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Martín Mombelli

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On the tensor product of bimodule categories over Hopf algebras

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Abstract Let H be a finite-dimensional Hopf algebra. We give a description of the tensor product of bimodule categories over $\text{Rep}(H)$. When the bimodule categories are invertible this description can be given explicitly. We present some consequences of this description in the case H is a pointed Hopf algebra.

Keywords Brauer-Picard group · Tensor category · Module category

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1 Introduction

The *Brauer-Picard groupoid* of finite tensor categories, introduced and studied in [7], is the 3-groupoid whose objects are finite tensor categories, a 1-morphism between two tensor categories $\mathcal{C}_1, \mathcal{C}_2$ are invertible $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule categories, 2-morphisms are equivalences of such bimodule categories and 3-morphisms are isomorphisms of such equivalences. Given a tensor category \mathcal{C} the *Brauer-Picard group* of \mathcal{C} , denoted by $\text{BrPic}(\mathcal{C})$, is the group of equivalence classes of invertible \mathcal{C} -bimodule categories.

The Brauer-Picard group of a tensor category has been used to classify its extensions by a finite group [7]. Also it has a close relation to certain structures appearing in mathematical physics, see [10]. In the work [7] the authors compute the Brauer-Picard group of categories Vect_G of finite-dimensional G -graded vector spaces, where G is an Abelian group.

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M. Mombelli (✉)

Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Medina Allende s/n,

(5000) Ciudad Universitaria, Córdoba, Argentina

e-mail: martin10090@gmail.com

url: <http://www.mate.uncor.edu/~mombelli>

M. Mombelli

e-mail: mombelli@mate.uncor.edu

It is natural to pursue the computation of the Brauer-Picard group of the tensor category of representations of an arbitrary finite-dimensional Hopf algebra H . To compute $\text{BrPic}(\text{Rep}(H))$ one has to be able to give an explicit description of tensor product of two $\text{Rep}(H)$ -bimodule categories.

It is well-known that any indecomposable exact $\text{Rep}(H)$ -bimodule category is equivalent to ${}_K\mathcal{M}$, the category of finite-dimensional left K -modules, where K is a right $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -simple left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebra. If S is another such $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebra one could ask about the decomposition of ${}_S\mathcal{M} \boxtimes_{\text{Rep}(H)} {}_K\mathcal{M}$ in indecomposable $\text{Rep}(H)$ -bimodule categories. For group algebras of finite Abelian groups this decomposition was explicitly given in [7], but for arbitrary Hopf algebras this problem seems more complicated. However, if both bimodule categories ${}_K\mathcal{M}$, ${}_S\mathcal{M}$ are invertible then ${}_S\mathcal{M} \boxtimes_{\text{Rep}(H)} {}_K\mathcal{M}$ is indecomposable and, under some additional assumptions, it is equivalent to ${}_{S \square_H K}\mathcal{M}$. We present some consequences of this result that will be useful to compute the Brauer-Picard group for pointed Hopf algebras over an Abelian group.

The contents of the paper are the following. Section 3 is dedicated to recall necessary definitions and facts on representations of tensor categories. In Sect. 4 we study the tensor product of bimodule categories over the category $\text{Rep}(H)$, where H is a finite-dimensional Hopf algebra and in Sect. 5 we restrict to the case when H is quasi-triangular, allowing us to give another proof of [9, Corollary 8.10] concerning about the fusion rules of module categories over a finite group.

2 Preliminaries and notation

Throughout the paper \mathbb{k} will denote an algebraically closed field of characteristic zero. All vector spaces will be considered over \mathbb{k} . For any Abelian category \mathcal{A} we shall denote by \mathcal{A}^{op} the *opposite Abelian category*, that is objects are the same but arrows are reversed. If A is an algebra we shall denote by ${}_A\mathcal{M}$ the category of finite-dimensional left A -modules.

If H is a Hopf algebra we shall denote by S_H its antipode. If K, S are left H -comodule algebras with coaction given by λ_K, λ_S we shall denote by ${}^H_K\mathcal{M}_S$ the category of (K, S) -bimodules V equipped with a left H -coaction $\delta : V \rightarrow H \otimes V$ such that δ is a morphism of (K, S) -bimodules, that is

$$(k \cdot v \cdot s)_{(-1)} \otimes (k \cdot v \cdot s)_{(0)} = k_{(-1)} v_{(-1)} s_{(-1)} \otimes k_{(0)} \cdot v_{(0)} \cdot s_{(0)},$$

for all $s \in S, k \in K, v \in V$.

If H is a Hopf algebra, H_0 is the coradical. If (K, λ) is a left H -comodule algebra and H_0 is a Hopf subalgebra, $K_0 = \lambda^{-1}(H_0 \otimes_{\mathbb{k}} K)$ is a left H_0 -comodule algebra.

2.1 Tensor categories

A *tensor category over \mathbb{k}* is a \mathbb{k} -linear Abelian rigid monoidal category. Hereafter all tensor categories will be assumed to be over a field \mathbb{k} . A *finite tensor category* [6] is a tensor category such that it has only a finite number of isomorphism classes of simple objects, Hom spaces are finite-dimensional \mathbb{k} -vector spaces, all objects have finite length, every simple object has a projective cover and the unit object is simple. All functors will be assumed to be \mathbb{k} -linear.

2.2 Quasi-triangular Hopf algebras

Let H be a Hopf algebra. A *quasi-triangular structure* on H is an invertible element $R \in H \otimes_{\mathbb{k}} H$ such that

$$(\Delta \otimes \text{id})(R) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13}R_{12}, \tag{2.1}$$

$$R^1 h_{(1)} \otimes R^2 h_{(2)} = h_{(2)} R^1 \otimes h_{(1)} R^2, \quad \text{for all } h \in H. \tag{2.2}$$

It is well known that $(S_H \otimes \text{id})(R) = R^{-1} = (\text{id} \otimes S_H)(R)$. For the inverse of R we shall use the notation $R^{-1} = R^{-1} \otimes R^{-2}$.

If (H, R) is a quasi-triangular Hopf algebra the category $\text{Rep}(H)$ is braided with braiding given by $c_{X,Y} : X \otimes_{\mathbb{k}} Y \rightarrow Y \otimes_{\mathbb{k}} X$, $c_{X,Y}(x \otimes y) = R^2 \cdot y \otimes R^1 \cdot x$ for all $X, Y \in \text{Rep}(H)$, $x \in X, y \in Y$. The inverse of c is given by $c_{X,Y}^{-1}(x \otimes y) = R^{-1} \cdot y \otimes R^{-2} \cdot x$ for all $X, Y \in \text{Rep}(H)$, $x \in X, y \in Y$.

3 Representations of tensor categories

Let \mathcal{C} be a tensor category. A *left representation* of \mathcal{C} , or a *left module category* over \mathcal{C} is an Abelian category \mathcal{M} equipped with an exact bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$, that we will sometimes refer as the *action*, natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y) \overline{\otimes} M \rightarrow X \otimes (Y \overline{\otimes} M)$, $\ell_M : \mathbf{1} \overline{\otimes} M \rightarrow M$ such that

$$m_{X,Y,Z \otimes M} m_{X \otimes Y, Z, M} = (\text{id}_X \otimes m_{Y,Z,M}) m_{X,Y \otimes Z, M} (a_{X,Y,Z} \otimes \text{id}_M), \tag{3.1}$$

$$(\text{id}_X \otimes \ell_M) m_{X, \mathbf{1}, M} = r_X \otimes \text{id}_M. \tag{3.2}$$

A left module category \mathcal{M} is *exact* [6], if for any projective object $P \in \mathcal{C}$ the object $P \overline{\otimes} M$ is projective in \mathcal{M} for all $M \in \mathcal{M}$. A *right module category* over \mathcal{C} is an Abelian category \mathcal{M} equipped with an exact bifunctor $\overline{\otimes} : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$ equipped with isomorphisms $\tilde{m}_{M,X,Y} : M \overline{\otimes} (X \otimes Y) \rightarrow (M \overline{\otimes} X) \overline{\otimes} Y$, $r_M : M \overline{\otimes} \mathbf{1} \rightarrow M$ such that

$$\tilde{m}_{M \overline{\otimes} X, Y, Z} \tilde{m}_{M, X, Y \otimes Z} (\text{id}_M \overline{\otimes} a_{X,Y,Z}) = (\tilde{m}_{M, X, Y} \otimes \text{id}_Z) \tilde{m}_{M, X \otimes Y, Z}, \tag{3.3}$$

$$(r_M \otimes \text{id}_X) \tilde{m}_{M, \mathbf{1}, X} = \text{id}_M \otimes \ell_X. \tag{3.4}$$

A $(\mathcal{C}, \mathcal{C}')$ -*bimodule category* is an Abelian category \mathcal{M} with left \mathcal{C} -module category and right \mathcal{C}' -module category structure together with natural isomorphisms $\{\gamma_{X,M,Y} : (X \overline{\otimes} M) \overline{\otimes} Y \rightarrow X \overline{\otimes} (M \overline{\otimes} Y), X \in \mathcal{C}, Y \in \mathcal{C}', M \in \mathcal{M}\}$ satisfying certain axioms. For details the reader is referred to [9, Prop. 2.12]. A $(\mathcal{C}, \mathcal{C}')$ -bimodule category is the same as left $\mathcal{C} \boxtimes \mathcal{C}'^{\text{op}}$ -module category. Here \boxtimes denotes Deligne's tensor product of Abelian categories [4]. For a bimodule category \mathcal{M} we shall denote by

$$\left\{ m_{X,Y,M}^l : (X \otimes Y) \overline{\otimes} M \rightarrow X \otimes (Y \overline{\otimes} M) : X, Y \in \mathcal{C}, M \in \mathcal{M} \right\} \quad \text{and} \\ \left\{ m_{M,X,Y}^r : M \overline{\otimes} (X \otimes Y) \rightarrow (M \overline{\otimes} X) \overline{\otimes} Y : X, Y \in \mathcal{C}, M \in \mathcal{M} \right\}$$

the left and right associativity isomorphisms respectively.

