μ, λ) only when special emphasis on the product is needed. The set of coinvariants of A is $A^{\operatorname{co} H} = \{a \in A : \lambda(a) = 1 \otimes a\}$.

If A is an H-comodule algebra via $\lambda : A \to H \otimes_{\Bbbk} A$, we shall say that a (right) ideal J is H-costable if $\lambda(J) \subseteq H \otimes_{\Bbbk} A$. We shall say that A is (right) H-simple if there is no nontrivial H-costable (right) ideal in A.

Let A be a left H-comodule algebra. We denote by ${}^{H}\mathcal{M}_A$ the category of left H-comodules, right A-modules M such that the right A-module structure $M \otimes_{\Bbbk} A \to M$ is an H-comodule map. If $P \in {}^{H}\mathcal{M}_A$ we will denote by $\operatorname{End}_A^H(P)$ the space of A-module and H-comodule endomorphisms.

A left coideal subalgebra of H is a subalgebra $K \subseteq H$ such that $\Delta(K) \subseteq H \otimes_{\Bbbk} K$. It is well-known that if H is finite-dimensional, K is a left coideal subalgebra of H and Q denotes the coalgebra H/HK^+, then the pair of functors $\mathcal{F} : {}^{H}\mathcal{M}_K \to {}^{Q}\mathcal{M}$, $\mathcal{F}(M) = M/MK^+$ and $\mathcal{G} : {}^{Q}\mathcal{M} \to {}^{H}\mathcal{M}_K$, $\mathcal{G}(V) = H \square_Q V$ give an equivalence of categories. See [Ma92], [Sk07].

For a coalgebra C we shall denote by C_0 the coradical and by $C_0 \subseteq C_1 \subseteq C_1 \cdots \subseteq C_n \subseteq \ldots$ the coradical filtration. For more details see [Sw69].

Let us recall that a coalgebra C is said to be $\mathbb{N}_0$-*graded*, or *graded* for short, if $C = \bigoplus_i C(i)$, such that $\Delta(C(n)) \subseteq \bigoplus_{i=0}^{n} C(i) \otimes_{\Bbbk} C(n-i)$ for all $n \in \mathbb{N}_0$ and $\varepsilon(C(n)) = 0$ for all $n \geq 1$. A coalgebra is *coradically graded* if it is graded and the coradical filtration is given by $C_n = \bigoplus_{i=0}^{n} C(i)$.

A Hopf algebra H is $\mathbb{N}_0$-*graded*, or *graded* for short, if it is $\mathbb{N}_0$-graded as an algebra and a coalgebra by the same grading.

A *filtration* on a Hopf algebra H is an algebra filtration $H^0 \subseteq H^1 \subseteq \cdots \subseteq H^m = H$ such that for all $n = 0 \ldots m$

$$\Delta(H^n) \subseteq \sum_{i=0}^{n} H^i \otimes_{\Bbbk} H^{n-i}.$$

If H is a filtered Hopf algebra such that H^0 is a Hopf subalgebra then the graded algebra associated to the filtration $\operatorname{gr} H = \bigoplus_{n \geq 0} H^n / H^{n-1}$, here $H^{-1} = 0$, is a graded Hopf algebra. The associated graded Hopf algebra to the coradical filtration will be denoted by $\operatorname{gr}_c H$. It is a well-known fact that $\operatorname{gr}_c H$ is a coradically graded Hopf algebra.

Let H be a graded Hopf algebra. A *lifting* of H is filtered Hopf algebra structure U on the same underlying vector space H such that $\operatorname{gr} U = H$.

Two liftings U and V of H are said to be *equivalent* if there is a Hopf algebra isomorphism $\phi : U \to V$ such that $\operatorname{gr} \phi = \operatorname{id}_H$. The set of equivalence classes of liftings of H will be denoted by $Lift(H)$.

2.1. **Twisting comodule algebras.** Let us recall that a Hopf 2-cocycle is a map $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$, invertible with respect to convolution, such that

$$(2.1) \qquad \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}),$$

$$(2.2) \qquad \sigma(x, 1) = \varepsilon(x) = \sigma(1, x),$$

for all $x, y, z \in H$. Using this cocycle there is a new Hopf algebra structure constructed over the same coalgebra H with the product described by

$$x \cdot_{[\sigma]} y = \sigma(x_{(1)}, y_{(1)})\sigma^{-1}(x_{(3)}, y_{(3)}) \ x_{(2)}y_{(2)}, \qquad x, y \in H.$$

We shall denote by H^σ this new Hopf algebra. We shall denote by $_\sigma H$ the space H with new multiplication given by

$$(2.3) \qquad x \cdot y = \sigma(x_{(2)}, y_{(2)}) \ x_{(1)}y_{(1)},$$

for all $x, y \in H$. The algebra $_\sigma H$ is a left H-comodule algebra with coaction given by the coproduct of H.

We shall say that σ is *cocentral* if for all $x, y \in H$

$$(2.4) \qquad \sigma(x_{(1)}, y_{(1)}) \ x_{(2)}y_{(2)} = \sigma(x_{(2)}, y_{(2)}) \ x_{(1)}y_{(1)}.$$

It is obvious that σ is cocentral if and only if $H^\sigma = H$. A useful way to check if a certain 2-cocycle σ is cocentral is to prove that the coproduct $_\sigma H :\to \ _\sigma H \otimes_{\Bbbk} H$ makes $_\sigma H$ a right H-comodule algebra. This observation will be used later without further mention.

If $\sigma : H \otimes H \to \Bbbk$ is a 2-cocycle and K is a left H-comodule algebra, then we can define a new product in K by

$$(2.5) \qquad a \cdot_\sigma b = \sigma(a_{(-1)}, b_{(-1)}) \ a_{(0)} \cdot b_{(0)},$$

$a, b \in K$. We shall denote by K_σ this new algebra. It is easy to see that K_σ is a left H^σ-comodule algebra. The following basic result will be useful later.

Lemma 2.1. *Let $\sigma : H \otimes H \to \Bbbk$ be a Hopf 2-cocycle and K be a left H-comodule algebra. There is an equivalence of categories $^H\mathcal{M}_K \simeq \ ^{H^\sigma}\mathcal{M}_{K_\sigma}$.*

Proof. If $M \in \ ^H\mathcal{M}_K$ we define a right K_σ-module structure by

$$m \cdot_\sigma k = \sigma(m_{(-1)}, k_{(-1)}) \ m_{(0)} \cdot k_{(0)},$$

for all $k \in K$, $m \in M$. We shall denote by M_σ the object M with this new action and the same left H-comodule structure. By a straightforward computation one can prove that $M_\sigma \in \ ^{H^\sigma}\mathcal{M}_{K_\sigma}$ and that the functor $M \longmapsto M_\sigma$ is an equivalence of categories. $\square$

In particular if $K \subseteq H$ is a left coideal subalgebra, $Q = H/HK^+$ and $\sigma : H \otimes H \to \Bbbk$ is a cocentral 2-cocycle then the categories $^H\mathcal{M}_{K_\sigma}$, $^Q\mathcal{M}$ are equivalent.

The following definition will be used later.

Definition 2.2. A Hopf 2-cocycle $\sigma : H \otimes H \to \Bbbk$ is *compatible* with a left coideal subalgebra $K \subseteq H$ if for any $x, y \in K$, $\sigma(x_{(2)}, y_{(2)})\, x_{(1)} y_{(1)} \in K$.

In the case when σ is compatible with K we denote ${}_\sigma K$ the left H-comodule algebra obtained from K using multiplication given as in (2.3). Any cocentral 2-cocycle is compatible with any left coideal subalgebra and in this case ${}_\sigma K = K_\sigma$.

2.2. **Hopf Galois extensions.** Let H be a Hopf algebra. A left H-Galois extension of an algebra R is an algebra A with a left H-comodule algebra structure, $A \to H \otimes_\Bbbk A$, $a \to a_{(-1)} \otimes a_{(0)}$, such that $A^{\operatorname{co} H} = R$ and the canonical map $can : A \otimes_R A \to H \otimes A$, $can(a \otimes b) = a_{(-1)} \otimes a_{(0)} b$ is a bijection. For more details on this subject the reader is referred to [SS04].

It is well known that if $\sigma : H \otimes_\Bbbk H \to \Bbbk$ is a 2-cocycle then ${}_\sigma H$ is a H-Galois extension of the field $\Bbbk$. In the finite-dimensional case the converse also holds, that is, if H is finite-dimensional and A is a finite-dimensional left H-Galois extension of the field then there is a 2-cocycle σ such that $A \simeq {}_\sigma H$.

3. Exact module categories over finite tensor categories

We recall the definitions of module categories over finite tensor categories. For more details the reader is referred to [O03a], [EO04].

A *module category* over a tensor category $\mathcal{C}$ is an abelian category $\mathcal{M}$ equipped with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$, $\ell_M : \mathbf{1} \otimes M \to M$ such that for any $X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$.

$$(3.1) \qquad (\operatorname{id} \otimes m_{Y,Z,M}) m_{X,Y \otimes Z,M}(a_{X,Y,Z} \otimes id) = m_{X,Y,Z \otimes M}\, m_{X \otimes Y,Z,M},$$

$$(3.2) \qquad\qquad (\operatorname{id} \otimes \ell_M) m_{X,\mathbf{1},Y} = r_X \otimes \operatorname{id}.$$

A *module functor* between module categories $\mathcal{M}$ and $\mathcal{M}'$ over a tensor category $\mathcal{C}$ is a pair $(\mathcal{F}, c)$, where $\mathcal{F} : \mathcal{M} \to \mathcal{M}'$ is a $\Bbbk$-linear functor and $c_{X,M} : \mathcal{F}(X \otimes M) \to X \otimes \mathcal{F}(M)$ is a natural isomorphism such that for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$:

$$(3.3) \qquad (\operatorname{id}_X \otimes c_{Y,M}) c_{X,Y \otimes M} \mathcal{F}(m_{X,Y,M}) = m_{X,Y,\mathcal{F}(M)}\, c_{X \otimes Y,M},$$

$$(3.4) \qquad\qquad \ell_{\mathcal{F}(M)}\, c_{\mathbf{1},M} = \mathcal{F}(\ell_M).$$

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be module categories over $\mathcal{C}$. We shall denote by $\operatorname{Hom}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ the category whose objects are module functors $(\mathcal{F}, c)$ from $\mathcal{M}_1$ to $\mathcal{M}_2$. A morphism between $(\mathcal{F}, c)$ and $(\mathcal{G}, d) \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is a natural transformation $\alpha : \mathcal{F} \to \mathcal{G}$ such that for any $X \in \mathcal{C}$, $M \in \mathcal{M}_1$:

$$(3.5) \qquad\qquad d_{X,M} \alpha_{X \otimes M} = (\operatorname{id}_X \otimes \alpha_M) c_{X,M}.$$

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exist module functors $F : \mathcal{M}_1 \to \mathcal{M}_2$ and $G : \mathcal{M}_2 \to \mathcal{M}_1$ and natural isomorphisms id $_{\mathcal{M}_1} \to F \circ G$, id $_{\mathcal{M}_2} \to G \circ F$ that satisfy (3.5).

The *direct sum* of two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ is the $\Bbbk$-linear category $\mathcal{M}_1 \times \mathcal{M}_2$ with coordinate-wise module structure.

A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories.

In this paper we further assume that all module categories have finitely many isomorphism classes of simple objects. The following definition of a very important class of module categories is due to P. Etingof and V. Ostrik.

Definition 3.1. [EO04] A module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$ is *exact* if for any projective $P \in \mathcal{C}$ and any $M \in \mathcal{M}$, the object $P \overline{\otimes} M$ is projective in $\mathcal{M}$.

Any semisimple finite module category over a finite tensor category $\mathcal{C}$ is exact. A direct sum of finite module categories is exact if and only if each summand is exact. Therefore, any exact module category over $\mathcal{C}$ is a finite direct product of exact indecomposable module categories over $\mathcal{C}$, see [EO04, Prop. 3.9].

3.1. **Module categories over Hopf algebras.** We recall some results concerning exact module categories over the category of representations of Hopf algebras. Let H be a finite-dimensional Hopf algebra.

If $\lambda : A \to H \otimes_{\Bbbk} A$ is a left H-comodule algebra then the category of finite-dimensional left A-modules $_A\mathcal{M}$ is a module category over $\mathrm{Rep}(H)$ with action $\overline{\otimes} : \mathrm{Rep}(H) \times {_A\mathcal{M}} \to {_A\mathcal{M}}$, $X \overline{\otimes} M = X \otimes_{\Bbbk} M$, for all $X \in \mathrm{Rep}(H), M \in {_A\mathcal{M}}$. The left A-module structure on $X \otimes_{\Bbbk} M$ is given by λ, that is, if $a \in A, x \in X, m \in M$ then $a \cdot (x \otimes m) = a_{(-1)} \cdot x \otimes a_{(0)} \cdot m$.

If A is right H-simple then $_A\mathcal{M}$ is an exact indecomposable module category over $\mathrm{Rep}(H)$, see [AM07, Prop. 1.20]. The exactness is a consequence of the beautiful results obtained by Skryabin [Sk07] on comodule algebras.

Remark 3.2. In particular the category $_A\mathcal{M}$ is an exact indecomposable module category over $\mathrm{Rep}(H)$ when

(1) $A = K_\sigma$, $K \subseteq H$ is a coideal subalgebra and $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ is a cocentral Hopf 2-cocycle,
(2) or, more generally, $A = {_\sigma K}$ where $K \subseteq H$ is a left coideal subalgebra and $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ is a 2-cocycle compatible with K.

Theorem 3.3. [AM07, Theorem 3.3] *If $\mathcal{M}$ is an exact idecomposable module category over $\mathrm{Rep}(H)$ then $\mathcal{M} \simeq {_A\mathcal{M}}$ for some right H-simple left comodule algebra A with $A^{\mathrm{co}\, H} = \Bbbk$.*   □

Two left H-comodule algebras A and B are *Morita equivariant equivalent*, and we shall denote it by $A \sim_M B$, if the module categories ${}_A\mathcal{M}$, ${}_B\mathcal{M}$ are equivalent as module categories over $\mathrm{Rep}(H)$.