If \mathcal{M} is a right \mathcal{C} -module category then \mathcal{M}^{op} denotes the opposite Abelian category with left \mathcal{C} action $\mathcal{C} \times \mathcal{M}^{\text{op}} \rightarrow \mathcal{M}^{\text{op}}$, $(M, X) \mapsto M \overline{\otimes} X^*$ and associativity isomorphisms $m_{X,Y,M}^{\text{op}} =$

$m_{Y^*, X^*, M}^{-1}$ for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. Similarly if \mathcal{M} is a left \mathcal{C} -module category. If \mathcal{M} is a $(\mathcal{C}, \mathcal{D})$ -bimodule category then \mathcal{M}^{op} is a $(\mathcal{D}, \mathcal{C})$ -bimodule category. See [9, Prop. 2.15].

A module functor between left \mathcal{C} -module categories \mathcal{M} and \mathcal{M}' over a tensor category \mathcal{C} is a pair (T, c) , where $T : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor and $c_{X, M} : T(X \overline{\otimes} M) \rightarrow X \overline{\otimes} T(M)$ is a family of natural isomorphism such that for any $X, Y \in \mathcal{C}, M \in \mathcal{M}$:

$$(\text{id}_X \otimes c_{Y, M})c_{X, Y \overline{\otimes} M} T(m_{X, Y, M}) = m_{X, Y, T(M)} c_{X \otimes Y, M}, \tag{3.5}$$

$$\ell_{T(M)} c_{1, M} = T(\ell_M). \tag{3.6}$$

We shall denote this functor by $(T, c) : \mathcal{M} \rightarrow \mathcal{M}'$. Sometimes we shall denote the family of isomorphisms c^T to emphasize the fact that they are related to the functor T .

Let \mathcal{M}_1 and \mathcal{M}_2 be left \mathcal{C} -module categories. The category whose objects are module functors $(\mathcal{F}, c) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ will be denoted by $\text{Func}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$. A morphism between (\mathcal{F}, c) and $(\mathcal{G}, d) \in \text{Func}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is a natural transformation $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ such that for any $X \in \mathcal{C}, M \in \mathcal{M}_1$:

$$d_{X, M} \alpha_{X \overline{\otimes} M} = (\text{id}_X \overline{\otimes} \alpha_M) c_{X, M}. \tag{3.7}$$

Two module categories \mathcal{M}_1 and \mathcal{M}_2 over \mathcal{C} are *equivalent* if there exist module functors $F : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ and $G : \mathcal{M}_2 \rightarrow \mathcal{M}_1$ and natural isomorphisms $\text{id}_{\mathcal{M}_1} \rightarrow F \circ G, \text{id}_{\mathcal{M}_2} \rightarrow G \circ F$ that satisfy (3.7).

The direct sum of two module categories \mathcal{M}_1 and \mathcal{M}_2 over a tensor category \mathcal{C} is the \mathbb{k} -linear category $\mathcal{M}_1 \times \mathcal{M}_2$ with coordinate-wise module structure. A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories. Any exact module category is equivalent to a direct sum of indecomposable exact module categories, see [6].

If $\mathcal{M}, \mathcal{M}'$ are right \mathcal{C} -modules, a module functor from \mathcal{M} to \mathcal{M}' is a pair (T, d) where $T : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor and $d_{M, X} : T(M \overline{\otimes} X) \rightarrow T(M) \overline{\otimes} X$ is a family of isomorphisms such that for any $X, Y \in \mathcal{C}, M \in \mathcal{M}$:

$$(d_{M, X} \otimes \text{id}_Y) d_{M \overline{\otimes} X, Y} T(m_{M, X, Y}) = m_{T(M), X, Y} d_{M, X \otimes Y}, \tag{3.8}$$

$$\ell_{T(M)} c_{1, M} = T(\ell_M). \tag{3.9}$$

If $\mathcal{M}, \mathcal{M}'$ are $(\mathcal{C}, \mathcal{D})$ -bimodule categories, a *bimodule functor* is the same as a module functor of $\mathcal{C} \boxtimes \mathcal{D}^{\text{op}}$ -module categories, that is a functor $F : \mathcal{M} \rightarrow \mathcal{M}'$ such that $(F, c) : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor of left \mathcal{C} -module categories, also $(F, d) : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor of right \mathcal{D} -module categories and

$$(\text{id}_X \otimes d_{M, Y}) c_{X, M \overline{\otimes}_Y F} (\gamma_{X, M, Y}) = \gamma_{X, F(M), Y} (c_{X, M} \otimes \text{id}_Y) d_{X \overline{\otimes}_Y M, Y}, \tag{3.10}$$

for all $M \in \mathcal{M}, X \in \mathcal{C}, Y \in \mathcal{D}$.

3.1 Tensor product of bimodule categories

Let $\mathcal{C}, \mathcal{C}', \mathcal{E}, \mathcal{E}'$ be tensor categories. If \mathcal{M} is a $(\mathcal{C}, \mathcal{E})$ -bimodule category and \mathcal{N} is an $(\mathcal{E}, \mathcal{C}')$ -bimodule category, the tensor product over \mathcal{E} is denoted by $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$. This category is a $(\mathcal{C}, \mathcal{C}')$ -bimodule category. For more details on the tensor product of module categories the reader is referred to [7, 9].

If \mathcal{M} is a $(\mathcal{C}, \mathcal{E})$ -bimodule category and \mathcal{N} is a $(\mathcal{C}, \mathcal{E}')$ -bimodule category then the category $\text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ has a structure of $(\mathcal{E}, \mathcal{E}')$ -bimodule category, see [9, Prop.3.18]. Let us briefly describe both structures. Let us denote

$$\overline{\otimes}^l : \mathcal{E} \times \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \rightarrow \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}), \overline{\otimes}^r : \mathcal{E}' \times \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \rightarrow \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$$

the left and right actions. If $X \in \mathcal{E}, Y \in \mathcal{E}', F \in \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ and $M \in \mathcal{M}$, then

$$(X \overline{\otimes}^l F)(M) = F(M \overline{\otimes} X), \quad (F \overline{\otimes}^r Y)(M) = F(M) \overline{\otimes} Y.$$

The module structures are the following. Let $X, X' \in \mathcal{E}, M \in \mathcal{M}$ and let $c_{X',M}^F : F(X' \overline{\otimes} M) \rightarrow X' \overline{\otimes} F(M)$ be the module functor structure of F . Then $c_{X',M}^{X \overline{\otimes}^l F} : (X \overline{\otimes} F)(X' \overline{\otimes} M) \rightarrow X' \overline{\otimes} (X \overline{\otimes} F)(M)$ is defined as the composition

$$F((X' \overline{\otimes} M) \overline{\otimes} X) \xrightarrow{F(\gamma_{X',M,X}^l)} F(X' \overline{\otimes} (M \overline{\otimes} X)) \xrightarrow{c_{X',M \overline{\otimes} X}^F} X' \overline{\otimes} F(M \overline{\otimes} X).$$

The associativity $m_{X,X',F}^l : (X \overline{\otimes} X') \overline{\otimes}^l F \rightarrow X \overline{\otimes}^l (X' \overline{\otimes}^l F)$ is the natural isomorphism

$$F(M \overline{\otimes} (X \overline{\otimes} X')) \xrightarrow{F(m_{M,X,X'}^r)} F((M \overline{\otimes} X) \overline{\otimes} X'),$$

for any $X, X' \in \mathcal{E}, M \in \mathcal{M}$. Also the map

$$c_{X,M}^{F \overline{\otimes}^r Y} : (F \overline{\otimes}^r Y)(X \overline{\otimes} M) \rightarrow X \overline{\otimes} (F \overline{\otimes}^r Y)(M)$$

is defined by the composition

$$F(X \overline{\otimes} M) \overline{\otimes} Y \xrightarrow{c_{X,M}^{F \otimes \text{id}_Y}} (X \overline{\otimes} F(M)) \overline{\otimes} Y \xrightarrow{\gamma_{X,F(M),Y}} X \overline{\otimes} (F(M) \overline{\otimes} Y).$$

Proposition 3.1 [9, Thm. 3.20] *If \mathcal{M} is a $(\mathcal{E}, \mathcal{C})$ -bimodule and \mathcal{N} is a $(\mathcal{C}, \mathcal{E}')$ -bimodule then there is a canonical equivalence of $(\mathcal{E}, \mathcal{E}')$ -bimodule categories:*

$$\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \simeq \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}). \tag{3.11}$$

3.2 The center of a bimodule category

The following definition was given in [8].

Definition 3.2 If \mathcal{M} is a \mathcal{C} -bimodule category the center of \mathcal{M} is the category $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ whose objects are pairs (M, ϕ^M) where $M \in \mathcal{M}$ and $\{\phi_X^M : X \overline{\otimes} M \rightarrow M \overline{\otimes} X : X \in \mathcal{C}\}$ is a family of natural isomorphisms such that

$$m_{M,X,Y}^r \phi_{X \otimes Y}^M = (\phi_X^M \otimes \text{id}_Y) \gamma_{X,M,Y}^{-1} (\text{id}_X \otimes \phi_Y^M) m_{X,Y,M}^l, \tag{3.12}$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. A morphism between two objects $(M, \phi^M), (N, \phi^N)$ in $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ is a morphism $f : M \rightarrow N$ in \mathcal{M} such that $(f \otimes \text{id}_X) \phi_X^M = \phi_X^N (\text{id}_X \otimes f)$ for all $X \in \mathcal{C}$.