Proposition 3.4. [AM07, Prop. 1.24] *The algebras A and B are Morita equivariant equivalent if and only if there exists $P \in {}^H\mathcal{M}_B$ such that $A \simeq \mathrm{End}_B(P)$ as H-comodule algebras.* $\qquad\square$

The left H-comodule structure on $\mathrm{End}_B(P)$ is given by $\lambda : \mathrm{End}_B(P) \to H \otimes_{\Bbbk} \mathrm{End}_B(P)$, $\lambda(T) = T_{(-1)} \otimes T_{(0)}$ where

$$(3.6) \qquad \langle \alpha, T_{(-1)} \rangle \, T_0(p) = \langle \alpha, T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle \, T(p_{(0)})_{(0)},$$

for any $\alpha \in H^*, T \in \mathrm{End}_B(P), p \in P$. It is easy to prove that $\mathrm{End}_B(P)^{\mathrm{co}\,H} = \mathrm{End}_B^H(P)$.

Let $K \subseteq H$ be a left coideal subalgebra and $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ be a cocentral 2-cocycle. Let us denote by Q the quotient H/HK^+.

Lemma 3.5. *Let A be a right H-simple left H-comodule algebra with trivial coinvariants Morita equivariant equivalent to K_σ. Then there exists an indecomposable object $P \in {}^Q\mathcal{M}$ such that $A \simeq \mathrm{End}_{K_\sigma}((H\square_Q P)_\sigma)$ and $\dim A = (\dim P)^2 \dim K$.*

Proof. The existence of the object P such that $A \simeq \mathrm{End}_{K_\sigma}((H\square_Q P)_\sigma)$ follows by Proposition 3.4 and Lemma 2.1. Since the coinvariants are trivial $A^{\mathrm{co}\,H} = \Bbbk$ then $\mathrm{End}_B^H(H\square_Q P) = \Bbbk$ and this implies that $H\square_Q P$ must be indecomposable thus P is also indecomposable. Since $H \simeq K \otimes Q$ as left K-modules, right Q-comodules, then $H\square_Q P \simeq K \otimes P$, whence $\dim A = (\dim P)^2 \dim K$. $\qquad\square$

Let us further assume that Q is a pointed cosemisimple coalgebra. Thus, if $P \in {}^Q\mathcal{M}$ is an indecomposable object then is 1-dimensional and there exists a group-like element $g \in Q$ and $0 \neq v \in P$ such that $P = \Bbbk < v >$ and $\delta(v) = g \otimes v$ is the left Q-comodule structure on P. In this case $H\square_Q P \simeq gK$.

Lemma 3.6. *Under the above assumptions if A is a left H-comodule algebra such that $A \sim_M K_\sigma$ then $A \simeq (gKg^{-1})_{\sigma^g}$, for some group-like element $g \in G(H)$ and $\sigma^g : H \otimes_{\Bbbk} H \to \Bbbk$ is the Hopf 2-cocycle given by*

$$\sigma^g(x, y) = \sigma(g^{-1}xg, g^{-1}yg),$$

for any $x, y \in H$.

Proof. From the above considerations $A \simeq \mathrm{End}_{K_\sigma}((gK)_\sigma)$. Any element $T \in \mathrm{End}_{K_\sigma}((gK)_\sigma)$ is determined by the value $T(g)$. Indeed, any element $gx \in gK$ can be written uniquely $gx = g \cdot_\sigma \left(\sigma^{-1}(g, x_{(1)}) \, x_{(2)}\right)$. Here, for any $x, y \in H$ we are denoting $x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)}) \, x_{(2)}y_{(2)}$ the product as in (2.5).

The maps $\phi \,:\, (gKg^{-1})_{\sigma^g} \;\to\; \mathrm{End}_{K_\sigma}((Kg)_\sigma)$, $\psi \,:\, \mathrm{End}_{K_\sigma}((Kg)_\sigma) \;\to\; (gKg^{-1})_{\sigma^g}$ given by

$$\phi(gy^{-1}g)(g \cdot x) = g \cdot (x \cdot y), \quad \psi(T) = g^{-1}T(g),$$

for any $x, y \in K$ $T \in \mathrm{End}_{K_\sigma}((Kg)_\sigma)$ are well-defined maps one the inverse of each other. It is straightforward to verify that both maps are H-comodule morphisms.

$\square$

Example 3.7. Let G be a finite group, $F \subseteq G$ be a subgroup and $H = \Bbbk G$ is the group algebra. The group algebra $\Bbbk F$ is a left coideal subalgebra of H. Let also $\psi \in Z^2(G, \Bbbk^\times)$ be a (necessarily cocentral) 2-cocycle. Let us denote by $\Bbbk^\psi F$ the twisted group algebra. An indecomposable object $P \in {}^Q\mathcal{M}$, $Q = \Bbbk G/(\Bbbk F)^+ \Bbbk G$ must be one-dimensional, say $P = <\Bbbk \overline{g}>$. If $A \sim_M \Bbbk^\psi F$ then there exists an element $g \in G$ such that $A \simeq \Bbbk^{\psi^g} F^g$, where F^g is the conjugate of F. This result is contained in [O03a, Thm 2].

4. Graded and filtered comodule algebras

Let us assume that H is a finite-dimensional filtered Hopf algebra with filtration given by $H^0 \subseteq H^1 \subseteq \cdots \subseteq H^m = H$ such that H^0 is a Hopf subalgebra.

Given a left H-comodule algebra (A, λ), we shall say that a filtration $A^0 \subseteq A^1 \subseteq \cdots \subseteq A^m$ is a *comodule algebra filtration* compatible with the filtration of H if it is an algebra filtration and for each $n = 0, \dots, m$

$$(4.1) \qquad\qquad \lambda(A^n) \subseteq \sum_{i=0}^{n} H^i \otimes_{\Bbbk} A^{n-i}.$$

Observe that A^0 is a left H^0-comodule algebra. Whenever no confusion arises we shall simply say that A is a *filtered H-comodule algebra*.

Any filtration on the Hopf algebra induces a compatible filtration on a comodule algebra.

Lemma 4.1. *Let $H^0 \subseteq H^1 \subseteq \cdots \subseteq H^m = H$ be a filtration on the Hopf algebra H and (A, λ) a comodule algebra. Define $A^n = \lambda^{-1}(H^n \otimes A)$ for all $n = 0 \dots m$, then $A^0 \subseteq \cdots \subseteq A^m = A$ is a filtration compatible with the filtration on H.*

Proof. Since λ is an algebra map, for any i, j we have that $A^i A^j \subseteq A^{i+j}$. Let be $0 \leq n \leq m$ and $a \in A^n$, then $\lambda(a) \in H^n \otimes A$, hence

$$(\mathrm{id}_H \otimes \lambda)\lambda(a) = (\Delta \otimes \mathrm{id}_A)\lambda(a) \in \sum_{i=0}^{n} H^i \otimes H^{n-i} \otimes A.$$

Whence $\lambda(a) \in \sum_{i=0}^{n} H^i \otimes A^{n-i}$.

$\square$

As a consequence any comodule algebra (A, λ) has a distinguished filtration compatible with the coradical filtration on H, *the Loewy series* on A; it is the filtration $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_m$ defined by $A_n = \lambda^{-1}(H_n \otimes A)$ for all $n \in \mathbb{N}$.

Definition 4.2. Let $H = \oplus_{i=0}^m H(i)$ be a graded Hopf algebra. We shall say that a left H-comodule algebra G, with comodule structure given by $\lambda : G \to H \otimes_\Bbbk G$, graded as an algebra $G = \oplus_{i=0}^m G(i)$ is a *graded comodule algebra* if for each $0 \leq n \leq m$

$$\lambda(G(n)) \subseteq \bigoplus_{i=0}^m H(i) \otimes_\Bbbk G(n-i).$$

A graded comodule algebra $G = \oplus_{i=0}^m G(i)$ is *Loewy-graded* if it is a graded comodule algebra and the Loewy series is given by $G_n = \oplus_{i=0}^n G(i)$.

Let (A, λ) be a filtered left H-comodule algebra, $A^0 \subseteq A^1 \subseteq \cdots \subseteq A^m$ compatible with the filtration $H^0 \subseteq H^1 \subseteq \cdots \subseteq H^m = H$. We can consider the graded algebra $\operatorname{gr} A = \bigoplus_{n \geq 0} A^n / A^{n-1}$, where $A^{-1} = 0$. The graded algebra associated to the Loewy series will be denoted by $\operatorname{gr}_l A$.

There is a well defined map $\overline{\lambda} : \operatorname{gr} A \to \operatorname{gr} H \otimes \operatorname{gr} A$ such that the following diagram commutes

$$
\begin{array}{ccc}
A^n & \xrightarrow{\quad \lambda \quad} & \bigoplus_{i=0}^n H^i \otimes_\Bbbk A^{n-i} \\
\downarrow & & \downarrow \\
A^n/A^{n-1} & \longrightarrow & \left(\sum_{i=0}^n H^i \otimes_\Bbbk A^{n-i} \right) / \sum_{j=0}^{n-1} H^j \otimes_\Bbbk A^{n-1-j} \\
\downarrow & & \downarrow{\scriptstyle \simeq} \\
\operatorname{gr} A(n) & \xrightarrow{\quad \overline{\lambda} \quad} & \sum_{i=0}^n \operatorname{gr} H(i) \otimes_\Bbbk \operatorname{gr} A(n-i).
\end{array}
$$

Lemma 4.3. *The space $(\operatorname{gr} A, \overline{\lambda})$ is a graded $\operatorname{gr} H$-comodule algebra and $(\operatorname{gr}_l A, \overline{\lambda})$ is a Loewy-graded $\operatorname{gr}_c H$-comodule algebra.*

Proof. This is a well-known fact. The proof can be found for example in [AD03], [Sw69]. □

The following result will be useful when we relate module categories over a Hopf algebra H and over $\operatorname{gr}_c H$.

Proposition 4.4. *Let $H^0 \subseteq H^1 \subseteq \cdots \subseteq H^m = H$ be a filtration on the Hopf algebra H such that H^0 is a semisimple Hopf subalgebra. Let A be a left H-comodule algebra and consider the filtration $A^i = \lambda^{-1}(H^i \otimes_\Bbbk A)$ for any $i = 0, \ldots, m$. The following assertions are equivalent.*

  (1) *A^0 is right H^0-simple.*
  (2) *A is right H-simple.*

Proof. Let us assume that A^0 is right H^0-simple. Let $J \subseteq A$ be a right ideal H-costable. Consider the filtration on J, $J^0 \subseteq J^1 \subseteq \dots J^m = J$, given by $J^i = \lambda^{-1}(H^i \otimes_{\Bbbk} J)$. The space $J^0 \subseteq A^0$ is a right ideal H^0-costable, thus $J^0 = 0$ or $J^0 = A^0$. In the second case $1 \in J$ and thus $J = A$. Let us assume that $J^0 = 0$. Follows from Lemma 4.1 that $\lambda(J^n) \subseteq \sum_{i=0}^{n} H^i \otimes_{\Bbbk} J^{n-i}$. Then $\lambda(J^1) \subseteq H_0 \otimes_{\Bbbk} J^1$ thus $J^1 = J^0$. Arguing inductively follows that $J^n = J^{n-1}$ for all n, thus $J = 0$.

Assume A is right H-simple. Since H^0 is semisimple, using [Li03, Thm. 3.1], we obtain that the Jacobson radical $J(A^0)$ is a H^0-costable ideal of A^0, hence $J(A^0)A$ is a right ideal H-costable in A. Since the radical is nilpotent the equality $J(A^0)A = A$ is impossible, therefore $J(A^0)A = 0$ and A^0 is semisimple. Thus, the inclusion $A^0 \subseteq A$ splits, that is, there is a left A^0-module $B \subseteq A$ such that $A = A^0 \oplus B$. Now, let $J \subseteq A^0$ be a nonzero right H^0-costable ideal. Then $JA = J \oplus JB \subseteq J \oplus B$. Since A is right H-simple we obtain that $JA = A$ which implies that $J = A^0$. $\qquad\square$

Under the same hypothesis as Proposition 4.4 we have the following result.

Corollary 4.5. *The following assertions are equivalent.*

(1) A^0 *is right* H^0-*simple.*
(2) $\operatorname{gr} A$ *is right* $\operatorname{gr} H$-*simple.*
(3) A *is right* H-*simple.*

Proof. (1) is equivalent to (3) is Proposition 4.4. But the same proof works for $\operatorname{gr} A$ since $(\operatorname{gr} A)^0 = A^0$. $\qquad\square$

5. Liftings and formal deformation of comodule algebras

For the rest of this section (H, Δ, m) will denote a graded Hopf algebra with grading $H = \oplus_{i=0}^{m} H(i)$.

Let G be a graded H-comodule algebra. If U is a lifting for H, we shall say that A is a *lifting* for G along U if $A = G$ as vector spaces and A is a filtered left U-comodule algebra and $\operatorname{gr} A \simeq G$ as left H-comodule algebras.

Two liftings A, B of G along U are said to be equivalent if there is an isomorphism of filtered U-comodule algebras $\psi : A \to B$ such that $\operatorname{gr} \psi = \operatorname{id}_G$. We shall denote by $Lift(A, U)$ the set of equivalence classes of liftings of A along U.

5.1. **Formal bialgebra deformation.** Let us recall graded bialgebra deformation from [DCY05]. See also [MW07]. For any $l \in \mathbb{N} \cup \{\infty\}$ consider the free $\Bbbk[t]/(t^{l+1})$-module $H[t]/(t^{l+1})$ as a graded vector space over $\Bbbk$ by $\deg t = 1$ and $\deg h = i$ if $h \in H(i)$. If $l = \infty$, $H[t]/(t^{l+1}) = H[t]$ and $\Bbbk[t]/(t^{l+1}) = \Bbbk[t]$.

An l-deformation of H is a collection $\big(H[t]/(t^{l+1}), m^l, \Delta^l\big)$, where

$$m^l : (H\otimes_{\Bbbk}H)[t]/(t^{l+1}) \to H[t]/(t^{l+1}), \quad \Delta^l : H[t]/(t^{l+1}) \to (H\otimes_{\Bbbk}H)[t]/(t^{l+1})$$

are $\Bbbk[t]/(t^{l+1})$-linear maps homogeneous of degree 0, such that $H[t]/(t^{l+1})$ is a bialgebra. This implies that there are maps $\Delta^l_s : H \to H\otimes_{\Bbbk}H$, $m^l_s : H\otimes_{\Bbbk}H \to H$ of degree $-s$ such that for all $x, y \in H$

$$\Delta^l(x) = \sum_{s=0}^{l} \Delta^l_s(x)\, t^s, \quad m^l(x\otimes y) = \sum_{s=0}^{l} m^l_s(x\otimes y)\, t^s.$$

It is also required that $\Delta^l_0 = \Delta$ and $m^l_0 = m$.

Two l-deformations $\big(H[t]/(t^{l+1}), m^l, \Delta^l\big)$ and $\big(H[t]/(t^{l+1}), \widetilde{m}^l, \widetilde{\Delta}^l\big)$ are isomorphic if there exists a bialgebra isomorphism $\phi : H[t]/(t^{l+1}) \to H[t]/(t^{l+1})$ homogeneous of degree 0. In this case there are maps $\phi_s : H \to H$ homogeneous of degree $-s$ such that for any $h \in H$, $\phi(h) = \sum_{s=0}^{l} \phi_s(h)t^s$. Here it is required that $\phi_0 = \mathrm{id}_H$.

The set of equivalence of l-deformations is denoted by $Iso^l(H)$. When $l = \infty$, this set is simply denoted by $Iso(H)$.

The following result is [DCY05, Thm. 2.2].

Theorem 5.1. *There exists a natural bijection between $Lift(H)$ and $Iso(H)$.*

5.2. **Formal deformation for comodule algebras.** Let us fix a l-deformation $\big(H[t]/(t^{l+1}), m^l, \Delta^l\big)$ of H that we shall denote by H_l for short. Let also (G, μ, λ) be a graded left H-comodule algebra.

Definition 5.2. An l-*deformation* of G is a collection $\big(G[t]/(t^{l+1}), \mu^l, \lambda^l\big)$ where $\mu^l : (G\otimes_{\Bbbk}G)[t]/(t^{l+1}) \to G[t]/(t^{l+1})$, $\lambda^l : G[t]/(t^{l+1}) \to (H\otimes_{\Bbbk}G)[t]/(t^{l+1})$ are $\Bbbk[t]/(t^{l+1})$-linear maps homogeneous of degree zero such that $G[t]/(t^{l+1})$ is a left H_l-comodule algebra. In particular there are maps $\lambda^l_s : G \to H\otimes_{\Bbbk}G$, $\mu^l_s : G\otimes_{\Bbbk}G \to G$ homogeneous of degree $-s$ such that for any $a, b \in G$

$$\lambda^l(a) = \sum_{s=0}^{l} \lambda^l_s(a)\, t^s, \quad \mu^l(a\otimes b) = \sum_{s=0}^{l} \mu^l_s(a\otimes b)\, t^s.$$

It is required that $\lambda^l_0 = \lambda$ and $\mu^l_0 = \mu$.

The associativity of the product, the coassociativity and the compatibility of the coaction of $G[t]/(t^{l+1})$ with the product imply that for any $0 \le n \le l$

$$\tag{5.1} \sum_{i+j=n} \mu^l_i(\mu^l_j\otimes\mathrm{id}_G) = \sum_{i+j=n} \mu^l_i(\mathrm{id}_G\otimes\mu^l_j),$$

$$\tag{5.2} \sum_{i+j=n} (\mathrm{id}_H\otimes\lambda^l_i)\lambda^l_j = \sum_{i+j=n} (\Delta^l_i\otimes\mathrm{id}_G)\lambda^l_j,$$

$$\tag{5.3} \sum_{i+j=n} \lambda^l_i\mu^l_j = \sum_{i+j+k+r=n} (m^l_i\otimes\mu^l_j)\, \tau_{23}\, (\lambda^l_k\otimes\lambda^l_r).$$

Two l-deformations $\left(G[t]/(t^{l+1}), \mu^l, \lambda^l\right)$, $\left(\widetilde{G}[t]/(t^{l+1}), \widetilde{\mu}^l, \widetilde{\lambda}^l\right)$ are equivalent if there exists an homogeneous map of degree zero $\phi : G[t]/(t^{l+1}) \to \widetilde{G}[t]/(t^{l+1})$ such that ϕ is a $\Bbbk[t]/(t^{l+1})$-linear isomorphism of H_l-comodule algebras. In particular there are homogeneous maps $\phi_s : G \to \widetilde{G}$ of degree $-s$ such that

$$\phi(a) = \sum_{s=0}^{l} \phi_s(a) \, t^s,$$

for all $a \in G$. We required that $\phi_0 = \mathrm{id}_G$. Denote by $Iso^l(G, H_l)$ the set of equivalence classes of l-deformations of G. The proof of the following Lemma is completely analogous to [DCY05, Lemma 2.1].

Lemma 5.3. *For $l > l'$ there exist a restriction map $res_{l,l'} : Iso^l(G, H_l) \to Iso^{l'}(G, H_{l'})$, and maps $r_l : Iso^\infty(G, H_\infty) \to Iso^l(G, H_l)$ such that*

$$Iso^\infty(G, H_\infty) \simeq \varprojlim_{l \in \mathbb{N}} Iso^l(G, H_l)$$

$\square$

The result [DCY05, Th. 2.2] can be extended to the comodule algebra setting in a natural way. Let U be a lifting of H corresponding to the l-deformation H_l under Theorem 5.1.

Theorem 5.4. *There is a natural bijection $Iso^\infty(G, H_\infty) \simeq Lift(G, U)$.*

The following technical Lemma will be crucial to find liftings of certain comodule algebras over pointed Hopf algebras.

Let F be a finite Abelian group, $H = \oplus_{i=0}^{m-1} H(i)$ be a finite-dimensional coradically graded Hopf algebra, $U^0 \subseteq U^1 \subseteq \cdots \subseteq U^{m-1} = U$ a filtered Hopf algebra U with $U^0 = \Bbbk F = H(0)$, such that $\mathrm{gr}\, U = H$. Let $g \in F$ and $x \in U^1 - U^0$ an element such that

$$(5.4) \qquad \Delta(x) = x \otimes 1 + g \otimes x, \quad fx = \chi(f)\, xf,$$

for any $f \in F$ where χ is a character for F.

Let (G, λ_0) be a Loewy-graded H-comodule with grading $G = \oplus_{i=0}^{m-1} G(i)$ such that $G(0) \simeq \Bbbk \widetilde{F}$ as H-comodules, where $\widetilde{F} \subseteq F$ is a subgroup, that is there exists a base $\{e_f : f \in \widetilde{F}\}$ of G such that the comodule structure are given by $\lambda(e_f) = f \otimes e_f$, for any $f \in \widetilde{F}$.

Let (A, λ) be a left U-comodule algebra and let us consider the filtration $A^i = \lambda^{-1}(U^i \otimes_{\Bbbk} A)$. Assume that $\mathrm{gr}\, A = G$.

Lemma 5.5. *Under the above assumptions, if $\overline{y} \in G(1)$ is an element such that*

$$\lambda_0(\overline{y}) = x \otimes 1 + g \otimes \overline{y}, \quad e_f \overline{y} = \chi(f)\, \overline{y} e_f,$$

for all $f \in \widetilde{F}$, then there exists an element $y \in A^1$ such that $\lambda(y) = x \otimes 1 + g \otimes y$, $e_f y = \chi(f)\, y e_f$ for all $f \in \widetilde{F}$ and the class of y in $A^1/A^0 = G(1)$ equals to $\overline{y}$.

Proof. Since we know that there are homogeneous maps $\lambda_s : G \to H \otimes_{\mathbb{k}} G$ of degree $-s$, such that the coaction of A is $\lambda = \sum_{s=0} \lambda_s$, then $\lambda(\overline{y}) = x \otimes 1 + g \otimes \overline{y} + \lambda_1(\overline{y})$ where $\lambda_1(\overline{y}) \in H(0) \otimes G(0)$. Let us write $\omega = \lambda_1(\overline{y})$. Set $\omega = \sum_{h \in F, f \in \widetilde{F}} \alpha_{h,f} \, h \otimes e_f$, where $\alpha_{h,f} \in \mathbb{k}$. Since $(\mathrm{id} \otimes \lambda)\lambda = (\Delta \otimes \mathrm{id})\lambda$ then

$$\sum_{h \in F, f \in \widetilde{F}} \alpha_{h,f} \, h \otimes h \otimes e_f = \sum_{h \in F, f \in \widetilde{F}} \alpha_{h,f} \, h \otimes f \otimes e_f + \sum_{h \in F, f \in \widetilde{F}} \alpha_{h,f} \, g \otimes h \otimes e_f,$$

whence we deduce that $\omega = \sum_{f \in \widetilde{F}, f \neq g} \beta_f \, (g - f) \otimes e_f$ for some $\beta_f \in \mathbb{k}$. If we define $a = \sum_{f \in \widetilde{F}, f \neq g} \beta_f \, e_f$ then $\omega = g \otimes a - \lambda(a)$, thus the element $y = \overline{y} + a$ satisfies that $\lambda(y) = x \otimes 1 + g \otimes y$.

The space $\mathcal{P} = \{ y \in A : \lambda(y) = \mu \, x \otimes 1 + g \otimes y, \mu \in \mathbb{k} \}$ is stable under the action of $\widetilde{F}$ given by

$$f \cdot y = e_f y e_{f^{-1}}.$$

Thus there exists an element $y \in \mathcal{P}$ such that for any $f \in \widetilde{F}$ $f \cdot y = \chi'(f) \, y$, for some $\chi' : \widetilde{F} \to \mathbb{k}$. Since for all $f \in \widetilde{F}$ $fx = \chi(f) \, xf$ then $\chi' = \chi$ and this ends the proof of the Lemma.

$\square$

6. Deformation bicomplex for comodule algebras

In [Y08] and [Y07] the author introduced a bicomplex to study algebraic deformations of H-comodule algebras A over a graded Hopf algebra H. The cohomology groups of the total complex are related to formal deformations of A over the same Hopf algebra H. In this section we introduce a certain bicomplex generalizing the one constructed by D. Yau in a way that cohomology groups of the total complex are related to formal deformations of A over a any lifting U of H.

6.1. **Bialgebra cohomology.** Let H be a bialgebra. For each $s \in \mathbb{N}$ and each $1 \leq i \leq s$ define the maps $m_i^s, \lambda_l^s, \lambda_r^s : H^{s+1} \to H^s$ and $\Delta_i^s, \delta_l^s, \delta_r^s : H^s \to H^{s+1}$ by

$$m_i^s(y^1 \otimes \ldots \otimes y^{s+1}) = y^1 \otimes \ldots \otimes y^i y^{i+1} \otimes \ldots \otimes y^{s+1},$$
$$\Delta_i^s(y^1 \otimes \ldots y^s) = y^1 \otimes \ldots \otimes y^i{}_{(1)} \otimes y_2^i \otimes \ldots \otimes y^s,$$
$$\lambda_l^s(x \otimes y^1 \otimes \ldots y^s) = x_{(1)} y^1 \otimes \ldots \otimes x_{(s)} y^s,$$
$$\lambda_r^s(y^1 \otimes \ldots y^s \otimes x) = y^1 x_{(1)} \otimes \ldots \otimes y^s x_{(s)},$$
$$\delta_l^s(y^1 \otimes \ldots y^s) = y^1{}_{(1)} \ldots y^s{}_{(1)} \otimes y^1{}_{(2)} \otimes \ldots \otimes y^s{}_{(2)},$$
$$\delta_r^s(y^1 \otimes \ldots y^s) = y^1{}_{(1)} \otimes \ldots \otimes y^s{}_{(1)} \otimes y^1{}_{(2)} \ldots y^s{}_{(2)},$$

for all $x, y^1, \ldots, y^s \in H$.

Let $C^{p,q}(H) = \mathrm{Hom}_{\mathbb{k}}(H^{\otimes p}, H^{\otimes q})$ and let

$$\partial_{p,q}^h : C^{p,q} \to C^{p+1,q}, \quad \partial_{p,q}^v : C^{p,q} \to C^{p,q+1},$$

be the maps defined by $\partial_{p,q}^h = \sum_{i=0}^{p+1} (-1)^i \, \partial_{p,q}^h[i]$, and $\partial_{p,q}^v = \sum_{i=0}^{q+1} (-1)^i \, \partial_{p,q}^v[i]$, where for any $f \in C^{p,q}(H)$

$$\partial_{p,q}^v[i](f) = \begin{cases} (\mathrm{id}\,_H \otimes f)\delta_l^p & \text{if } i = 0 \\ \Delta_i^q \circ f & \text{if } 1 \leq i \leq q \\ (f \otimes \mathrm{id}\,_H)\delta_r^p & \text{if } i = q + 1 \end{cases}$$

and

$$\partial_{p,q}^h[i](f) = \begin{cases} \lambda_l^q(\mathrm{id}\,_H \otimes f) & \text{if } i = 0 \\ f \circ m_i^p & \text{if } 1 \leq i \leq p \\ \lambda_r^q(f \otimes \mathrm{id}\,_H) & \text{if } i = p + 1 \end{cases}$$

We will use the truncated complex, that is $C^{p,q}(H) = 0$ if $pq = 0$.