Lemma 3.3 [9, Lemma 7.8] *If \mathcal{M} is a \mathcal{C} -bimodule category the center $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ is a $\mathcal{Z}(\mathcal{C})$ -bimodule category.*

Let us briefly explain the left and right actions that we shall denote them by $\overline{\otimes}_l$ and $\overline{\otimes}_r$, respectively. For any $X \in \mathcal{C}$, $M \in \mathcal{M}$ define

$$(X, c_X)\overline{\otimes}_l(M, \phi^M) = (X\overline{\otimes}M, \phi^{X\overline{\otimes}M}) \text{ and}$$

$$(M, \phi^M)\overline{\otimes}_r(X, c_X) = (M\overline{\otimes}X, \phi^{M\overline{\otimes}X})$$

where

$$\phi_Y^{X\overline{\otimes}M} = \gamma_{X,M,Y}^{-1}(\text{id}_X \otimes \phi_Y^M) m_{X,Y,M}^l (c_{YX} \otimes \text{id}_M) (m_{Y,X,M}^l)^{-1}, \tag{3.13}$$

$$\phi_Y^{M\overline{\otimes}X} = m_{M,X,Y}^r (\text{id}_M \otimes c_{YX}) (m_{M,Y,X}^r)^{-1} (\phi_Y^M \otimes \text{id}_X) \gamma_{Y,M,X}^{-1}, \tag{3.14}$$

for all $Y \in \mathcal{C}$.

3.3 Module categories over Hopf algebras

Assume that H is a finite-dimensional Hopf algebra and let (\mathcal{A}, λ) be a left H -comodule algebra. The category ${}_{\mathcal{A}}\mathcal{M}$ is a representation of $\text{Rep}(H)$. The action $\overline{\otimes} : \text{Rep}(H) \times {}_{\mathcal{A}}\mathcal{M} \rightarrow {}_{\mathcal{A}}\mathcal{M}$ is given by $V\overline{\otimes}M = V \otimes_{\mathbb{k}} M$ for all $V \in \text{Rep}(H)$, $M \in {}_{\mathcal{A}}\mathcal{M}$. The left \mathcal{A} -module structure on $V \otimes_{\mathbb{k}} M$ is given by the coaction λ . When \mathcal{A} is right H -simple, that is, it has no non-trivial right ideal H -costable, then the category ${}_{\mathcal{A}}\mathcal{M}$ is exact. Reciprocally, if \mathcal{M} is an exact indecomposable module category over $\text{Rep}(H)$ then there exists a left H -comodule algebra \mathcal{A} right H -simple with trivial coinvariants such that $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ as $\text{Rep}(H)$ -modules, see [1, Theorem 3.3].

Definition 3.4 If (\mathcal{A}, ρ) is a right H -comodule algebra then $(\mathcal{A}^{\text{op}}, \bar{\rho})$ is a left H -comodule algebra, where \mathcal{A}^{op} denotes the opposite algebra and $\bar{\rho} : \mathcal{A} \rightarrow H \otimes \mathcal{A}$ is defined by $\bar{\rho}(a) = S_H(a_{(1)}) \otimes a_{(0)}$, where $\rho(a) = a_{(0)} \otimes a_{(1)}$ for all $a \in \mathcal{A}$. We shall denote this left H -comodule algebra by $\bar{\mathcal{A}}$.

Lemma 3.5 *There is an equivalence $({}_{\mathcal{A}}\mathcal{M})^{\text{op}} \simeq {}_{\bar{\mathcal{A}}}\mathcal{M}$ as left $\text{Rep}(H)$ -modules.*

Proof Define $(F, c) : ({}_{\mathcal{A}}\mathcal{M})^{\text{op}} \rightarrow {}_{\bar{\mathcal{A}}}\mathcal{M}$ by $F(M) = M^*$ for any $M \in {}_{\mathcal{A}}\mathcal{M}$. If $f \in M^*$, $m \in M$, $a \in \mathcal{A}$ then $(a \cdot f)(m) = f(a \cdot m)$. For any $X \in \text{Rep}(H)$, $M \in ({}_{\mathcal{A}}\mathcal{M})^{\text{op}}$ the maps $c_{X,M} : F(X\overline{\otimes}M) \rightarrow X\overline{\otimes}F(M)$ are the identities. One can easily verify that this functor defines an equivalence of module categories. \square

Proposition 3.6 [1, Prop. 1.23] *If \mathcal{A} and \mathcal{A}' are right H -simple left H -comodule algebras, there is an equivalence of categories*

$$\text{Fun}_{\text{Rep}(H)}({}_{\mathcal{A}}\mathcal{M}, {}_{\mathcal{A}'}\mathcal{M}) \simeq {}_{\mathcal{A}'}^H\mathcal{M}_{\mathcal{A}}. \tag{3.15}$$

We shall explain briefly the proof of this Proposition. Any module functor $(F, c^F) : {}_{\mathcal{A}}\mathcal{M} \rightarrow {}_{\mathcal{A}'}\mathcal{M}$ is exact [6], thus there is exists an object $P \in {}_{\mathcal{A}'}\mathcal{M}_{\mathcal{A}}$ such that $F(M) = P \otimes_{\mathcal{A}} M$. The object P has a left H -comodule structure given by

$$\lambda : P \rightarrow H \otimes_{\mathbb{k}} P, \quad \lambda(p) = c_{H,\mathcal{A}}^F(p \otimes 1 \otimes 1),$$

for all $p \in P$.

For any finite-dimensional Hopf algebra H we shall denote by $\text{diag}(H)$ the left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebra with H as the underlying algebra structure and comodule structure:

$$\lambda : \text{diag}(H) \rightarrow H \otimes_{\mathbb{k}} H^{\text{cop}} \otimes_{\mathbb{k}} \text{diag}(H), \quad \lambda(h) = h_{(1)} \otimes h_{(3)} \otimes h_{(2)},$$

for all $h \in H$. The category ${}_H\mathcal{M}$ is a $\text{Rep}(H)$ -bimodule category with obvious structure. The proof of the following result is easy and omitted.

Lemma 3.7 *There is an equivalence of $\text{Rep}(H)$ -bimodule categories*

$${}_H\mathcal{M} \simeq_{\text{diag}(H)} \mathcal{M}.$$

4 Tensor product of bimodule categories over Hopf algebras

Let A, B be finite-dimensional Hopf algebras. A $(\text{Rep}(B), \text{Rep}(A))$ -bimodule category is the same as a left $\text{Rep}(A^{\text{cop}} \otimes B)$ -module category, see [4, Prop. 5.5]. Thus any exact $(\text{Rep}(B), \text{Rep}(A))$ -bimodule category is equivalent to the category ${}_S\mathcal{M}$ of left S -modules, where S is a finite-dimensional right $A^{\text{cop}} \otimes B$ -simple left $A^{\text{cop}} \otimes B$ -comodule algebra, see [1, Thm. 3.3]. The main purpose of this section is to understand the tensor product of $(\text{Rep}(B), \text{Rep}(A))$ -bimodule categories.

Set $\pi_A : A \otimes B \rightarrow A, \pi_B : A \otimes B \rightarrow B$ the algebra maps

$$\pi_A(x \otimes y) = \epsilon(y)x, \quad \pi_B(x \otimes y) = \epsilon(x)y,$$

for all $x \in A, y \in B$. If S is a left $A^{\text{cop}} \otimes B$ -comodule algebra the actions of the tensor categories $\text{Rep}(A), \text{Rep}(B)$ are as follows. If $M \in {}_S\mathcal{M}, X \in \text{Rep}(B), Y \in \text{Rep}(A)$ then

$$X \overline{\otimes} M = X \otimes_{\mathbb{k}} M, \quad M \overline{\otimes} Y = Y \otimes_{\mathbb{k}} M,$$

where the left action of S is:

$$s \cdot (x \otimes m) = \pi_B(s_{(-1)}) \cdot x \otimes s_{(0)} \cdot m, \quad s \cdot (y \otimes m) = \pi_A(s_{(-1)}) \cdot y \otimes s_{(0)} \cdot m,$$

for all $s \in S, x \in X, y \in Y, m \in M$. We state the following lemma that will be useful later.

Lemma 4.1 *For any $h \in A \otimes B$*

$$\pi_B(h_{(1)}) \otimes \pi_A(h_{(2)}) = \pi_B(h_{(2)}) \otimes \pi_A(h_{(1)}). \tag{4.1}$$

We shall give to the category ${}^B_K\mathcal{M}_S$ the following $\text{Rep}(A)$ -bimodule category structure. If $X, Y \in \text{Rep}(A), P \in {}^B_K\mathcal{M}_S$ then

$$X \overline{\otimes}^l P = P \otimes_S (X \otimes_{\mathbb{k}} S), \quad P \overline{\otimes}^r Y = Y \otimes_{\mathbb{k}} P.$$

Here the left S -module structure on $X \otimes_{\mathbb{k}} S$ is given by:

$$s \cdot (x \otimes t) = \pi_A(s_{(-1)}) \cdot x \otimes s_{(0)} t, \tag{4.2}$$

for all $s, t \in S, x \in X$. The object $X \overline{\otimes}^l P$ belongs to the category ${}^B_K \mathcal{M}_S$ with the following structure:

$$r \cdot (p \otimes x \otimes t) \cdot s = r \cdot p \otimes x \otimes ts, \quad \delta_1(p \otimes x \otimes s) = p_{(-1)} \pi_B(s_{(-1)}) \otimes p_{(0)} \otimes x \otimes s_{(0)},$$

for all $x \in X, r \in K, s, t \in S, p \in P$. The object $P \overline{\otimes}^r Y$ belongs to the category ${}^B_K \mathcal{M}_S$ with the following structure:

$$r \cdot (y \otimes p) \cdot s = \pi_A(r_{(-1)}) \cdot y \otimes r_{(0)} \cdot p \cdot s, \quad \delta_2(y \otimes p) = p_{(-1)} \otimes y \otimes p_{(0)},$$

for all $r \in K, s, t \in S, p \in P, y \in Y$. We shall denote the category ${}^B_K \mathcal{M}_S$ with the above described $\text{Rep}(A)$ -bimodule category by $\mathcal{M}(A, B, K, S)$ to emphasize the presence of this extra structure.

Proposition 4.2 *The category $\mathcal{M}(A, B, K, S)$ is a $\text{Rep}(A)$ -bimodule category.*

Proof The map $\delta_1 : X \overline{\otimes}^l P \rightarrow B \otimes_k X \overline{\otimes}^l P$ is well defined. Indeed, if $x \in X, s, t \in S, p \in P$ then

$$\begin{aligned} \delta_1(p \cdot t \otimes x \otimes s) &= p_{(-1)} \pi_B(t_{(-1)} s_{(-1)}) \otimes p_{(0)} \cdot t_{(0)} \otimes x \otimes s_{(0)} \\ &= p_{(-1)} \pi_B(t_{(-1)} s_{(-1)}) \otimes p_{(0)} \otimes \pi_A(t_{(0)} \text{ }_{(-1)}) \cdot x \otimes t_{(0)} \text{ }_{(0)} s_{(0)} \\ &= p_{(-1)} \pi_B(t_{(0)} \text{ }_{(-1)} s_{(-1)}) \otimes p_{(0)} \otimes \pi_A(t_{(-1)}) \cdot x \otimes t_{(0)} \text{ }_{(0)} s_{(0)} \\ &= \delta_1(p \otimes t \cdot (x \otimes s)). \end{aligned}$$

The third equality follows from (4.1). It can be proven by a straightforward computation that both objects $X \overline{\otimes}^l P, P \overline{\otimes}^r Y$ are in the category ${}^B_K \mathcal{M}_S$. Let $X, Y \in \text{Rep}(A), P \in {}^B_K \mathcal{M}_S$, define

$$m^l_{X,Y,P} : (X \otimes_k Y) \overline{\otimes}^l P \rightarrow X \overline{\otimes}^l (Y \overline{\otimes}^l P),$$

$$m^r_{M,X,Y} : P \overline{\otimes}^r (X \otimes_k Y) \rightarrow (P \overline{\otimes}^r X) \overline{\otimes}^r Y$$

by

$$m^l_{X,Y,P}(p \otimes x \otimes y \otimes s) = p \otimes y \otimes 1 \otimes x \otimes s, \quad m^r_{M,X,Y}(x \otimes y \otimes p) = y \otimes x \otimes p,$$

for all $x \in X, y \in Y, p \in P, s \in S$. One can verify easily that both maps belong to the category ${}^B_K \mathcal{M}_S$ and they satisfy axioms (3.1), (3.2) and (3.3), (3.4) respectively. The maps $\gamma_{X,P,Y} : (X \overline{\otimes}^l P) \overline{\otimes}^r Y \rightarrow X \overline{\otimes}^l (P \overline{\otimes}^r Y), \gamma(y \otimes p \otimes x \otimes s) = y \otimes p \otimes x \otimes s$ are morphisms in the category ${}^B_K \mathcal{M}_S$ and they satisfy the requirements of [9, Prop. 2.12], hence $\mathcal{M}(A, B, K, S)$ is a $\text{Rep}(A)$ -bimodule category. \square

Theorem 4.3 *Let K, S be two right $A^{\text{cop}} \otimes_k B$ -simple left $A^{\text{cop}} \otimes_k B$ -comodule algebras. The equivalence (3.15) establishes an equivalence*

$$\mathcal{M}(A, B, K, S) \simeq \text{Fun}_{\text{Rep}(B)}({}_S \mathcal{M}, {}_K \mathcal{M})$$

of $\text{Rep}(A)$ -bimodule categories.

Proof Define $\Phi : \mathcal{M}(A, B, K, S) \rightarrow \text{Fun}_{\text{Rep}(B)}({}_S\mathcal{M}, {}_K\mathcal{M})$ by

$$\Phi(P)(N) = P \otimes_S N,$$

for all $P \in {}^B_R\mathcal{M}_S, N \in {}_S\mathcal{M}$. We shall define on the functor Φ structures of left and right $\text{Rep}(A)$ -module functor. The natural isomorphisms $c_{X,P} : \Phi(X \overline{\otimes}^l P) \rightarrow X \overline{\otimes}^l \Phi(P)$ are defined by

$$(c_{X,P})_N : (P \otimes_S (X \otimes_{\mathbb{k}} S)) \otimes_S N \rightarrow P \otimes_S (X \otimes_{\mathbb{k}} N),$$

$$(c_{X,P})_N(p \otimes x \otimes s \otimes n) = p \otimes x \otimes s \cdot n,$$

for all $N \in {}_S\mathcal{M}, p \in P, x \in X, s \in S, n \in N$. The natural isomorphisms $d_{P,Y} : \Phi(P \overline{\otimes} Y) \rightarrow \Phi(P) \overline{\otimes} Y$ is defined by

$$(d_{P,Y})_N : (Y \otimes_{\mathbb{k}} P) \otimes_S N \rightarrow Y \otimes_{\mathbb{k}} (P \otimes_S N), (d_{P,Y})_N(y \otimes p \otimes n) = y \otimes p \otimes n,$$

for all $p \in P, y \in Y, n \in N$. It is easy to prove that the maps $c_{X,P}, d_{P,Y}$ are well-defined and make the functor Φ a left and right $\text{Rep}(A)$ -module functor, respectively. \square

Using the previous Theorem, equivalence (3.11) and Lemma 3.5 we obtain:

Corollary 4.4 *Let K be a right $A^{\text{cop}} \otimes_{\mathbb{k}} B$ -simple left $A^{\text{cop}} \otimes_{\mathbb{k}} B$ -comodule algebra and L a right $B^{\text{cop}} \otimes_{\mathbb{k}} A$ -simple left $B^{\text{cop}} \otimes_{\mathbb{k}} A$ -comodule algebra. There is an equivalence of $\text{Rep}(A)$ -bimodule categories:*

$${}_L\mathcal{M} \boxtimes_{\text{Rep}(B)} {}_K\mathcal{M} \simeq \mathcal{M}(A, B, K, \overline{L}).$$

Recall that \overline{L} is the opposite algebra of L with left $A^{\text{cop}} \otimes_{\mathbb{k}} B$ -comodule structure $l \mapsto (S_A^{-1} \otimes S_B)(\tau(l_{(-1)})) \otimes l_{(0)}$ where $l \mapsto l_{(-1)} \otimes l_{(0)}$ is the left $B^{\text{cop}} \otimes A$ -comodule structure and $\tau : B \otimes_{\mathbb{k}} A \rightarrow A \otimes_{\mathbb{k}} B$ is the map $\tau(b \otimes a) = a \otimes b$.

Keep in mind that K, S are finite-dimensional left $A^{\text{cop}} \otimes_{\mathbb{k}} B$ -comodule algebras. Using the map $\pi_B : A \otimes_{\mathbb{k}} B \rightarrow B$ the algebras K, S are left B -comodule algebras, thus \overline{S} is a right B -comodule algebra:

$$\overline{S} \rightarrow \overline{S} \otimes_{\mathbb{k}} B, \quad s \mapsto s_{(0)} \otimes S_B^{-1}(\pi_B(s_{(-1)})), \tag{4.3}$$

for all $s \in S$. Hence it makes sense to consider the co-tensor product $\overline{S} \square_B K$. It is clear that $\overline{S} \square_B K$ is a subalgebra of $\overline{S} \otimes_{\mathbb{k}} K$. The following result is [2, Lemma 2.2]. We shall give the proof for the sake of completeness.

Lemma 4.5 *$S \otimes_{\mathbb{k}} K$ is a left B -comodule with coaction $\rho : S \otimes_{\mathbb{k}} K \rightarrow B \otimes S \otimes_{\mathbb{k}} K$ given by*

$$\rho(s \otimes k) = \pi_B(s_{(-1)}) \pi_B(k_{(-1)}) \otimes s_{(0)} \otimes k_{(0)},$$

for all $s \in S, k \in K$. Moreover $(S \otimes K)^{\text{co} B} = \overline{S} \square_B K$.

Proof Let $\sum s \otimes k \in \overline{S} \square_B K$. Abusing of the notation, from now on we shall omit the summation symbol. Then

$$s_{(0)} \otimes S_B^{-1}(\pi_B(s_{(-1)})) \otimes k = s \otimes \pi_B(k_{(-1)}) \otimes k_{(0)}.$$

Thus we deduce that

$$\pi_B(s_{(0) (-1)}) \otimes s_{(0) (0)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes k = \pi_B(s_{(-1)}) \otimes s_{(0)} \otimes \pi_B(k_{(-1)}) \otimes k_{(0)}.$$

Then

$$\rho(s \otimes k) = \pi_B(s_{(0) (-1)}) \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes s_{(0) (0)} \otimes k = 1 \otimes s \otimes k.$$

Now, let $s \otimes k \in (\mathcal{S} \otimes K)^{\text{co}B}$ then

$$1 \otimes s \otimes k = \pi_B(s_{(-1)}) \pi_B(k_{(-1)}) \otimes s_{(0)} \otimes k_{(0)}.$$

From this equality we deduce that $1 \otimes s_{(0)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes k$ is equal to

$$\pi_B(s_{(-1)}) \pi_B(k_{(-1)}) \otimes s_{(0) (0)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(0) (-1)})) \otimes k_{(0)}.$$

Then $s_{(0)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes k = s \otimes \pi_B(k_{(-1)}) \otimes k_{(0)}$, so $s \otimes k \in \overline{S} \square_B K$. □

Define the map $\lambda : \overline{S} \square_B K \rightarrow A \otimes_{\mathbb{k}} A^{\text{cop}} \otimes \overline{S} \square_B K$ by

$$\lambda(s \otimes k) = \mathcal{S}_A(\pi_A(s_{(-1)})) \otimes \pi_A(k_{(-1)}) \otimes s_{(0)} \otimes k_{(0)},$$

for all $s \otimes k \in \overline{S} \square_B K$.