6.2. Deformation bicomplex for comodule algebras.
Let (A, μ, λ) be a left H-comodule. For any $n \in \mathbb{N}$ and $a^i \in A$ define the maps

$$\lambda_1^n : A^{\otimes n} \to H \otimes_{\Bbbk} A^{\otimes n}, \quad \lambda_2^n : A^{\otimes n} \to H^{\otimes n} \otimes_{\Bbbk} A$$

by

$$\lambda_1^n(a^1 \otimes \ldots \otimes a^n) = a^1{}_{(-1)} \ldots a^n{}_{(-1)} \otimes a^1{}_{(0)} \otimes \ldots \otimes a^n{}_{(0)},$$
$$\lambda_2^n(a^1 \otimes \ldots \otimes a^n) = a^1{}_{(-1)} \otimes \ldots \otimes a^n{}_{(-1)} \otimes a^1{}_{(0)} \ldots a^n{}_{(0)},$$

for all $a^1 \ldots a^n \in A$. Define also an A-bimodule structure on $H^{\otimes n} \otimes_{\Bbbk} A$ via the maps

$$\beta_l^n : A \otimes_{\Bbbk} H^{\otimes n} \otimes_{\Bbbk} A \to H^{\otimes n} \otimes_{\Bbbk} A, \quad \beta_r^n : H^{\otimes n} \otimes_{\Bbbk} A \otimes_{\Bbbk} A \to H^{\otimes n} \otimes_{\Bbbk} A,$$

by

$$\beta_l^n(a \otimes y^1 \otimes \ldots \otimes y^n \otimes b) = \lambda_l^n(a_{(-1)} \otimes y^1 \otimes \ldots \otimes y^n) \otimes a_{(0)}b,$$
$$\beta_r^n(y^1 \otimes \ldots \otimes y^n \otimes a \otimes b) = \lambda_r^n(y^1 \otimes \ldots \otimes y^n \otimes b_{(-1)}) \otimes ab_{(0)},$$

for all $y^1, \ldots, y^n \in H$, $a, b \in A$. For $n = 0$ we define $\beta_l^0 = \mu = \beta_r^0$. We also shall need for any $n \in \mathbb{N}$ and $1 \leq i \leq n$ the maps

$$\mu_i^n : A^{\otimes n+1} \to A^{\otimes n+1}, \quad \mu_i^n(a^1 \otimes \ldots \otimes a^{n+1}) = a^1 \otimes \ldots \otimes a^i a^{i+1} \otimes \ldots \otimes a^{n+1},$$

for all $a^1, \ldots, a^{n+1} \in A$.

Let us recall the bicomplex $C^{p,q}(A)$ defined by

$$C^{p,q}(A) = \begin{cases} 0 & \text{if } p = 0 \\ \mathrm{Der}\,(A) & \text{if } (p, q) = (1, 0) \\ \mathrm{Hom}_{\Bbbk}(A^{\otimes p}, H^{\otimes q} \otimes_{\Bbbk} A) & \text{otherwise.} \end{cases}$$

Define the horizontal and vertical differentials

$$\partial_{p,q}^h : C^{p,q}(A) \to C^{p+1,q}(A), \quad \partial_{p,q}^v : C^{p,q}(A) \to C^{p,q+1}(A),$$

by

$$\partial_{p,q}^h = \sum_{i=0}^{p+1} (-1)^i \, \partial_{p,q}^h[i], \quad \partial_{p,q}^v = \sum_{i=0}^{q+1} (-1)^i \, \partial_{p,q}^v[i].$$

Where for any $f \in C^{p,q}(A)$

$$\partial_{p,q}^h[i](f) = \begin{cases} \beta_l^q(id_A \otimes f) & \text{if } i = 0 \\ f \circ \mu_i^p & \text{if } 1 \leq i \leq p \\ \beta_r^q(f \otimes \text{id}_A) & \text{if } i = p+1, \end{cases}$$

and

$$\partial_{p,q}^v[i](f) = \begin{cases} (\text{id}_H \otimes f)\lambda_1^p & \text{if } i = 0 \\ (\Delta_i^q \otimes \text{id}_A)f & \text{if } 1 \leq i \leq q \\ (\text{id}_{H \otimes q} \otimes \lambda)f & \text{if } i = q+1. \end{cases}$$

We now define a new complex. For each p, q set $\mathcal{C}^{p,q}(A, H) = C^{p,q}(A) \oplus C^{p,q+1}(H)$. Define the horizontal and vertical differentials by

$$d_{p,q}^h : \mathcal{C}^{p,q}(A, H) \to \mathcal{C}^{p+1,q}(A, H), \quad d_{p,q}^v : \mathcal{C}^{p,q}(A, H) \to \mathcal{C}^{p,q+1}(A, H)$$

by the formulas

$$d_{p,q}^h(\phi, f) = (\partial_{p,q}^h \phi \, , \, \partial_{p,q+1}^h f),$$

and

$$d_{p,q}^v(\phi, f) = (\partial_{p,q}^v \phi + (-1)^q \, (f \otimes \text{id}_A)\lambda_2^p \, , \, \partial_{p,q+1}^v f).$$

Here we abuse of the notation by using the same symbol for denoting the differentials coming from the bialgebra cohomology of H and those coming from the cohomology of the comodule algebra A.

Lemma 6.1. $\mathcal{C}^{*,*}(A, H) = (\mathcal{C}^{p,q}(A, H), d_{p,q}^h, (-1)^p d_{p,q}^v)$ is a bicomplex.

Proof. We have to verify that the following equations hold: $d^h \circ d^h = 0$, $d^v \circ d^v = 0$ and $d^h \circ d^v = d^v \circ d^h$. The first one is obvious. Let us prove the other two.

It can be easily checked, case by case, that for any $0 \leq i \leq q+2$ and any $0 \leq j \leq p+1$ the equalities

$$\partial_{p,q+1}^v[i]((f \otimes \text{id}_A)\lambda_2^p) = (\partial_{p,q}^v[i](f) \otimes \text{id}_A)\lambda_2^p,$$

$$\partial_{p,q+1}^h[j]\big((f \otimes \text{id}_A)\lambda_2^p\big) = \big(\partial_{p,q+1}^h[j](f) \otimes \text{id}\big)\lambda_2^{p+1}$$

hold. This implies that

(6.1) $$\partial_{p,q+1}^v((f \otimes \text{id}_A)\lambda_2^p) = (\partial_{p,q}^v(f) \otimes \text{id}_A)\lambda_2^p$$

and

(6.2) $$\partial_{p,q+1}^h\big((f \otimes \text{id}_A)\lambda_2^p\big) = \big(\partial_{p,q+1}^h(f) \otimes \text{id}\big)\lambda_2^{p+1}.$$

Let $p, q \in \mathbb{N}$ and $(\phi, f) \in \mathcal{C}^{p,q}(A, H)$ then

$$\begin{aligned} d_{p,q+1}^v d_{p,q}^v(\phi, f) &= d_{p,q+1}^v\big(\partial_{p,q}^v \phi + (-1)^q \, (f \otimes \text{id}_A)\lambda_2^p \, , \, \partial_{p,q}^v f\big) \\ &= \big(\partial_{p,q+1}^v \partial_{p,q}^v \phi + (-1)^q \partial_{p,q+1}^v((f \otimes \text{id}_A)\lambda_2^p) + \\ & \quad (-1)^{q+1}(\partial_{p,q}^v f \otimes \text{id}_A)\lambda_2^p, \partial_{p,q+1}^v \partial_{p,q}^v f\big). \end{aligned}$$

Follows from (6.1) that this last expression is zero. By a straightforward computation one proves that $d^v_{p+1,q}d^h_{p,q} = d^h_{p,q+1}d^v_{p,q}$ using (6.2).

$\square$

The total complex $(\mathrm{Tot}^*(\mathcal{C}^{*,*}(A,H)), d)$ of the bicomplex $\mathcal{C}^{*,*}(A,H)$ is defined by

$$\mathrm{Tot}^n(\mathcal{C}^{*,*}(A,H)) = \bigoplus_{p+q=n} C^{p,q}(A) \oplus \bigoplus_{p+q=n} C^{p,q+1}(H),$$

and the differentials $d^n : \mathrm{Tot}^n(\mathcal{C}^{*,*}(A,H)) \to \mathrm{Tot}^{n+1}(\mathcal{C}^{*,*}(A,H))$ are defined by

$$d^n|_{\mathcal{C}^{n-i,i}(A,H)} = d^h_{n-i,i} + (-1)^p\, d^v_{n-i,i},$$

for each $0 \le i \le n$. Borrowing the notation of D. Yau [Y08], we will denote the cohomology groups of this complex by $H^*_{ca}(A,H)$.

Let $(\phi, f) \in \mathcal{C}^{2,0}(A,H) = C^{2,0}(A) \oplus C^{2,1}(H)$, $(\psi, g) \in \mathcal{C}^{1,1}(A,H) = C^{1,1}(A) \oplus C^{1,2}(H)$. The element $(\phi, f) + (\psi, g) \in Z^2_{ca}(A,H)$ if and only if

$$(6.3) \qquad \partial^h_{2,0}\phi = 0, \quad \partial^v_{1,1}\psi - (g\otimes\mathrm{id}_A)\lambda = 0$$

$$(6.4) \qquad \partial^h_{2,1}f = 0 = \partial^v_{1,2}g, \quad \partial^v_{2,1}f + \partial^h_{1,2}g = 0,$$

$$(6.5) \qquad \partial^v_{2,0}\phi + (f\otimes\mathrm{id}_A)\lambda^2_2 + \partial^h_{1,1}\psi = 0.$$

The next result follows from a simple computation.

Proposition 6.2. *Let H be a graded Hopf algebra, (G, μ, λ) a graded left H-comodule algebra. Let $\left(H[t]/(t^{l+1}), m^l, \Delta^l\right)$ be an l-deformation of H and $\left(G[t]/(t^{l+1}), \mu^l, \lambda^l\right)$ an l-deformation of G compatible with the deformation of H. Then $(\mu^l_1 + m^l_1) + (\lambda^l_1, \Delta^l_1) \in Z^2_{ca}(G,H)$.* $\square$

7. Comodule algebras over pointed Hopf algebras

Let H be a Hopf algebra and R be a braided Hopf algebra in $^H_H\mathcal{YD}$.

Definition 7.1. A *braided left comodule algebra* over R is an algebra $B \in {}^H_H\mathcal{YD}$ together with a linear map $\delta_B : B \to R\otimes_\Bbbk B$, $\delta_B(x) = x^{[-1]}\otimes x^{[0]}$, such that δ_B is a morphism in the category $^H_H\mathcal{YD}$, (B, δ_B) is a left R-comodule, and such that for all $x, y \in B$

$$(7.1) \qquad \delta_B(xy) = x^{[-1]}(x^{[0]}{}_{(-1)} \cdot y^{[-1]})\otimes x^{[0]}{}_{(0)}y^{[0]}.$$

7.1. Comodule algebras over bosonizations. We shall give a simple recipe to produce comodule algebras over a Hopf algebra constructed from a bosonization.

Let H_0 be a Hopf algebra and $R \in {}^{H_0}_{H_0}\mathcal{YD}$ be a braided Hopf algebra. Let us recall that the bosonization, or Radford biproduct, $H = R\#H_0$, is the Hopf algebra over the vector space $R\otimes_{\Bbbk}H_0$ with product and coproduct given by

$$(r\#g)(s\#f) = rg_{(1)} \cdot s\#g_{(2)}f,$$

$$\Delta(r\#g) = r_{(1)}\#(r_{(2)})_{(-1)}g_{(1)}\otimes(r_{(2)})_{(0)}\#g_{(2)},$$

for all $r, s \in R$, $f, g \in H$.

Let B be a braided comodule algebra over R and F be a left H_0-comodule algebra. We shall give to the tensor product $B\otimes_{\Bbbk}F$ a natural structure of $R\#H_0$-comodule algebra. Let us define the product and left $R\#H_0$-comodule structure by:

$$\delta(x\otimes f) = x^{[-1]}\# x^{[0]}{}_{(-1)}f_{(-1)}\otimes x^{[0]}{}_{(0)}\otimes f_{(0)},$$

$$(x\otimes f)(y\otimes g) = x(f_{(-1)} \cdot y)\otimes f_{(0)}g,$$

for all $x, y \in B$, $f, g \in F$. We shall denote by $B\#F$ the space $B\otimes_{\Bbbk}F$ together this new product and the coaction δ.

Proposition 7.2. *The following assertions hold:*

    (1) *$B\#F$ is a left $R\#H_0$-comodule algebra.*
    (2) *If $F^{\mathrm{co}\,H_0} = \Bbbk$ and $B^{\mathrm{co}\,R} = \Bbbk$ then we have that $(B\#F)^{\mathrm{co}\,R\#H_0} = \Bbbk$.*

Proof. (1) follows by a straightforward computation.

Let $\sum x_i\otimes f_i \in (B\#F)^{\mathrm{co}\,R\#H_0}$. Then

$$\delta(\sum x_i\otimes f_i) = \sum x_i^{[-1]}\# x_i^{[0]}{}_{(-1)}f_{i(-1)}\otimes x_i^{[0]}{}_{(0)}\otimes f_{i(0)} = \sum 1\otimes1\otimes x_i\otimes f_i.$$

Applying the counit on the second tensorand we obtain that

$$\sum x_i^{[-1]}\otimes x_i^{[0]}\otimes f_i = \sum 1\otimes x_i\otimes f_i,$$

whence $x_i \in B^{\mathrm{co}\,R} = \Bbbk$, which implies that $f_i \in F^{\mathrm{co}\,H_0} = \Bbbk$. This proves (2). $\qquad\square$

7.2. Loewy-graded comodule algebras over pointed Hopf algebras.

For the rest of this section G will denote a finite group, $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ will denote a Yetter-Drinfeld module over G and $\mathfrak{B}(V)$ the associated Nichols algebra, see [AS02] and references therein. The Nichols algebra is a $\mathbb{N}_0$-graded braided Hopf algebra with grading $\mathfrak{B}(V) = \oplus_{i=0}^m\mathfrak{B}^i(V)$. Let us denote by $\delta : V \to \Bbbk[G]\otimes_{\Bbbk}V$ the coaction.

We denote the bosonization $H = \mathfrak{B}(V)\#\Bbbk[G]$. The Hopf algebra H is a coradically graded Hopf algebra, with grading given by $H(n) = \mathfrak{B}^n(V)\otimes_{\Bbbk}\Bbbk[G]$ for any $n = 0\ldots m$. We will denote by $\theta : H \to \mathfrak{B}(V)$ and $p : H \to \Bbbk[G]$ the canonical projections. For any $h \in H$ we have that $h = \theta(h_{(1)})p(h_{(2)})$. The space $\mathfrak{B}(V)$ are the coinvariants of H under the coaction $(\mathrm{id}\otimes p)\Delta$, that is $\mathfrak{B}(V) = \{h \in h : (\mathrm{id}\otimes p)\Delta(h) = h\otimes1\}$.

Let (A, λ) be a Loewy-graded left H-comodule algebra with gradation given by $A = \bigoplus_{i=0}^{m} A(i)$. Let $\pi : A \to A(0)$ be the canonical projection.