Lemma 4.6 $(\overline{S} \square_B K, \lambda)$ is a left $A \otimes_{\mathbb{k}} A^{\text{cop}}$ -comodule algebra.

Proof Let us prove first that λ is well-defined. Let $s \otimes k \in \overline{S} \square_B K$, then $s_{(0)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes k = s \otimes \pi_B(k_{(-1)}) \otimes k_{(0)}$, hence

$$\begin{aligned} s_{(0) (0)} \otimes s_{(0) (-1)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes k_{(-1)} \otimes k_{(0)} \\ = s_{(0)} \otimes s_{(-1)} \otimes \pi_B(k_{(-1)}) \otimes k_{(0) (-1)} \otimes k_{(0) (0)}. \end{aligned}$$

Therefore

$$\begin{aligned} s_{(0) (0)} \otimes \mathcal{S}_A(\pi_A(s_{(0) (-1)})) \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes \pi_A(k_{(-1)}) \otimes k_{(0)} \\ = s_{(0)} \otimes \mathcal{S}_A(\pi_A(s_{(-1)})) \otimes \pi_B(k_{(-1)}) \otimes \pi_A(k_{(0) (-1)}) \otimes k_{(0) (0)}. \end{aligned}$$

Thus $\mathcal{S}_A(\pi_A(s_{(-1)})) \otimes \pi_A(k_{(0) (-1)}) \otimes s_{(0)} \otimes \pi_B(k_{(-1)}) \otimes k_{(0) (0)}$ is equal to

$$\mathcal{S}_A(\pi_A(s_{(0) (-1)})) \otimes \pi_A(k_{(-1)}) \otimes s_{(0) (0)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(-1)})) \otimes k_{(0)}.$$

Using (4.1) we get that $\mathcal{S}_A(\pi_A(s_{(-1)})) \otimes \pi_A(k_{(0) (-1)}) \otimes s_{(0)} \otimes \pi_B(k_{(-1)}) \otimes k_{(0) (0)}$ is equal to $\mathcal{S}_A(\pi_A(s_{(-1)})) \otimes \pi_A(k_{(-1)}) \otimes s_{(0) (0)} \otimes \mathcal{S}_B^{-1}(\pi_B(s_{(0) (-1)})) \otimes k_{(0)}$. Hence $\lambda(\overline{S} \square_B K) \subseteq A^{\text{cop}} \otimes A \otimes \overline{S} \square_B K$. It follows straightforward that λ is an algebra map. □

Lemma 4.5 implies that the category $\overline{S} \square_B K \cdot \mathcal{M}$ is a $\text{Rep}(A)$ -bimodule category. In what follows we shall study the relation between this $\text{Rep}(A)$ -bimodule category and $\mathcal{M}(A, B, K, S)$.

Let $\mathcal{F} : \overline{S} \square_B K \mathcal{M} \rightarrow \mathcal{M}(A, B, K, S)$, $\mathcal{G} : \mathcal{M}(A, B, K, S) \rightarrow \overline{S} \square_B K \mathcal{M}$ be the functors defined by

$$\mathcal{F}(N) = (\overline{S} \otimes_{\mathbb{k}} K) \otimes_{\overline{S} \square_B K} N, \quad \mathcal{G}(P) = P^{\text{co}B}, \tag{4.4}$$

for all $N \in \overline{S} \square_B K \mathcal{M}$, $P \in \mathcal{M}(A, B, K, S)$. This pair of functors were considered first in [5], see also [2]. The (K, S) -bimodule structure on $\mathcal{F}(N)$ is given as follows:

$$k' \cdot (s \otimes k \otimes n) \cdot s' = s s' \otimes k' k \otimes n, \quad \text{for all } s, s' \in S, k, k' \in K, n \in N.$$

Define the map $\delta : \mathcal{F}(N) \rightarrow B \otimes_{\mathbb{k}} \mathcal{F}(N)$ by

$$\delta(s \otimes k \otimes n) = \pi_B(k_{(-1)}) \pi_B(s_{(-1)}) \otimes s_{(0)} \otimes k_{(0)} \otimes n,$$

for all $s \in S, k \in K, n \in N$. It follows from $(S \otimes K)^{\text{co}B} = \overline{S} \square_B K$ that δ is well-defined. Also $\mathcal{F}(N) \in \mathcal{M}(A, B, K, S)$, details are left to the reader. The action of $\overline{S} \square_B K$ on $\mathcal{G}(P)$ is given by

$$(s \otimes k) \cdot p = k \cdot p \cdot s, \quad \text{for all } s \otimes k \in \overline{S} \square_B K.$$

Proposition 4.7 *The functors \mathcal{F}, \mathcal{G} are left and right $\text{Rep}(A)$ -module functors.*

Proof First we shall prove that \mathcal{F} is a module functor. Let $N \in \overline{S} \square_B K \mathcal{M}$ and $X \in \text{Rep}(A)$. Define

$$c_{X,N} : (\overline{S} \otimes_{\mathbb{k}} K) \otimes_{\overline{S} \square_B K} (X \otimes_{\mathbb{k}} N) \rightarrow ((\overline{S} \otimes_{\mathbb{k}} K) \otimes_{\overline{S} \square_B K} N) \otimes_S (X \otimes_{\mathbb{k}} S)$$

by $c_{X,N}(s \otimes k \otimes x \otimes n) = 1 \otimes k \otimes n \otimes x \otimes s$, for all $x \in X, s \in S, k \in K, n \in N$. □

Claim 4.1 *The map $c_{X,N}$ is well-defined.*

Proof of claim First observe that for any $x \in X, s, t \in S$ we have that

$$x \otimes st = s_{(0)} \cdot (S_A(\pi_A(s_{(-1)}))) \cdot x \otimes t. \tag{4.5}$$

Recall that the action of S on $X \otimes_{\mathbb{k}} S$ is given in (4.2). Let $s' \otimes k' \in \overline{S} \square_B K, x \in X, s \in S, k \in K, n \in N$. Then

$$\begin{aligned} c_{X,N}((s \otimes k) \cdot (s' \otimes k') \otimes x \otimes n) &= c_{X,N}(s' s \otimes k k' \otimes x \otimes n) = 1 \otimes k k' \otimes n \otimes x \otimes s' s \\ &= 1 \otimes k k' \otimes n \otimes s'_{(0)} \cdot (S_A(\pi_A(s'_{(-1)}))) \cdot x \otimes s \\ &= s'_{(0)} \otimes k k' \otimes n \otimes S_A(\pi_A(s'_{(-1)})) \cdot x \otimes s \end{aligned}$$

The second equality follows from (4.5). On the other hand the element $c_{X,N}(s \otimes k \otimes S_A(\pi_A(s'_{(-1)}))) \cdot x \otimes (s'_{(0)} \otimes k')$ is equal to

$$\begin{aligned} &= 1 \otimes k \otimes (s'_{(0)} \otimes k') \cdot n \otimes S_A(\pi_A(s'_{(-1)})) \cdot x \otimes s \\ &= s'_{(0)} \otimes k k' \otimes n \otimes S_A(\pi_A(s'_{(-1)})) \cdot x \otimes s. \end{aligned}$$

This finishes the proof of the claim. □

Clearly $c_{X,N}$ is a (K, S) -bimodule homomorphism and also a B -comodule homomorphism. Equations (3.5) and (3.6) are satisfied. Thus (\mathcal{F}, c) is a module functor.

If $N \in \overline{S} \square_{BK} \mathcal{M}$ and $Y \in \text{Rep}(A)$ define

$$d_{N,Y} : (\overline{S} \otimes_{\mathbb{k}} K) \otimes_{\overline{S} \square_{BK}} (Y \otimes_{\mathbb{k}} N) \rightarrow Y \otimes_{\mathbb{k}} ((\overline{S} \otimes_{\mathbb{k}} K) \otimes_{\overline{S} \square_{BK}} N)$$

by $d_{N,Y}(s \otimes k \otimes y \otimes n) = \pi_A(k_{(-1)}) \cdot y \otimes s \otimes k_{(0)} \otimes n$ for all $s \in S, k \in K, n \in N, y \in Y$. It follows from a straightforward computation that the maps $d_{N,Y}$ are well-defined isomorphisms in the category $\mathcal{M}(A, B, K, S)$ and they satisfy (3.8), (3.9). Hence (\mathcal{F}, d) is a module functor.

Now, let us prove that \mathcal{G} is a module functor. Let $P \in \mathcal{M}(A, B, K, S), X, Y \in \text{Rep}(A)$. Set $c'_{X,P} : (P \otimes_S X \otimes_{\mathbb{k}} S)^{\text{co}B} \rightarrow X \otimes_{\mathbb{k}} P^{\text{co}B}$ the map defined by

$$c'_{X,P}(p \otimes x \otimes s) = S_A(\pi_A(s_{(-1)})) \cdot x \otimes p \cdot s_{(0)},$$

for all $p \otimes x \otimes s \in (P \otimes_S X \otimes_{\mathbb{k}} S)^{\text{co}B}$. Define also $d'_{p,Y} : (Y \otimes_{\mathbb{k}} P)^{\text{co}B} \rightarrow Y \otimes_{\mathbb{k}} P^{\text{co}B}$ by

$$d'_{p,Y}(y \otimes p) = y \otimes p, \text{ for all } y \otimes p \in (Y \otimes_{\mathbb{k}} P)^{\text{co}B}.$$

One can easily prove that (\mathcal{G}, c') is a module functor of left $\text{Rep}(A)$ -module categories and (\mathcal{G}, d') is a module functor of right $\text{Rep}(A)$ -module categories. □

5 Tensor product of module categories over a quasi-triangular Hopf algebra

In this section H will denote a finite-dimensional quasi-triangular Hopf algebra. We shall describe the tensor product of module categories over $\text{Rep}(H)$.

5.1 Module categories over a braided tensor category

First, let us recall some general considerations about the tensor product of module categories over a braided tensor category. Let \mathcal{C} be a braided tensor category with braiding $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ for all $X, Y \in \mathcal{C}$. Let \mathcal{M} be a right \mathcal{C} -module. Then \mathcal{M} has a left \mathcal{C} -module structure $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ given by $Y \overline{\otimes}^{\text{rev}} M := M \overline{\otimes} Y$ for all $Y \in \mathcal{C}, M \in \mathcal{M}$ and the associativity constraints $m_{X,Y,M}^{\text{rev}} : (X \otimes Y) \overline{\otimes}^{\text{rev}} M \rightarrow X \overline{\otimes}^{\text{rev}} (Y \overline{\otimes}^{\text{rev}} M)$ are given by

$$m_{X,Y,M}^{\text{rev}} = m_{M,Y,X}(\text{id}_M \otimes c_{X,Y}),$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. This category is indeed a left \mathcal{C} -module category, see [9, Lemma 7.2], that we shall denote by \mathcal{M}^{rev} . Equipped with these two structures \mathcal{M} is a \mathcal{C} -bimodule category. For details see [9, Prop. 7.1].

Remark 5.1 In particular if \mathcal{M} is a right \mathcal{C} -module and \mathcal{N} are left \mathcal{C} -module then \mathcal{M} is a bimodule category using the reverse right action, and the tensor product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is a left \mathcal{C} -module category.

If \mathcal{M} is a \mathcal{C} -bimodule category then the center $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ has two left $\mathcal{Z}(\mathcal{C})$ -module structures: the one denoted by $\overline{\otimes}_l$ explained in Sect. 3.2 given by (3.14) and the reverse action of the right action $\overline{\otimes}_r$ presented in (3.13). Both right actions give the same module category. This result will be useful later.

Proposition 5.2 *There is an equivalence of left $\mathcal{Z}(\mathcal{C})$ -module categories between $(\mathcal{Z}_{\mathcal{C}}(\mathcal{M}), \overline{\otimes}_r^{\text{rev}})$ and $(\mathcal{Z}_{\mathcal{C}}(\mathcal{M}), \overline{\otimes}_l)$.*

Proof Define $(\mathcal{F}, d) : (\mathcal{Z}_{\mathcal{C}}(\mathcal{M}), \overline{\otimes}_l) \rightarrow (\mathcal{Z}_{\mathcal{C}}(\mathcal{M}), \overline{\otimes}_r^{\text{rev}})$ the module functor as follows. The functor \mathcal{F} on objects is the identity, that is $\mathcal{F}(M, \phi^M) = (M, \phi^M)$ for any $(M, \phi^M) \in \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$. If $(X, c_X) \in \mathcal{Z}(\mathcal{C})$ define

$$d_{X,M} : (X, c_X) \overline{\otimes}_l (M, \phi^M) \rightarrow (M, \phi^M) \overline{\otimes}_r (X, c_X), \quad d_{M,X} = \phi_X^M.$$

The maps ϕ_X^M are morphisms in the category $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$. Indeed, we must prove that for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$

$$(\phi_X^M \otimes \text{id}_Y) \phi_Y^{X \otimes M} = \phi_Y^{M \otimes X} (\text{id}_Y \otimes \phi_X^M). \tag{5.1}$$

Using (3.12) one can see that the right hand side of (5.1) equals to

$$m_{M,X,Y}^r (\text{id}_M \otimes c_{YX}) \phi_{Y \otimes X}^M (m_{Y,X,M}^l)^{-1}, \tag{5.2}$$

and the left hand side of (5.1) equals to

$$m_{M,X,Y}^r \phi_{X \otimes Y}^M (c_{YX} \otimes \text{id}_M) (m_{Y,X,M}^l)^{-1}. \tag{5.3}$$

Follows from the naturality of ϕ that the expressions (5.2), (5.3) are equal. Let us prove now that the functor (\mathcal{F}, d) is a module functor. Equation (3.5) amounts to

$$(\phi_Y^M \otimes \text{id}_X) \phi_X^{Y \otimes M} m_{X,Y,M}^l = m_{M,Y,X}^r (c_{X,Y} \otimes \text{id}_M) \phi_{X \otimes Y}^M, \tag{5.4}$$

for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. Equation (5.4) can be checked by a direct computation. □

5.2 Tensor product of module categories over a quasi-triangular Hopf algebra

Let H be a finite-dimensional quasi-triangular Hopf algebra with R-matrix R . Any left $\text{Rep}(H)$ -module category is a $\text{Rep}(H)$ -bimodule category as explained in the beginning of Sect. 5.1. Given two left H -comodule algebras K, S our aim now is to describe the left $\text{Rep}(H)$ -module category over the tensor product ${}_K \mathcal{M} \boxtimes_{\text{Rep}(H)} {}_S \mathcal{M}$ using the left module category $\text{Fun}_{\text{Rep}(H)}({}_S \mathcal{M}, {}_K \mathcal{M})$ and Proposition 3.6.

Proposition 5.3 *Let K, S be two left H -comodule algebras. The category ${}^H_K \mathcal{M}_S$ is a left $\text{Rep}(H)$ -module.*

Proof Define $\overline{\otimes} : \text{Rep}(H) \times {}^H_K \mathcal{M}_S \rightarrow {}^H_K \mathcal{M}_S$ by

$$X \overline{\otimes} P := P \otimes_S (X \otimes_{\mathbb{k}} S),$$

for all $X \in \text{Rep}(H), P \in {}^H_K \mathcal{M}_S$. Here the left action of S on $X \otimes_{\mathbb{k}} S$ is given by the coaction of S . The object $P \otimes_S (X \otimes_{\mathbb{k}} S) \in {}^H_K \mathcal{M}_S$ with structure given by

$$\begin{aligned} \delta_P(p \otimes x \otimes s) &= p_{(-1)} R^2 s_{(-1)} \otimes p_{(0)} \otimes R^1 \cdot x \otimes s_{(0)}, \\ r \cdot (p \otimes x \otimes s) &= r \cdot p \otimes x \otimes s, \quad (p \otimes x \otimes s) \cdot l = p \otimes x \otimes sl, \end{aligned}$$

for all $p \in P, r \in K, s, l \in S$. Follows straightforward that these maps are well defined. Clearly $P \otimes_S (X \otimes_K S)$ is a (K, S) -bimodule and δ_P is a K -module morphism. The associativity isomorphisms

$$m_{X,Y,P} : P \otimes_S (X \otimes_K Y) \otimes_K S \rightarrow (P \otimes_S (Y \otimes_K S)) \otimes_S X \otimes_K S$$

are given by

$$m_{X,Y,P}(p \otimes (x \otimes y) \otimes s) = (p \otimes R^{-1} \cdot y \otimes 1) \otimes (R^{-2} \cdot x \otimes s),$$

for all $p \in P, x \in X, y \in Y, s \in S$. The maps $m_{X,Y,P}$ are well defined morphisms in the category ${}^H_K \mathcal{M}_S$. Indeed, let $l \in S$ then

$$\begin{aligned} m_{X,Y,P}(p \otimes l_{(-1)} \cdot (x \otimes y) \otimes l_{(0)} s) &= p \otimes R^{-1} l_{(-1)} \cdot y \otimes 1 \otimes R^{-2} l_{(-2)} \cdot x \otimes l_{(0)} s \\ &= p \otimes l_{(-2)} R^{-1} \cdot y \otimes 1 \otimes l_{(-1)} R^{-2} \cdot x \otimes l_{(0)} s \\ &= p \otimes l_{(-1)} R^{-1} \cdot y \otimes l_{(0)} \otimes R^{-2} \cdot x \otimes s \\ &= p \cdot l \otimes R^{-1} \cdot y \otimes 1 \otimes R^{-2} \cdot x \otimes s \\ &= m_{X,Y,P}(p \cdot l \otimes x \otimes y \otimes s). \end{aligned}$$