Let us define
$$\mathfrak{B}_A = \{a \in A : (\mathrm{id} \otimes \pi)\lambda(a) \in H \otimes 1\}.$$

This space should be thought as a kind of *diagram* [AS02], for the comodule algebra A. Indeed, if $(A, \lambda) = (H, \Delta)$ then $\mathfrak{B}_A = \mathfrak{B}(V)$ is the diagram corresponding to H.

For each $n \in \mathbb{N}$ define $\mathfrak{B}_A(n) = \mathfrak{B}_A \bigcap A(n)$.

Proposition 7.3. *The following statements holds.*

(1) $\mathfrak{B}_A(0) = \Bbbk 1$, $\mathfrak{B}_A = \bigoplus_{n=0} \mathfrak{B}_A(n)$.
(2) $\mathfrak{B}_A \subseteq A$ *is a left H-subcomodule subalgebra, in particular* $\lambda(\mathfrak{B}_A(n)) \subseteq \oplus_{i=0}^{n} H(n-i) \otimes_{\Bbbk} \mathfrak{B}_A(i)$.
(3) *There exists an injective H-comodule algebra homomorphism $\iota :$ $\mathfrak{B}_A \to \mathfrak{B}(V)$ such that $\iota(\mathfrak{B}_A(n)) \subseteq \mathfrak{B}^n(V)$ for any $n = 0 \dots m$.*
(4) *If $\mathfrak{B}_A(1) = 0$ then $\mathfrak{B}_A = \Bbbk 1$.*
(5) *The space $\iota(\mathfrak{B}_A(1)) \subseteq V$ is a $\Bbbk G$-subcomodule.*
(6) *The multiplication map $\mu : \mathfrak{B}_A \otimes_{\Bbbk} A_0 \to A$ is injective.*

Proof. (1). The first assertion is clear. Let $a \in \mathfrak{B}_A$, then $a = \sum_{i=0} a_i$, where $a_i \in A(i)$. Since $(\mathrm{id} \otimes \pi)\lambda(a) \in H \otimes_{\Bbbk} 1$ and the spaces $H(i)$ are disjoint we conclude that $(\mathrm{id} \otimes \pi)\lambda(a_i) \in H(i) \otimes_{\Bbbk} 1$ for each i, hence $a_i \in \mathfrak{B}_A(i)$.

(2). It is clear that $\mathfrak{B}_A$ is a subalgebra since λ and π are algebra maps. Let $a \in \mathfrak{B}_A$, then $a_{(-1)} \otimes \pi(a_{(0)}) = h \otimes 1$, for some $h \in H$. Then

$$(7.2) \qquad \Delta(h) \otimes 1 = a_{(-2)} \otimes a_{(-1)} \otimes \pi(a_{(0)}) = a_{(-1)} \otimes a_{(0)(-1)} \otimes \pi(a_{(0)(0)}).$$

Therefore $a_{(-1)} \otimes a_{(0)} \in H \otimes \mathfrak{B}_A$.

(3). For $a \in \mathfrak{B}_A$ define the map $\iota : \mathfrak{B}_A \to H$ by $\iota(a) = h \in H$, where h is the unique element in H such that $a_{(-1)} \otimes \pi(a_{(0)}) = h \otimes 1$. Using (7.2) it is easy to see that ι is an H-comodule map.

Let us prove that ι is injective. Let $a \in \mathfrak{B}_A$ such that $\iota(a) = 0$, thus $a_{(-1)} \otimes \pi(a_{(0)}) = 0$. Let us write $a = \sum_n a_n$ where $a_n \in A(n)$. In this case $\lambda(a) = \sum_n \lambda(a_n) = \sum_n \sum_{i=0}^{n} b_i^n$, where $b_i^n \in H(i) \otimes_{\Bbbk} A(n-i)$, for $i = 0 \dots n$. Since $a_{(-1)} \otimes \pi(a_{(0)}) = \sum_n b_n^n = 0$ this means that for each n, $b_n^n = 0$, thus $\lambda(a_n) \in H_{n-1} \otimes_{\Bbbk} A$, whence $a_n \in A_{n-1}$, which is impossible unless $a_n = 0$ since for all n: $A(n) \bigcap A_{n-1} = 0$. Thus ι is injective.

Observe that $\lambda\pi = (p \otimes \pi)\lambda$, thus if $a \in \mathfrak{B}_A$ then there exists an element $h \in H$ such that $a_{(-1)} \otimes \pi(a_{(0)}) = h \otimes 1$. Hence

$$(\mathrm{id} \otimes p)\Delta)(h) \otimes 1 = a_{(-2)} \otimes p(a_{(-1)}) \otimes \pi(a_{(0)}) = a_{(-1)} \otimes (p \otimes \pi)\lambda(a_{(0)})$$

$$= a_{(-1)} \otimes \lambda\pi(a_{(0)}) = h \otimes 1 \otimes 1.$$

Thus $(\mathrm{id} \otimes p)\Delta(h) = h \otimes 1$, that is $h \in \mathfrak{B}(V)$. Since $H(n) = \mathfrak{B}^n(V) \# \Bbbk G$ and ι is an H-comodule map then $\iota(\mathfrak{B}_A(n)) \subseteq \mathfrak{B}^n(V)$. This ends the proof of (3).

(4). Let us assume that $\mathfrak{B}_A(1) = 0$. We can (and will) assume that $\mathfrak{B}_A \subseteq \mathfrak{B}(V)$. Let $a \in \mathfrak{B}_A(2)$ then $\Delta(a) \in H_0 \otimes_{\Bbbk} \mathfrak{B}_A(2) \oplus H(2) \otimes_{\Bbbk} \mathfrak{B}_A(0)$, thus $a \in H_1$, hence $a = 0$. Using the same argument we can prove inductively that $\mathfrak{B}_A(n) = 0$ for all $n \geq 2$.

(5). Let us consider $W = \mathfrak{B}_A(1)$ as a $\Bbbk G$-comodule via $(p \otimes \mathrm{id}_A)\lambda$. Then $W = \oplus_{g \in G} W_g$, where $W_g = \{w \in \mathfrak{B}_A(1) : p(a_{(-1)}) \otimes a_0 = g \otimes a\}$. If $a \in W_g$ then $\lambda(a) = g \otimes a + v_a \otimes 1$, for some $v_a \in V$. Note that $\iota(a) = v_a$. In this case, $\delta(v_a) = g \otimes v_a$, thus $\iota(W)$ is a subcomodule of V.

Now, let us prove (6). Consider the map $\phi : A \to \mathfrak{B}(V) \otimes_{\Bbbk} A_0$ given by $\phi(a) = \theta(a_{(-1)}) \otimes \pi(a_{(0)})$ and let $\mu : \mathfrak{B}_A \otimes_{\Bbbk} A_0 \to A$ be the multiplication map. Then $\phi \circ \mu = \iota \otimes \mathrm{id}_{A_0}$, indeed if $x \in \mathfrak{B}_A, a \in A_0$ then

$$\phi(xa) = \theta(x_{(-1)}a_{(-1)}) \otimes \pi(x_{(-1)}a_{(0)}) = \theta(x_{(-1)}a_{(-1)}) \otimes \pi(x_{(-1)})a_{(0)}$$
$$= \theta(\iota(x)a_{(-1)}) \otimes a_{(0)} = \iota(x) \otimes a.$$

Since ι is injective this implies that μ is injective. $\qquad\qquad\square$

Remark 7.4. The space $\iota(\mathfrak{B}_A)$ is a left homogeneous coideal subalgebra of $\mathfrak{B}(V)$. The algebra $\mathfrak{B}_A$ need not be generated by $\mathfrak{B}_A(1)$. However, in some special cases, for example when $\mathfrak{B}(V)$ is a quantum linear space, $\mathfrak{B}_A$ is always generated in degree one.

Let us further assume that (A, λ) is right H-simple. Follows by Proposition 4.4 that $A(0)$ is right $\Bbbk[G]$-simple and therefore there exists a subgroup $F \subseteq G$ and a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$ such that $A(0) \simeq \Bbbk^\psi[F]$ as $\Bbbk[G]$-comodule algebras. That is, there is a basis $\{e_f\}_{f \in F}$ of $A(0)$ such that

$$e_f e_{f'} = \psi(f, f')\, e_{ff'}, \quad \lambda(e_f) = f \otimes e_f,$$

for all $f, f' \in F$.

In the case when ψ is trivial there is an action of F on $\mathfrak{B}_A$ given as follows. If $x \in \mathfrak{B}_A$ and $f \in F$ then

$$f \cdot x := e_f x e_{f^{-1}} \in \mathfrak{B}_A.$$

With this action and the trivial coaction $\delta : \mathfrak{B}_A \to \Bbbk[F] \otimes_{\Bbbk} \mathfrak{B}_A$, given by $\delta(x) = 1 \otimes x$, $\mathfrak{B}_A$ is a Yetter-Drinfeld module over $\Bbbk[F]$. Thus, we can consider the smash product $\mathfrak{B}_A \# A(0)$ as in Proposition 7.2.

The following proposition tells us that the comodule algebra A can be recovered from $\mathfrak{B}_A$ and A_0.

Proposition 7.5. *Assume that the group G is a cyclic group, then $A \simeq \mathfrak{B}_A \# A(0)$ as H-comodule algebras.*

Proof. Since G is cyclic then ψ must be trivial and A_0 is isomorphic to the group algebra $\Bbbk F$, where $F \subseteq G$ is a subgroup. Thus, we can consider the product $\mathfrak{B}_A \# A(0)$.

By Proposition 7.3 (6) the multiplication map $\mu : \mathfrak{B}_A \# A_0 \to A$ is injective and clearly is an H-comodule algebra map. Let us prove that μ is surjective.

Let Q denote the quotient coalgebra $H/(\Bbbk F)^+ H$ and $\gamma : H \to Q$ the canonical projection. Since $A \in {}^H_{\Bbbk F}\mathcal{M}$, then $A \simeq H\square_Q \overline{A}$, where $\overline{A} = A/(\Bbbk F)^+ A$ and the left Q-comodule structure on $\overline{A}$ is given by $(\gamma \otimes \mathrm{id}_A)\lambda$. The left H-comodule structure on $H\square_Q \overline{A}$ is given by the coproduct of H.

There is an isomorphism $\alpha : \Bbbk F \otimes Q \xrightarrow{\simeq} H$ of left $\Bbbk F$-modules, right Q-comodules. Thus there is a linear isomorphism $A \simeq \Bbbk F \otimes \overline{A}$.

It is easy to see that $(H\square_Q \overline{A})_0 = \Bbbk F \otimes \overline{1}$. Let us denote by $\pi : H\square_Q \overline{A} \to (H\square_Q \overline{A})_0$ the projection. The image of the injective map $\beta : \overline{A} \to H\square_Q \overline{A}$ given by $\beta(\overline{a}) = \alpha(1 \otimes \overline{a}_{(-1)}) \otimes \overline{a}_{(0)}$ composed with the isomorphism $A \simeq H\square_Q \overline{A}$ is inside $\mathfrak{B}_A$. Indeed, for any $\overline{a} \in \overline{A}$ the element

$$\alpha(1 \otimes \overline{a}_{(-1)})_{(1)} \otimes \pi(\alpha(1 \otimes \overline{a}_{(-1)})_{(2)} \otimes \overline{a}_{(0)}) \in H \otimes 1.$$

This implies that $\dim \overline{A} \leq \dim \mathfrak{B}_A$. Whence, $\dim A \leq \mathfrak{B}_A \mid F \mid$, thus μ is surjective and therefore a bijection. $\qquad\square$

Remark 7.6. Assume we are under the hypothesis of Proposition 7.5. It is immediate to verify that under the identification of $A(0) \simeq \Bbbk F \hookrightarrow \Bbbk G$, the space $\iota(\mathfrak{B}_A) \subseteq \mathfrak{B}(V)$ is stable under the action of F.

As a remarkable consequence of the above proposition we have the following result. Let U be a Hopf algebra such that $\mathrm{gr}_c U = \mathfrak{B}(V) \# \Bbbk G$ for a cyclic group G.

Lemma 7.7. *If A is a U-simple left U-comodule algebra with trivial coinvariants, then $\dim A$ divides $\dim U$.* $\qquad\square$

This result is related with a conjecture made by Y. Zhu for semisimple Hopf algebras and it is related to Kaplansky's sixth conjecture. See [Z, Conjecture 4.1].

8. Some classification results

Let H be a pointed Hopf algebra with coradical the group algebra of a cyclic group C_n, such that $\mathrm{gr}_c H = \mathfrak{B}(V) \# \Bbbk[C_n]$, where $V \in {}^{\Bbbk C_n}_{\Bbbk C_n}\mathcal{YD}$ is a Yetter-Drinfeld module and $\mathfrak{B}(V)$ the associated Nichols algebra.

We propose the following strategy to classify module categories over the category $\mathrm{Rep}(H)$. First we determine all Loewy-graded right $\mathfrak{B}(V) \# \Bbbk[G]$-simple left $\mathfrak{B}(V) \# \Bbbk[G]$-comodule algebras A, with trivial coinvariants. By Proposition 7.5 this comodule algebras are determined by the subalgebra $\mathfrak{B}_A$ and A_0, where $\mathfrak{B}_A = \oplus_i \mathfrak{B}_A(i)$ is a homogeneous left coideal subalgebra of $\mathfrak{B}(V)$ stable under the action of A_0. Since A_0 is a $\Bbbk C_n$-simple left $\Bbbk C_n$-comodule algebra, then $A_0 = \Bbbk C_d$ for some divisor d of n. In our examples $\mathfrak{B}_A$ is generated as an algebra by $\mathfrak{B}_A(1)$, so really $\mathfrak{B}_A$ depends on a subspace

$W \subseteq V$ stable under the action of C_d. To end we shall find all possible liftings of this comodule algebras.

In this section we present the classification of exact module categories over some Hopf algebras, where the dimension of V is 1 or 2. More precisely we will show the classification for the Taft Hopf algebras T_q, over the Radford Hopf algebras $\mathbf{r}_q$, over the book Hopf algebras and over the small quantum groups $u_q(\mathfrak{sl}_2)$.

Let us recall the definition of these Hopf algebras and some other Hopf algebras that will be used later. Some notations are taken from [AS98]. Let $n \in \mathbb{N}$ and q be a n-th primitive root of unity.

- The *Taft Hopf algebra* $T_q = \Bbbk\langle g, x|\ gx = q\,xg,\ g^n = 1,\ x^n = 0\rangle$, with coproduct determined by $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$.
- The algebra $\widehat{T(q)} = \Bbbk\langle g, x|\ gx = q\,xg,\ g^{n^2} = 1,\ x^n = 0\rangle$. The coproduct is determined by $\Delta(x) = x \otimes 1 + g \otimes x$, $\Delta(g) = g \otimes g$.
- The *Radford Hopf algebra* is $\mathbf{r}(q) = \Bbbk\langle g, x|\ gx = q\,xg,\ g^{n^2} = 1,\ x^n = 1 - g^n\rangle$. The coproduct is determined by $\Delta(g) = g \otimes g$, $\Delta(x) = g \otimes x + x \otimes 1$.
- The *book Hopf algebras* $\mathcal{H}(1, q) = \Bbbk\langle g, x, y|\ g^n = 1,\ gx = q\,xg,\ gy = q^{-1}\,yg,\ xy = q\,yx,\ x^n = 0 = y^n\rangle$. With coproduct determined by $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g^{-1} \otimes x$, $\Delta(y) = y \otimes 1 + g^{-1} \otimes y$.
- The *Frobenius-Lusztig kernel* $u_q(\mathfrak{sl}_2) = \Bbbk\langle g, x, y|\ gx = q^2\,xg,\ gy = q^{-2}\,yg,\ g^n = 1,\ x^n = 0 = y^n,\ xy - yx = g - g^{-1}\rangle$. The coproduct is determined by $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(y) = y \otimes 1 + g^{-1} \otimes y$.

8.1. Module categories over $\mathrm{Rep}(T_q)$.

In this section we recover, using different techniques, the results obtained in [EO04, Thm 4.10].

Let d be a divisor of n. Set $n = dm$. For any $\xi \in \Bbbk$ define the algebra $\mathcal{A}(d, \xi)$ generated by elements h, w subject to relations

$$h^d = 1, \quad hw = q^m\,wh, \quad y^n = \xi\,1.$$

The algebra $\mathcal{A}(d, \xi)$ is a left T_q-comodule algebra with coaction determined by

$$\lambda(h) = g^m \otimes h, \ \lambda(w) = x \otimes 1 + g \otimes w.$$

Let us denote by $K(d)$ the left coideal subalgebra of T_q generated by g^m and x.

Lemma 8.1. *For any $\xi \in \Bbbk$ there is an isomorphism $\mathcal{A}(d, \xi) \simeq K(d)_{\sigma_\xi}$ for some cocentral 2-cocycle σ_ξ. In particular the algebras $\mathcal{A}(d, \xi)$ are right T_q-simple left T_q-comodule algebras such that $\mathcal{A}(d, \xi)^{\mathrm{co}\,T_q} = \Bbbk$.*

Proof. The algebra $\mathcal{A}(n,\xi)$ is a left T_q-Galois extension and $\mathcal{A}(d,\xi) \subset \mathcal{A}(n,\xi)$ is a subalgebra and a left T_q-submodule. Thus there exists a 2-cocycle such that $\mathcal{A}(n,\xi) \simeq {}_{\sigma_\xi}T_q$. This cocycle is cocentral since $\mathcal{A}(n,\xi)$ is also a right T_q-comodule algebra with structure map $\rho : \mathcal{A}(n,\xi) \to \mathcal{A}(n,\xi)\otimes_{\Bbbk}T_q$ given by

$$\rho(h) = h\otimes g, \quad \rho(w) = w\otimes 1 + h\otimes x.$$

The other statements are straightforward. $\qquad\square$

Lemma 8.2.   (i) *Categories ${}_{\mathcal{A}(d,\xi)}\mathcal{M}$ are semisimple for any divisor d of n and any $\xi \in \Bbbk^\times$.*

  (ii) *For any $\xi \in \Bbbk^\times$ the rank of ${}_{\mathcal{A}(d,\xi)}\mathcal{M}$ is $\frac{n}{d}$.*

  (iii) *The algebras $\mathcal{A}(d,\xi)$, $\mathcal{A}(d',\xi')$ are Morita equivariant equivalent if and only if $d = d'$, $\xi = \xi'$.*

Proof. (i) and (ii) are Straightforward. Let us assume that $\mathcal{A}(d,\xi) \sim_M \mathcal{A}(d,\xi')$. From (ii) we get that $d = d'$. Let us denote by Q the quotient coalgebra $Q = T_q/K(d)^+T_q \simeq \Bbbk C_m$. Using Lemma 3.6 and Lemma 8.1 there exists a group-like element $f \in G(T_q)$ such that $\mathcal{A}(d,\xi') \simeq (fK(d)f^{-1})_{\sigma_\xi}$. Thus $\mathcal{A}(d,\xi') \simeq \mathcal{A}(d,\xi)$, whence $\xi = \xi'$. $\qquad\square$

Proposition 8.3. *If $\mathcal{M}$ is an exact indecomposable module category over $\mathrm{Rep}(T_q)$ then $\mathcal{M} \simeq {}_{\Bbbk C_d}\mathcal{M}$ or $\mathcal{M} \simeq {}_{\mathcal{A}(d,\xi)}\mathcal{M}$ for some divisor d of n and $\xi \in \Bbbk$.*

Proof. Let G be a Loewy-graded left T_q-comodule algebra. Let d be a divisor of n such that $G_0 = \Bbbk C_d$. Let us assume that $G \neq G_0$. Using Proposition 7.5 $G \simeq \mathfrak{B}_G\#\Bbbk C_d$. Since $G \neq G_0$ then $\mathfrak{B}_G(1) \neq 0$. The only possibility is that $G = \mathcal{A}(d,0)$.

Let (A,λ) be a T_q-simple left T_q-comodule algebra such that A is a lifting of $\mathcal{A}(d,0)$ along T_q for some divisor d of n. So in this case $A_0 = \Bbbk C_d$ is the group algebra of the cyclic group of d elements generated by h where $\lambda(h^i) = g^{mi}\otimes h^i$, for any $i = 0\dots d-1$, where $dm = n$.

By Lemma 5.5 there is an element $y \in \mathcal{A}_1 - \mathcal{A}_0$ such that $\lambda(y) = x\otimes 1 + g\otimes y$ and and $hy = q^m yh$. Since A is a lifting for $\mathcal{A}(d,0)$, the set $\{h^iy^j : i = 0\dots d-1, j = 0\dots n-1\}$ is a basis for A. Here $A_0 = \Bbbk\{h^i : i = 0\dots d-1\}$.

Since $\lambda(y^n) = x^n\otimes 1 + g^n\otimes y^n = 1\otimes y^n$ then $y^n \in A_0$ and there exists an $\xi \in \Bbbk$ such that $y^n = \xi 1$ and we have a projection $\mathcal{A}(d,\xi) \twoheadrightarrow A$ which is an isomorphism because both algebras have the same dimension. $\qquad\square$

8.2. Module categories over $\mathrm{Rep}(\mathbf{r}_q)$.

Let C_{n^2} be the cyclic group of n^2 elements generated by g. Let $V \in {}^{\Bbbk C_{n^2}}_{\Bbbk C_{n^2}}\mathcal{YD}$ be the one-dimensional module generated by w with action and coaction determined by

$$g \cdot w = q\, w, \quad \delta(w) = g\otimes w.$$

The following result is straightforward.

Lemma 8.4. $\mathrm{gr}_c\mathbf{r}(q) = \widehat{T(q)}$ *and* $\widehat{T(q)} \simeq \mathfrak{B}(V)\#\Bbbk C_{n^2}$. $\qquad\square$

We shall classify exact module categories over the Radford Hopf algebras in a similar way as for the Taft Hopf algebras. No new difficulty arise since the dimension of the the vector space V is 1.

Let us define families of left $\mathbf{r}(q)$-module algebras, right $\mathbf{r}(q)$-simple. Let d be a divisor of n^2 and set $n^2 = dm$. Let also $\xi \in \Bbbk^\times$.

(a) The group algebra $\Bbbk C_d = \Bbbk\langle h : h^d = 1\rangle$, with coaction determined by $\lambda(h) = g^m \otimes h$.

(b) Algebras $\mathcal{A}(d) = \Bbbk\langle h, w : h^d = 1, w^n = 1, hw = q^m\, wh\rangle$ and coaction determined by $\lambda(h) = g^m \otimes h$, $\lambda(w) = x \otimes 1 + g \otimes w$.

(c) If there is an integer a such that $na = d$, then $\mathcal{B}(a, \xi) = \Bbbk\langle h, w : h^d = 1, w^n = 1 + \xi\, h^a, hw = q^m\, wh\rangle$ and coaction determined by $\lambda(h) = g^m \otimes h$, $\lambda(w) = x \otimes 1 + g \otimes w$.

For any d divisor of n^2 and any $\xi \in \Bbbk$ the algebras $\mathcal{A}(d)$ and $\mathcal{B}(a, \xi)$ are left $\mathbf{r}(q)$-module algebras, right $\mathbf{r}(q)$-simple with trivial coinvariants. The algebras listed above are non-isomorphic as comodule algebras.

Lemma 8.5. *The algebras $\mathcal{A}(d)$ and $\mathcal{B}(a, \xi)$ are twistings of left coideal subalgebras of $\mathbf{r}(q)$ by a compatible Hopf 2-cocycle, pair-wise non Morita equivariant equivalent.*

Proof. Algebras $\mathcal{A}(n)$ and $\mathcal{B}(n, \xi)$ are $\mathbf{r}(q)$-Galois extensions of the field and $\mathcal{A}(d) \subseteq \mathcal{A}(n)$, $\mathcal{B}(a, \xi) \subseteq \mathcal{B}(n, \xi)$ are $\mathbf{r}(q)$-subcomodules subalgebras. Let us prove now that these algebras are non Morita equivariant equivalent. For this we will need the following result:

Claim 8.1. *Let H be a pointed Hopf algebra. Let $J \subseteq H$ be a left coideal subalgebra and $\sigma : H \otimes_\Bbbk H \to \Bbbk$ a compatible 2-cocycle, such that the graded algebra $\mathrm{gr}_l(_\sigma J)$ is isomorphic to a left coideal subalgebra K of $\mathrm{gr}_c H$ and the coalgebra quotient $Q = \mathrm{gr}_c H/(\mathrm{gr}_c H)K^+$ is pointed cosemisimple. Then if $P \in {}^H\mathcal{M}_{\sigma J}$ is an indecomposable object there exists a group-like element $g \in H_0$ such that $P \simeq g \cdot {}_\sigma J$.*

Idea of the proof. Consider the filtration $P_i = \delta^{-1}(H_i \otimes_\Bbbk P)$. The associated graded vector space $\mathrm{gr}\, P$ is an object in the category ${}^{\mathrm{gr}_c H}\mathcal{M}_K \simeq {}^Q\mathcal{M}$. Since P is indecomposable, the so is $\mathrm{gr}\, P$. Therefore there exists a group-like element $g \in H$ such that $\mathrm{gr}\, P = gK$. $\qquad\square$

Let A be a left $\mathbf{r}(q)$-comodule algebra. Let us assume, for instance, that $A \sim_M \mathcal{B}(a, \xi)$ for some a such that $na = d$. Then there exists an object $P \in {}^{r(q)}\mathcal{M}_{\mathcal{B}(a,\xi)}$ such that $A \simeq \mathrm{End}_{\mathcal{B}(a,\xi)}(P)$. Since in our case we are under the hypothesis of Claim 8.1, $P = g \cdot \mathcal{B}(a, \xi)$, and arguing as in Lemma 3.6 we obtain that $A \simeq \mathcal{B}(a, \xi)$, thus if A is an algebra in the list it must be equal to $\mathcal{B}(a, \xi)$. The same argument can be used to prove that if $A \sim_M \mathcal{A}(d)$ then $A \simeq \mathcal{A}(d)$. $\qquad\square$

Proposition 8.6. *If $\mathcal{M}$ is an exact indecomposable module category over $\mathrm{Rep}(\mathbf{r}(q))$ then $\mathcal{M} \simeq {}_{\Bbbk C_m}\mathcal{M}$, $\mathcal{M} \simeq {}_{\mathcal{A}(d)}\mathcal{M}$ or $\mathcal{M} \simeq {}_{\mathcal{B}(a,\xi)}\mathcal{M}$.*

Proof. Let (G, λ_0) be a Loewy-graded $\widehat{T(q)}$-comodule algebra. We assume that $G \neq G_0$ and $G_0 = \Bbbk\langle h : h^d = 1\rangle$ for some d divisor of n^2. Since $\widehat{T(q)} \simeq \mathfrak{B}(V)\#\Bbbk C_{n^2}$, V is one-dimensional and $\mathfrak{B}_G \subseteq \mathfrak{B}(V)$ is generated by $\mathfrak{B}_G(1)$, thus G is generated by h and an element w subject to relations

$$h^d = 1, \quad hw = q^m\,wh, \quad w^n = 0,$$

Where $dm = n^2$. The coaction is determined by

$$\lambda_0(h) = g^m{\otimes}h, \quad \lambda_0(w) = x{\otimes}1 + g{\otimes}w.$$

Let (A, λ) be a lifting of G along $\mathbf{r}(q)$. In particular $A_0 = G_0$. Using again Lemma 5.5 it is easy to see that A is generated by elements h and w with $\lambda(h) = g^m{\otimes}h$, $\lambda(w) = x{\otimes}1 + g{\otimes}w$ and $hw = q^m\,wh$.