This proves that $m_{X,Y,P}$ is well-defined. The proof that $m_{X,Y,P}$ is a (K, S) -bimodule morphism is straightforward. Let us prove that $m_{X,Y,P}$ is a comodule map. If $\tilde{P} = P \otimes_S (Y \otimes_K S)$ then $\delta_{\tilde{P}}(m_{X,Y,P}(p \otimes (x \otimes y) \otimes s))$ equals to

$$p_{(-1)} J^2 r^2 s_{(-1)} \otimes p_{(0)} \otimes J^1 R^{-1} \cdot y \otimes 1 \otimes r^1 R^{-2} \cdot x \otimes s_{(0)},$$

for any $p \in P, x \in X, y \in Y, s \in S$. Here $R = R^1 \otimes R^2 = J^1 \otimes J^2 = r^1 \otimes r^2$. On the other hand $(\text{id}_H \otimes m_{X,Y,P}) \delta_P(p \otimes (x \otimes y) \otimes s)$ is equal to

$$\begin{aligned} &= p_{(-1)} R^2 s_{(-1)} \otimes m_{X,Y,P}(p_{(0)} \otimes R^1_{(-1)} \cdot x \otimes R^1_{(2)} \cdot y \otimes s_{(0)}) \\ &= p_{(-1)} R^2 s_{(-1)} \otimes p_{(0)} \otimes r^{-1} R^1_{(2)} \cdot y \otimes 1 \otimes r^{-2} R^1_{(1)} \cdot x \otimes s_{(0)} \\ &= p_{(-1)} R^2 s_{(-1)} \otimes p_{(0)} \otimes R^1_{(1)} r^{-1} \cdot y \otimes 1 \otimes R^1_{(2)} r^{-2} \cdot x \otimes s_{(0)}. \end{aligned}$$

The third equality follows from (2.2). Both terms are equal if and only if

$$J^1 R^{-1} \otimes r^1 R^{-2} \otimes J^2 r^2 = R^1_{(-1)} r^{-1} \otimes R^1_{(2)} r^{-2} \otimes R^2,$$

and this follows by (2.1). The associativity of m follows from the Yang-Baxter equation: $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$. □

We shall denote the category ${}^H_K \mathcal{M}_S$ with the structure of left $\text{Rep}(H)$ -module category explained in Proposition 5.2 by $\mathcal{M}(R, K, S)$ to emphasize the fact that the R -matrix is involved in the module category structure.

Theorem 5.4 *Let K, S be two right H -simple left H -comodule algebras. The equivalence (3.15) establishes an equivalence*

$$\mathcal{M}(R, K, S) \simeq \text{Fun}_{\text{Rep}(H)}({}_S \mathcal{M}, {}_K \mathcal{M})$$

of $\text{Rep}(H)$ -modules.

Proof Define $(\Phi, c) : \mathcal{M}(R, K, S) \rightarrow \text{Fun}_{\text{Rep}(H)}({}_S\mathcal{M}, {}_K\mathcal{M})$ by

$$\Phi(P)(N) = P \otimes_S N$$

for all $P \in {}^H_K\mathcal{M}_S$, $N \in {}_S\mathcal{M}$. The natural transformations $c_{X,P} : \Phi(X \otimes P) \rightarrow X \otimes \Phi(P)$ are defined by

$$(c_{X,P})_N : (P \otimes_S (X \otimes_{\mathbb{k}} S)) \otimes_S N \rightarrow P \otimes_S (X \otimes_{\mathbb{k}} N),$$

$$(c_{X,P})_N (p \otimes x \otimes s \otimes n) = p \otimes x \otimes s \cdot n,$$

for all $X \in \mathcal{C}$, $P \in \mathcal{M}(R, K, S)$, $N \in {}_S\mathcal{M}$, $x \in X$, $p \in P$, $n \in N$, $s \in S$. The functor (Φ, c) is a module functor and is an equivalence of module categories. \square

Corollary 5.5 *There is an equivalence of left $\text{Rep}(H)$ -modules:*

$$({}_S\mathcal{M})^{\text{op}} \boxtimes_{\text{Rep}(H)} {}_K\mathcal{M} \simeq \mathcal{M}(R, K, S). \tag{5.5}$$

5.3 Fusion rules for $\text{Rep}(\mathbb{k}G)$ -modules

Let G be a finite group. Using the equivalence (5.5) we can give another proof of [9, Corollary 8.10] concerning about the tensor product of indecomposable exact module categories over $\text{Rep}(\mathbb{k}G)$. The Hopf algebra $\mathbb{k}G$ is quasi-triangular with trivial R-matrix $1 \otimes 1$.

For any subgroup $F \subseteq G$ and $\psi \in Z^2(F, \mathbb{k}^\times)$ the twisted group algebra $\mathbb{k}_\psi F$ is a right $\mathbb{k}G$ -simple left $\mathbb{k}G$ -comodule algebra. Let $F_i \subseteq G$ be subgroups and $\psi_i \in Z^2(F_i, \mathbb{k}^\times)$ for $i = 1, 2$. Let $S \subseteq G$ be a set of representative classes of the double cosets $F_2 \backslash G / F_1$. For any $s \in S$ define $F_s = s^{-1} F_1 s \cap F_2$ and $\psi_s \in Z^2(F_s, \mathbb{k}^\times)$ the 2-cocycle defined by

$$\psi_s(x, y) = \psi_1(sxs^{-1}, sy s^{-1}) \psi_2(x, y),$$

for any $x, y \in F_s$.

Proposition 5.6 [9, Corollary 8.10] *There is an equivalence*

$$\mathbb{k}_{\psi_1 F_1} \mathcal{M} \boxtimes_{\text{Rep}(\mathbb{k}G)} \mathbb{k}_{\psi_2 F_2} \mathcal{M} \simeq \bigoplus_{s \in S} \mathbb{k}_{\psi_s F_s} \mathcal{M}. \tag{5.6}$$

Proof Let $V \in {}^{\mathbb{k}G}_{\mathbb{k}_{\psi_2 F_2}} \mathcal{M}_{\mathbb{k}_{\psi_1 F_1}}$ with coaction given by $\delta : V \rightarrow \mathbb{k}G \otimes V$. Then $V = \bigoplus_{g \in G} V_g$ where $V_g = \{v \in V : \delta(v) = g \otimes v\}$. For any $s \in S$ define

$$V_{(s)} = \bigoplus_{g \in F_1 s F_2} V_g,$$

thus $V = \bigoplus_{s \in S} V_{(s)}$ and each vector space $V_{(s)}$ is a subobject of V in the category ${}^{\mathbb{k}G}_{\mathbb{k}_{\psi_2 F_2}} \mathcal{M}_{\mathbb{k}_{\psi_1 F_1}}$.

The subspace V_s carries a structure of $\mathbb{k}_{\psi_s F_s}$ as follows. For any $h \in F_s$, $v \in V_s$ define

$$h \triangleright v = h \cdot (v \cdot shs^{-1}).$$

Define the functor $\mathcal{F} : {}^{\mathbb{k}G}_{\mathbb{k}_{\psi_2 F_2}} \mathcal{M}_{\mathbb{k}_{\psi_1 F_1}} \rightarrow \bigoplus_{s \in S} \mathbb{k}_{\psi_s F_s} \mathcal{M}$, $\mathcal{F}(V) = \bigoplus_{s \in S} V_s$ and for any $s \in S$ the vector space V_s has the action of $\mathbb{k}_{\psi_s F_s}$ as explained before. The functor \mathcal{F} is indeed a module functor.

Let $V \in {}^{\mathbb{k}G}_{\mathbb{k}\psi_2 F_2} \mathcal{M}_{\overline{\mathbb{k}\psi_1 F_1}}$ and assume that $V = V_{(s)}$ for some $s \in S$. It is not difficult to see that

$$(X \overline{\otimes} V)_{(s)} = X \overline{\otimes} V = V \otimes_{\overline{\mathbb{k}\psi_1 F_1}} (X \otimes_{\mathbb{k}} \overline{\mathbb{k}\psi_1 F_1})$$

for any $X \in \text{Rep}(\mathbb{k}G)$, hence

$$\mathcal{F}(X \overline{\otimes} V) = \oplus_{f \in F_1} V_{sf} \otimes_{\mathbb{k}\psi_1 F_1} (X \otimes_{\mathbb{k}} \mathbb{k}\psi_1 F_1)$$

as vector spaces. Define $c_{X,V} : \mathcal{F}(X \overline{\otimes} V) \rightarrow X \otimes_{\mathbb{k}} \mathcal{F}(V)$ by

$$c_{X,V}(v \otimes x \otimes f) = f \cdot x \otimes v \cdot f,$$

for any $x \in X, v \in V, f \in F_1$. It follows from a straightforward computation that the map $c_{X,V}$ is well-defined and (3.5), (3.6) are satisfied. Now, define $\mathcal{G} : \bigoplus_{s \in S} {}^{\mathbb{k}\psi_s F_s} \mathcal{M} \rightarrow {}^{\mathbb{k}G}_{\mathbb{k}\psi_2 F_2} \mathcal{M}_{\overline{\mathbb{k}\psi_1 F_1}}$ as follows. If $W \in {}^{\mathbb{k}\psi_s F_s} \mathcal{M}$ for some $s \in S$ then

$$\mathcal{G}(W) = (\mathbb{k}F_1 \otimes_{\mathbb{k}} \mathbb{k}F_2) \otimes_{\mathbb{k}\psi_s F_s} W.$$

The right action of $\mathbb{k}\psi_s F_s$ on the tensor product $\mathbb{k}F_1 \otimes_{\mathbb{k}} \mathbb{k}F_2$ is

$$(x \otimes y) \cdot f = \psi_1(x^{-1}, s f s^{-1}) \psi_2(y, f) s^{-1} f^{-1} s x \otimes y f,$$

for all $x \in F_1, y \in F_2, f \in F_s$.

For any $x, f \in F_1, y, g \in F_2, w \in W$ define

$$g \cdot (x \otimes y \otimes w) = \psi_2(g, y) (x \otimes g y \otimes w),$$

$$(x \otimes y \otimes w) \cdot f = \psi_1(f, x^{-1}) (x f^{-1} \otimes y \otimes w),$$

$$\delta(x \otimes y \otimes w) = y s x \otimes (x \otimes y \otimes w).$$

Equipped with these maps the object $\mathcal{G}(W)$ is an object in the category ${}^{\mathbb{k}G}_{\mathbb{k}\psi_1 F_2} \mathcal{M}_{\overline{\mathbb{k}\psi_2 F_1}}$. □

6 Applications for computing the Brauer-Picard group

6.1 The Brauer-Picard group of a tensor category

Let $\mathcal{C}_1, \mathcal{C}_2$ be finite tensor categories. The following definitions were given in [7].