Since $\lambda(w^n) = (1 - g^n){\otimes}1 + g^n{\otimes}w^n$, whence $w^n \in A_0$. Thus, there exists $\xi_i \in \Bbbk$, $i = 0\ldots d - 1$, such that $w^n = \sum_{i=0}^{d-1} \xi_i\,h^i$. Therefore

$$\sum_{i=0}^{d-1} \xi_i\,g^n{\otimes}h^i + (1 - g^n){\otimes}1 = \sum_{i=0}^{d-1} \xi_i\,g^{mi}{\otimes}h^i.$$

This implies that $\xi_0 = 1$. If m does not divides n then $\xi_i = 0$ for all $i = 1\ldots d - 1$, if there is an integer a such that $ma = n$ then $\xi_i = 0$ for all $i \neq a$ and ξ_a is arbitrary. In the first case $A \simeq \mathcal{A}(d)$ and in the second case $A \simeq \mathcal{B}(a, \xi_a)$. $\qquad\square$

8.3. **Module categories over** $\mathrm{Rep}(\mathcal{H}(1, q))$**.** We shall assume that $n > 2$.
Let $V = \Bbbk\{x, y\}$ denote the 2-dimensional Yetter-Drinfeld module over $\Bbbk C_n$, where C_n is the cyclic group generated by g, with action $\cdot : \Bbbk C_n{\otimes}_{\Bbbk} V \to V$ and coaction $\delta : V \to \Bbbk C_n{\otimes}_{\Bbbk} V$ determined by

$$g \cdot x = q\,x, \quad g \cdot y = q^{-1}\,y, \quad \delta(x) = g^{-1}{\otimes}x, \quad \delta(y) = g^{-1}{\otimes}y.$$

It is not difficult to prove that $\mathcal{H}(1, q) = \mathfrak{B}(V)\#\Bbbk C_n$. So $\mathcal{H}(1, q)$ is a coradically graded Hopf algebra with gradation given by $\mathcal{H}(1, q)(i) = \mathfrak{B}^i(V){\otimes}_{\Bbbk}\Bbbk C_n$.

Lemma 8.7. *Let $K = \oplus_{i=0}^{n-1} K(i) \subseteq \mathfrak{B}(V)$ be an homogeneous left coideal subalgebra of $\mathcal{H}(1, q)$, that is K is graded as an algebra, $K(j) \subseteq \mathfrak{B}^j(V)$ and $\Delta(K(j)) \subseteq \oplus_{i=0}^{j} \mathcal{H}(1, q)(i){\otimes}_{\Bbbk}K(j - i)$ for all $j = 0\ldots n - 1$. Then K is generated as an algebra by $K(1)$.*

Proof. If $\dim K(1) = 2$ then $K = \mathfrak{B}(V)$ and the claim is obviously true. Let us assume that $\dim K(1) = 1$, and let $w = ax + by$ be a nonzero element of $K(1)$. We will prove that for any $m = 1\ldots n - 1$, $K(m)$ is the 1-dimensional vector space generated by w^m.

Let $\theta \in K(m)$, then since $K(m) \subseteq \mathfrak{B}^m(V)$, $\theta = \sum_{i=0}^{m} \alpha_i\,y^i x^{m-i}$ for some $\alpha_i \in \Bbbk$.

Let us denote by χ_i^n the quantum Gaussian coefficients (or q-binomial coefficients), that is

$$\chi_i^n = \frac{(q^{n-i+1} - 1) \dots (q^n - 1)}{(q - 1) \dots (q^i - 1)}$$

It is well-known that

$$\Delta(x^k) = \sum_{j=0}^{k} \chi_j^k \, g^{-j} x^{k-j} \otimes x^j, \quad \Delta(y^k) = \sum_{j=0}^{k} \chi_j^k \, y^{k-j} g^{-j} \otimes y^j.$$

This implies that

$$(8.1) \qquad \Delta(\theta) = \sum_{i=0}^{m} \sum_{j,l} \alpha_i \chi_j^{k-i} \chi_l^i \left(y^{i-l} g^{-j-l} x^{k-i-j} \otimes y^l x^j \right).$$

Since $\Delta(\theta) \in \oplus_{i=0}^{m} \mathcal{H}(1,q)(m-i) \otimes K(i)$, there exists an element $v \in \mathcal{H}(1,q)(m-1)$ such that the summand of $\Delta(\theta)$ that belongs to $\mathcal{H}(1,q)(m-1) \otimes K(1)$ equals $v \otimes w$. Using equation (8.1) we obtain that

$$a \, v = \sum_{i=0}^{m-1} \alpha_i \chi_1^{m-i} \chi_0^i \, y^i g^{-1} x^{m-i-1},$$

$$b \, v = \sum_{i=1}^{m} \alpha_i \chi_0^{m-i} \chi_1^i \, y^{i-1} g^{-1} x^{m-i}.$$

When $a = 0$, or $b = 0$, the above equations immediately imply that θ is a scalar multiple of y^m, or x^m respectively. Let us assume that $ab \neq 0$. In this case we have that

$$\frac{b}{a} \sum_{i=0}^{m-1} \alpha_i \, \chi_1^{m-i} \chi_0^i \, y^i g^{-1} x^{m-i-1} = \sum_{i=1}^{m} \alpha_i \, \chi_0^{m-i} \chi_1^i \, y^{i-1} g^{-1} x^{m-i}.$$

Comparing coefficients we obtain that for any $i = 1 \dots m-1$

$$\frac{b}{a} \alpha_i \chi_1^{m-i} = \alpha_{i+1} \chi_1^{i+1}.$$

From this equation we obtain that $\alpha_i = \gamma \chi_i^m a^{m-i} b^i$, for some $\gamma \in \Bbbk$, thus $\theta = \gamma \, (ax + by)^m$. $\qquad \square$

I. Heckenberger pointed out to me that the above result is a very special property that holds for quantum linear spaces, clearly not valid for arbitrary Nichols algebras.

Let us define collections of left $\mathcal{H}(1,q)$-comodule algebras right $\mathcal{H}(1,q)$-simple. The algebras in the list are pair-wise non isomorphic as comodule algebras. Let $d \in \mathbb{N}$ be a divisor of n and $dm = n$.

(a) The group algebra $\Bbbk C_d = \Bbbk \langle h : h^d = 1 \rangle$, with coaction determined by $\lambda(h) = g^m \otimes h$.

(b) For any $\xi \in \Bbbk$, the algebras $\mathcal{A}_0(d,\xi) = \Bbbk\langle h,w : h^d = 1, w^n = \xi 1, hw = q^m\, wh\rangle$ and coaction determined by $\lambda(h) = g^m\otimes h$, $\lambda(w) = x\otimes 1 + g^{-1}\otimes w$.

(c) For any $\xi \in \Bbbk$, the algebras $\mathcal{A}_1(d,\xi) = \Bbbk\langle h,w : h^d = 1, w^n = \xi 1, hw = q^m\, wh\rangle$ and coaction determined by $\lambda(h) = g^m\otimes h$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$.

(d) For any $\xi,\mu \in \Bbbk$, $\mu \neq 0$, the algebras $\mathcal{A}(\xi,\mu) = \Bbbk\langle w : w^n = \xi 1\rangle$ with coaction determined by $\lambda(w) = (\mu\,x + y)\otimes 1 + g^{-1}\otimes w$.

(e) If n is even, then for any $\xi,\mu \in \Bbbk$, $\mu \neq 0$, the algebras $\mathcal{A}'(\xi,\mu) = \Bbbk\langle h,w : h^2 = 1, w^n = \xi 1, hw = q^{\frac{n}{2}}\,wh\rangle$ with coaction determined by $\lambda(h) = g^{\frac{n}{2}}\otimes h$, $\lambda(w) = (\mu\,x + y)\otimes 1 + g^{-1}\otimes w$.

(f) For any $\xi,\mu \in \Bbbk$, the algebras $\mathcal{D}(d,\xi,\mu) = \Bbbk\langle h,w,z : h^d = 1, w^n = \xi 1, z^n = \mu 1, hz = q^m\,zh, hw = q^{-m}\,wh, zw - q\,wz = 0\rangle$. The coaction is determined by $\lambda(h) = g^m\otimes h$, $\lambda(z) = x\otimes 1 + g^{-1}\otimes z$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$.

(g) For any $\xi,\mu,\eta \in \Bbbk$, the algebras $\mathcal{D}_1(\xi,\mu,\eta) = \Bbbk\langle h,w,z : h^n = 1, w^n = \xi 1, z^n = \mu 1, hz = q\,zh, hw = q^{-1}\,wh, zw - q\,wz = \eta\,h^{n-2}\rangle$. The coaction is determined by $\lambda(h) = g\otimes h$, $\lambda(z) = x\otimes 1 + g^{-1}\otimes z$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$.

(h) Assume that $n = 2k$, $1 < k \in \mathbb{N}$. For any $\xi,\mu \in \Bbbk$, the algebras $\mathcal{D}_2(\xi,\mu,\eta) = \Bbbk\langle h,w,z : h^k = 1, w^n = \xi 1, z^n = \mu 1, hz = q^2\,zh, hw = q^{-2}\,wh, zw - q\,wz = \eta\,h^{k-1}\rangle$. The coaction is determined by $\lambda(h) = g^2\otimes h$, $\lambda(z) = x\otimes 1 + g^{-1}\otimes z$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$.

Lemma 8.8. *The algebras listed above are of the form $_\sigma K$, where $K \subseteq \mathcal{H}(1,q)$ is a left coideal subalgebra and σ is a compatible Hopf 2-cocycle. Moreover the algebras listed above are pair-wise non Morita equivariant equivalent.*

Proof. The proof is completely analogous to the proof of Lemma 8.5. $\square$

Proposition 8.9. *Let $\mathcal{M}$ be an exact indecomposable module category over $\mathrm{Rep}(\mathcal{H}(1,q))$, then $\mathcal{M} \simeq {}_K\mathcal{M}$, where K is one of the H-comodule algebras listed above.*

Proof. Let (G,λ_0) be a Loewy-graded $\mathcal{H}(1,q)$-comodule algebra such that $G_0 = \Bbbk C_d$ for some divisor d of n and $G \neq G_0$. Follows from Proposition 7.5 that $G \simeq K\#G_0$ where $K \subseteq \mathfrak{B}(V)$ is an homogeneous left coideal subalgebra. Since $K(1) \neq 0$, follows from Lemma 8.7 that the only four possibilities for K are the following. $K(1) = \Bbbk < x >$, $K(1) = \Bbbk < y >$, $K(1) = \Bbbk < \mu x + y >$ for some $0 \neq \mu \in \Bbbk$, or $K = \mathfrak{B}(V)$. Notice that if $K(1) = \Bbbk < \mu x + y >$, since K is invariant under the action of G_0 and $g \cdot x = q\,x, g \cdot y = q^{-1}\,y$ the only possibilities are that $G_0 = \Bbbk 1$, thus $d = 1$, or that $G_0 \simeq \Bbbk C_2$.

Let (A,λ) be a lifting of G along $\mathcal{H}(1,q)$. In the first three cases it is easy to prove that $A \simeq \mathcal{A}_0(d,\xi)$, $A \simeq \mathcal{A}_1(d,\xi)$ or $A \simeq \mathcal{A}(\xi,\mu)$ for some $\xi,\mu \in \Bbbk$.

Let us assume that (A, λ) is a lifting of $\mathfrak{B}(V)\#\Bbbk C_d$ along $\mathcal{H}(1, q)$. Using the same arguments as in the proof of Proposition 8.3 one can prove that A is generated by elements h, w, z subject to relations $h^d = 1$, $w^n = \xi 1$, $z^n = \mu 1$, $hz = q^m zh$, $hw = q^{-m} wh$ for some $\xi, \mu \in \Bbbk$, and the coaction is determined by $\lambda(h) = g^m\otimes h$, $\lambda(z) = x\otimes 1 + g^{-1}\otimes z$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$. Since $\lambda(zw - qwz) = g^{-2}\otimes(zw - qwz)$, then $zw - qwz \in A_0$. Thus there exists $\eta_i \in \Bbbk$ such that $zw - qwz = \sum_{i=0}^d \eta_i h^i$. Therefore

$$\sum_{i=0}^d \eta_i\, g^{mi}\otimes h^i = \sum_{i=0}^d \eta_i\, g^{-2}\otimes h^i.$$

We conclude that there are two possibilities: if there is no $i = 0 \ldots d - 1$ such that $g^{-2} = g^{mi}$, in which case all $\eta_i = 0$ and $zw - qwz = 0$, thus $A \simeq \mathcal{D}(d, \xi, \mu)$. The other case is when there exists one $i = 0 \ldots d - 1$ such that $g^{-2} = g^{mi}$ thus $\eta_j = 0$ for all j except when $j = i$. Also $n \mid mi + 2$. Since $i < d$, $mi + 2 < n + 2$ then $n = mi + 2$, but since m is a divisor of n, this implies that m divides 2, hence $m = 1$ or $m = 2$. In the first case $d = n$ and $A \simeq \mathcal{D}_1(\xi, \mu, \eta)$ and in the second case $A \simeq \mathcal{D}_2(\xi, \mu, \eta)$. $\qquad\square$

8.4. **Module categories over** $\mathrm{Rep}(u_q(\mathfrak{sl}_2))$. For convenience we shall work over a twist equivalent Hopf algebra rather than with $u_q(\mathfrak{sl}_2)$ itself.

Let $2 < n \in \mathbb{N}$ and $q \in \Bbbk$ be an n-th primitive root of 1. Let $\mathcal{H}_q$ be the algebra generated by elements g, x, y subject to relations

$$gx = q^2\, xg,\ \ gy = q^{-2}\, yg,\ \ g^n = 1,\ x^n = 0,\ \ y^n = 0,\ \ xy - q^2\, yx = 1 - g^{-2}.$$

The algebra $\mathcal{H}_q$ is a Hopf algebra with coproduct determined by $\Delta(g) = g\otimes g$, $\Delta(x) = x \otimes 1 + g^{-1} \otimes x$, $\Delta(y) = y \otimes 1 + g^{-1} \otimes y$.

It is easy to verify that $\mathcal{H}_q \simeq u_q(\mathfrak{sl}_2)$ as Hopf algebras, where the isomorphism is given by multiplying by the group-like element g and that $\mathrm{gr}_c\mathcal{H}_q \simeq \mathfrak{B}(V)\#\Bbbk C_n$ as Hopf algebras, where $V = \Bbbk\{x, y\}$ is the 2-dimensional Yetter-Drinfeld module over $\Bbbk C_n$ with structure maps $\delta : V \to \Bbbk C_n\otimes_\Bbbk V$, $\cdot : \Bbbk C_n\otimes_\Bbbk V \to V$ given by

$$\delta(x) = g^{-1}\otimes x,\ \delta(y) = g^{-1}\otimes y,\ g \cdot x = q^2\, x,\ g \cdot y = q^{-2}\, y.$$

It is well-known that the category of representations of $H = \mathfrak{B}(V)\#\Bbbk C_n$ is tensor equivalent to $\mathrm{Rep}(\mathcal{H}_q)$, see for example [Ma08]. Therefore we shall describe module categories over $\mathrm{Rep}(H)$. Let us define a family of left H-comodule algebras right H-simple. Let $d \in \mathbb{N}$ be a divisor of n and $dm = n$.

(a) The group algebra $\Bbbk C_d = \Bbbk\langle h : h^d = 1\rangle$, with coaction determined by $\lambda(h) = g^m\otimes h$.

(b) For any $\xi \in \Bbbk$, the algebras $\mathcal{A}_0(d, \xi) = \Bbbk\langle h, w : h^d = 1, w^n = \xi 1, hw = q^m\, wh\rangle$ and coaction determined by $\lambda(h) = g^m\otimes h$, $\lambda(w) = x\otimes 1 + g^{-1}\otimes w$.

(c) For any $\xi \in \Bbbk$, the algebras $\mathcal{A}_1(d, \xi) = \Bbbk\langle h, w : h^d = 1, w^n = \xi 1, hw = q^m\, wh\rangle$ and coaction determined by $\lambda(h) = g^m\otimes h$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$.

(d) For any $\xi, \mu \in \Bbbk$, $\mu \neq 0$, the algebras $\mathcal{A}(\xi, \mu) = \Bbbk\langle w : w^n = \xi 1\rangle$ with coaction determined by $\lambda(w) = (\mu\,x + y)\otimes 1 + g^{-1}\otimes w$.

(e) If n is even, for any $\xi, \mu \in \Bbbk$, $\mu \neq 0$, the algebras $\mathcal{A}'(\xi, \mu) = \Bbbk\langle h, w : h^2 = 1, w^n = \xi 1, hw = q^{\frac{n}{2}}\,wh\rangle$ with coaction determined by $\lambda(h) = g^{\frac{n}{2}}\otimes h$, $\lambda(w) = (\mu\,x + y)\otimes 1 + g^{-1}\otimes w$.

(f) If $n = 4m$, for any $\xi, \mu \in \Bbbk$, $\mu \neq 0$, the algebras $\mathcal{A}''(\xi, \mu) = \Bbbk\langle h, w : h^4 = 1, w^n = \xi 1, hw = q^m\,wh\rangle$ with coaction determined by $\lambda(h) = g^m\otimes h$, $\lambda(w) = (\mu\,x + y)\otimes 1 + g^{-1}\otimes w$.

(g) For any $\xi, \mu \in \Bbbk$, the algebras $\mathcal{B}(d, \xi, \mu) = \Bbbk\langle h, w, z : h^d = 1$, $w^n = \xi 1$, $z^n = \mu 1$, $hw = q^m\,wh$, $hz = q^m\,zh$, $zw - q^2\,wz = 1\rangle$, with coaction determined by $\lambda(h) = g^m\otimes h$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$, $\lambda(z) = x\otimes 1 + g^{-1}\otimes z$.

(h) If there exists $a \in \mathbb{N}$ such that $ma + 2 = n$, then for any $\xi, \mu, \eta \in \Bbbk$, with $\eta \neq 0$, set the algebras $\mathcal{C}(d, \xi, \mu, \eta, a) = \Bbbk\langle h, w, z : h^d = 1$, $w^n = \xi 1$, $z^n = \mu 1$, $hw = q^m\,wh$, $hz = q^m\,zh$, $zw - q^2\,wz = 1 + \eta\,h^a\rangle$, with coaction determined by $\lambda(h) = g^m\otimes h$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$, $\lambda(z) = x\otimes 1 + g^{-1}\otimes z$.

Lemma 8.10. *The algebras listed above are of the form $_\sigma K$, where $K \subseteq H$ is a left coideal subalgebra and σ is a compatible 2-cocycle. Moreover the algebras listed above are pair-wise non Morita equivariant equivalent.*

Proof. The proof is completely analogous to the proof of Lemma 8.5. $\square$

Proposition 8.11. *If $\mathcal{M}$ is an indecomposable exact module category over $\mathrm{Rep}(u_q(\mathfrak{sl}_2))$ then $\mathcal{M}$ is equivalent to one of the following categories: $_{\Bbbk C_d}\mathcal{M}$, $_{\mathcal{A}_0(d,\xi)}\mathcal{M}$, $_{\mathcal{A}_1(d,\xi)}\mathcal{M}$, $_{\mathcal{A}(\xi,\mu)}\mathcal{M}$, $_{\mathcal{A}'(\xi,\mu)}\mathcal{M}$, $_{\mathcal{A}''(\xi,\mu)}\mathcal{M}$, $_{\mathcal{B}(d,\xi,\mu)}\mathcal{M}$, $_{\mathcal{C}(d,\xi,\mu,\eta,a)}\mathcal{M}$.*

Proof. Let (G, λ_0) be a Loewy-graded $\mathfrak{B}(V)\#\Bbbk C_n$-comodule algebra. We assume that $G \neq G_0$ and $G_0 = \Bbbk\langle h : h^d = 1\rangle$ for some d divisor of n. The space $\mathfrak{B}_G \subseteq \mathfrak{B}(V)$ is an homogeneous left coideal subalgebra generated as an algebra by $\mathfrak{B}_G(1)$. This can be proven using the same arguments as in the proof of Lemma 8.7.

Assume that $\dim \mathfrak{B}_G(1) = 2$, then evidently $\mathfrak{B}_G = \mathfrak{B}(V)$. Let (A, λ) be a lifting of (G, λ_0) along H. Arguing as in Proposition 8.9 we obtain that A is generated by elements h, w, z subject to relations

$$h^d = 1, \quad w^n = \xi 1, \quad z^n = \xi 1, \quad hw = q^m\,wh, \quad hz = q^m\,zh.$$

Where the coaction is determined by $\lambda(h) = g^m\otimes h$, $\lambda(z) = x\otimes 1 + g^{-1}\otimes z$, $\lambda(w) = y\otimes 1 + g^{-1}\otimes w$. Since

$$(8.2) \qquad \lambda(zw - q^2\,wz) = (1 - g^{-2})\otimes 1 + g^{-2}\otimes(zw - q^2\,wz),$$

then there exists $\alpha_i \in \Bbbk$, $i = 0 \ldots d - 1$ such that $zw - q^2\,wz = \sum_i \alpha_i\,h^i$. Using equation (8.2) we obtain that

$$\sum_i \alpha_i\,g^{mi}\otimes h^i = (1 - g^{-2})\otimes 1 + \sum_i \alpha_i\,g^{-2}\otimes h^i.$$

This implies that $\alpha_0 = 1$. If there is no $1 \leq i \leq d-1$ such that $mi = n-2$ then $\alpha_i = 0$ for all $i = 1 \ldots d-1$ and in this case $A \simeq \mathcal{B}(d, \xi, \mu)$. If there is a natural number a such that $ma = n-2$ then $\alpha_i = 0$ for all $i \neq a$ and α_a is arbitrary. In this case $A \simeq \mathcal{C}(d, \xi, \mu, \alpha_a, a)$.

If $\dim \mathfrak{B}_G(1) = 1$, we have three possibilities: $\mathfrak{B}_G(1) = \Bbbk < x >$, $\mathfrak{B}_G(1) = \Bbbk < y >$ or $\mathfrak{B}_G(1) = \Bbbk < \mu x + y >$ for some $\mu \in \Bbbk^\times$. In the first two cases $A \simeq \mathcal{A}_0(d, \xi)$ or $A \simeq \mathcal{A}_1(d, \xi)$ respectively. In the third case, since $g \cdot x = q^2 x$ and $g \cdot y = q^{-2} y$, the only possibilities are that $G_0 = \Bbbk 1$, $G_0 = \Bbbk C_2$ or $G_0 = \Bbbk C_4$ and then $A \simeq \mathcal{A}(\xi, \mu)$, $A \simeq \mathcal{A}'(\xi, \mu)$ or $A \simeq \mathcal{A}''(\xi, \mu)$ for appropriate $\xi, \mu \in \Bbbk$.

$\square$

9. SOME FURTHER COMMENTS

1. Notice that if H is one of the Hopf algebras in the following list:
   (i) a group algebra $\Bbbk G$ of a finite group G,
   (ii) a Taft Hopf algebra T_q,
   (iii) a Radford Hopf algebra $\mathbf{r}(q)$,
   (iv) the Frobenius-Lusztig kernel $u_q(\mathfrak{sl}_2)$,

then exact module categories over $\mathrm{Rep}(H)$ are of the form $_{\sigma K}\mathcal{M}$, where $K \subseteq H$ is a left coideal subalgebra and σ is a compatible Hopf 2-cocycle. It would be interesting to know if this statement is true for any finite-dimensional pointed Hopf algebra with abelian coradical.

2. Using the same strategy as for the book Hopf algebras and for the Radford Hopf algebras it is likely that a classification for a bosonization of quantum linear spaces can be worked out easily. Also the results by Etingof and Ostrik on the classification of exact module categories over finite supergroups $\wedge V \# \Bbbk G$, see [EO04, section 4.2], can be obatined using our techniques, at least when G is a cyclic group.

3. In principle the techniques developed here would help to classify exact module categories over $\mathrm{Rep}(u_q(\mathfrak{sl}_n))$, since in the paper [KLS08] the authors classified all homogeneous coideal subalgebras of $u_q(\mathfrak{sl}_n)$.

REFERENCES

[AM07]   N. ANDRUSKIEWITSCH and J.M. MOMBELLI, *On module categories over finite-dimensional Hopf algebras*, J. Algebra **314** (2007), 383–418.

[AS98]   N. ANDRUSKIEWITSCH and H.-J. SCHNEIDER, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), 658–691.

[AS02]   N. ANDRUSKIEWITSCH and H.-J. SCHNEIDER, *Pointed Hopf Algebras*, in New directions in Hopf algebras, 1–68, Math. Sci. Res. Inst. Publ. 43, Cambridge Univ. Press, Cambridge, 2002.

[AD03]   N. Andruskiewitsch and S. Dăscălescu, *Co-Frobenius Hopf algebras and the coradical filtration*, Math. Z. 243, 145154 (2003)

[Be67]   Bénabou, J., *Introduction to bicategories*, Reports of the Midwest Category Seminar (1967), Lecture Notes in Math. **47**, pp. 1–77, Springer, Berlin.

[BEK00] J. Böckenhauer, D. E. Evans and Y. Kawahigashi, *Chiral Structure of Modular Invariants for Subfactors,* Commun. Math. Phys. **210** (2000), 733–784.

[BO01]   R. Bezrukavnikov and V. Ostrik, *On tensor categories attached to cells in affine Weyl groups II*, `math.RT/0102220`.

[DCY05] Y. Du, X.-W. Chen, and Y. Ye, *On graded bialgebra deformations*, math.QA/0506030.

[CS07]   R. Coquereaux and G. Schieber, *Orders and dimensions for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ module categories and boundary conformal field theories on a torus*, J. of Mathematical Physics, **48**, 043511 (2007).

[CS08]   R. Coquereaux and G. Schieber, *From conformal embeddings to quantum symmetries: an exceptional $SU(4)$ example*, Journal of Physics- Conference Series Volume **103** (2008), 012006.

[EO04]   P. Etingof and V. Ostrik, *Finite tensor categories*, Mosc. Math. J. **4** (2004), no. 3, 627–654.

[ENO05] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, Ann. Math. **162**, 581–642 (2005).

[ENO08] P. Etingof, D. Nikshych and V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, preprint `arXiv:0809.3031`.

[FS01]   J. Fuchs and C. Schweigert, *Category theory for conformal boundary conditions*, in Vertex Operator Algebras in Mathematics and Physics, (2000). Preprint `math.CT/0106050`.

[KLS08] V.K. Kharchenko and A.V. Lara Sagahon, *Right coideal subalgebras in $U_q(\mathfrak{sl}_{n+1})$*, J. Algebra **319** (2008) 2571–2625.

[KO02]   A. Kirillov Jr. and V. Ostrik, *On a q-analogue of the McKay correspondence and the ADE classification of sl_2 conformal field theories*, Adv. Math. **171** (2002), no. 2, 183–227.

[Li03]   V. Linchenko, *Nilpotent subsets of Hopf module algebras*, in Groups, Rings, Lie and Hopf Algebras, Y. Bahturin (Ed.) (2003).

[Ma92]   A. Masuoka, *Freeness of Hopf algebras over coideal subalgebras*, Comm. in Algebra 20 **5**, 1353–1373 (1992).

[Ma08]   A. Masuoka, *Abelian and non-abelian second cohomologies of quantized enveloping algebras*, J. Algebra **320** (2008), 1–47.

[M07]   M. Mombelli, *Dynamical twists in Hopf algebras*, Int. Math. Res. Not. (2007) vol.2007.

[MW07]   M. Mastnak and S. Witherspoon, *Bialgebra cohomology, pointed Hopf algebras, and deformations*, preprint `math.RA/0704.2771`.

[N07]   D. Nikshych, *Non group-theoretical semisimple Hopf algebras from group actions on fusion categories*, Selecta Math. **14** (2008), 145–161.

[O03a]   V. Ostrik, *Module categories, Weak Hopf Algebras and Modular invariants*, Transform. Groups, 2 **8**, 177–206 (2003).

[O03b]   V. Ostrik, *Module categories over the Drinfeld double of a Finite Group*, Int. Math. Res. Not. **2003**, no. 27, 1507–1520.

[O08]   V. Ostrik, *Module Categories Over Representations of $SL_q(2)$ in the Non-Semisimple Case*, Geom. funct. anal. Vol. **17** (2008), 2005–2017.

[Sk07]   S. Skryabin, *Projectivity and freeness over comodule algebras*, Trans. Am. Math. Soc. **359**, No. 6, 2597-2623 (2007).

[SS04]   H-J. Schneider and P. Schauenburg, *Galois type extensions and Hopf algebras*, preprint (2004), 89 pages.

[Sw69]   M. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

[Y07]   D. YAU, *Cohomology and deformation of module-algebras*, J. Pure Appl. Algebra **209** (2007), 781–792.

[Y08]   D. YAU, *Deformation bicomplex of module-algebras*, Homology Homotopy Appl. **10**, Number 1 (2008), 97–128.

[Z]   Y. ZHU, *The dimension of irreducible modules for transitive module algebras*, Comm. Alg. 29 **7**, 2877–2886 (2001).

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