Definition 6.1

(a) An exact $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule category \mathcal{M} is *invertible* if there are bimodule equivalences

$$\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}_1} \mathcal{M} \simeq \mathcal{C}_2, \quad \mathcal{M} \boxtimes_{\mathcal{C}_2} \mathcal{M}^{\text{op}} \simeq \mathcal{C}_1.$$

(b) The Brauer-Picard groupoid $\underline{\text{BrPic}}$ is the 3-groupoid whose objects are finite tensor categories, 1-morphisms from \mathcal{C}_1 to \mathcal{C}_2 are invertible $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule categories, 2-morphisms are equivalences of such bimodule categories, and 3-morphisms are isomorphisms of such equivalences. Forgetting the 3-morphisms and the 2-morphisms and identifying 1-morphisms one obtains the groupoid BrPic . The group $\text{BrPic}(\mathcal{C})$ of auto-morphisms of \mathcal{C} in BrPic is called the *Brauer-Picard group of \mathcal{C}* .

6.2 Invertible module categories over a braided tensor category

Let \mathcal{C} be a braided tensor category. Any left \mathcal{C} -module category is a \mathcal{C} -bimodule category using the reverse action as explained in Sect. 5.1.

Definition 6.2 We shall say that an exact \mathcal{C} -module category \mathcal{M} is *invertible* if there is a bimodule equivalence

$$\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{C}.$$

The group of invertible \mathcal{C} -module categories will be denoted by $\text{InvMod}(\mathcal{C})$.

Proposition 6.3 *Let \mathcal{C} be a tensor category. There is an isomorphism of groups $\text{BrPic}(\mathcal{C}) \simeq \text{InvMod}(\mathcal{Z}(\mathcal{C}))$.*

Proof Denote by $\mathcal{Z} : \text{Bimod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C})$ the center functor. As a consequence of [9, Thm. 7.13, Lemma 7.14] and Proposition 5.2 this functor restricts to an isomorphism of groups. \square

6.3 Invertible $\text{Rep}(H)$ -bimodule categories

In this section we study the tensor product of invertible module categories over the representation categories of Hopf algebras using the tools developed in the previous sections.

Let H be a finite-dimensional Hopf algebra. Recall that if \mathcal{M} is a $\text{Rep}(H)$ -bimodule category, then there exists a left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebra K , right $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -simple with trivial coinvariants such that $\mathcal{M} \simeq {}_K \mathcal{M}$ as $\text{Rep}(H)$ -bimodule categories.

Theorem 6.4 *Let K, S be left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebras right $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -simple with trivial coinvariants. Assume also that*

- (i) $S \otimes_{\mathbb{k}} K$ is free as a left $S \square_H K$ -module,
- (ii) the module category ${}_{S \square_H K} \mathcal{M}$ is exact, and
- (iii) ${}_S \mathcal{M}, {}_K \mathcal{M}$ are invertible $\text{Rep}(H)$ -bimodule categories.

Then, there is an equivalence of $\text{Rep}(H)$ -bimodule categories

$${}_S \mathcal{M} \boxtimes_{\text{Rep}(H)} {}_K \mathcal{M} \simeq {}_{S \square_H K} \mathcal{M}. \tag{6.1}$$

Proof By Corollary 4.4 there exists an equivalence of $\text{Rep}(H)$ -bimodule categories

$${}_S \mathcal{M} \boxtimes_{\text{Rep}(H)} {}_K \mathcal{M} \simeq \mathcal{M}(H, H, K, \overline{S}).$$

Since invertible bimodule categories are indecomposable, then the category $\mathcal{M}(H, H, K, \overline{S})$ is an indecomposable bimodule category. Consider the functor $\mathcal{F} : {}_{S \square_H K} \mathcal{M} \rightarrow \mathcal{M}(H, H, K, \overline{S})$ explained in (4.4). Since $S \otimes_{\mathbb{k}} K$ is free as a left $S \square_H K$ -module then \mathcal{F} is full and faithful. The full subcategory of $\mathcal{M}(H, H, K, \overline{S})$ consisting of objects $\mathcal{F}(N)$ where $N \in {}_{S \square_H K} \mathcal{M}$ is an exact submodule category and since $\mathcal{M}(H, H, K, \overline{S})$ is indecomposable, \mathcal{F} must be an equivalence, see [11, p. 91]. \square

The left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebra $\text{diag}(H)$ can be thought as a coideal subalgebra in $H \otimes_{\mathbb{k}} H^{\text{cop}}$. The map $\iota : \text{diag}(H) \rightarrow H \otimes_{\mathbb{k}} H^{\text{cop}}$ given by $\iota(h) = h_{(-1)} \otimes h_{(2)}$ is an injective comodule algebra map. Let Q be the coalgebra quotient $(H \otimes_{\mathbb{k}} H^{\text{cop}}) / (H \otimes_{\mathbb{k}} H^{\text{cop}}) \text{diag}(H)^+$.

Corollary 6.5 *Let K, S be left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebras right $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -simple with trivial coinvariants such that conditions (i) and (ii) of Theorem 6.4 are fulfilled and ${}_S \mathcal{M} \boxtimes_{\text{Rep}(H)} {}_K \mathcal{M} \simeq \text{Rep}(H)$. Then, there is an isomorphism of $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebras*

$$S \square_H K \simeq \text{End}_{\text{diag}(H)}(H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V), \tag{6.2}$$

for some $V \in \mathcal{Q} \mathcal{M}$. Moreover

$$(S \square_H K)^{\text{co } H \otimes_{\mathbb{k}} H^{\text{cop}}} = \text{End}^Q(V). \tag{6.3}$$

Proof By Theorem 6.4 the module categories ${}_S \square_H K \mathcal{M}, \text{diag}(H) \mathcal{M}$ are equivalent. It follows from [1, Lemma 1.26] that there exists an object $P \in {}^{H \otimes_{\mathbb{k}} H^{\text{cop}}} \mathcal{M}_{\text{diag}(H)}$ such that

$$S \square_H K \simeq \text{End}_{\text{diag}(H)}(P).$$

The left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule structure on $\text{End}_{\text{diag}(H)}(P)$ is given by $\lambda : \text{End}_{\text{diag}(H)}(P) \rightarrow H \otimes_{\mathbb{k}} H^{\text{cop}} \otimes_{\mathbb{k}} \text{End}_{\text{diag}(H)}(P)$, $\lambda(T) = T_{(-1)} \otimes T_{(0)}$ where

$$\langle \alpha, T_{(-1)} \rangle T_0(p) = \langle \alpha, T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)}, \tag{6.4}$$

for any $\alpha \in (H \otimes_{\mathbb{k}} H^{\text{cop}})^*$, $T \in \text{End}_{\text{diag}(H)}(P)$, $p \in P$.

There is an equivalence of categories ${}^{H \otimes_{\mathbb{k}} H^{\text{cop}}} \mathcal{M}_{\text{diag}(H)} \simeq \mathcal{Q} \mathcal{M}$. The functors $\Psi : {}^{H \otimes_{\mathbb{k}} H^{\text{cop}}} \mathcal{M}_{\text{diag}(H)} \rightarrow \mathcal{Q} \mathcal{M}$, $\Phi : \mathcal{Q} \mathcal{M} \rightarrow {}^{H \otimes_{\mathbb{k}} H^{\text{cop}}} \mathcal{M}_{\text{diag}(H)}$ defined by

$$\Psi(M) = M / (H \otimes_{\mathbb{k}} H^{\text{cop}}) \text{diag}(H)^+, \quad \Phi(V) = (H \otimes_{\mathbb{k}} H^{\text{cop}}) \square_Q V,$$

$M \in {}^{H \otimes_{\mathbb{k}} H^{\text{cop}}} \mathcal{M}_{\text{diag}(H)}$, $V \in \mathcal{Q} \mathcal{M}$ give an equivalence of categories. The left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule structure on $(H \otimes_{\mathbb{k}} H^{\text{cop}}) \square_Q V$,

$$\delta : (H \otimes_{\mathbb{k}} H^{\text{cop}}) \square_Q V \rightarrow H \otimes_{\mathbb{k}} H^{\text{cop}} \otimes_{\mathbb{k}} (H \otimes_{\mathbb{k}} H^{\text{cop}}) \square_Q V$$

and the right $\text{diag}(H)$ -action are given by

$$\delta(h \otimes t \otimes v) = h_{(1)} \otimes t_{(2)} \otimes h_{(1)} \otimes t_{(1)} \otimes v, \quad (h \otimes t \otimes v) \cdot x = h x_{(1)} \otimes t x_{(2)} \otimes v,$$

for all $x \in H$, $h \otimes t \otimes v \in (H \otimes_{\mathbb{k}} H^{\text{cop}}) \square_Q V$. This proves isomorphism (6.2). Isomorphism (6.3) follows from $\text{End}_{\text{diag}(H)}(P)^{\text{co } H} = \text{End}_{\text{diag}(H)}^H(P)$. □

Corollary 6.6 *Assume H is pointed. Let K, S be left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule algebras as in Corollary 6.5. Assume also that*

$$(S \square_H K)_0 = S_0 \square_{H_0} K_0. \tag{6.5}$$

Then ${}_{S_0} \mathcal{M}, {}_{K_0} \mathcal{M}$ are invertible $\text{Rep}(H_0)$ -bimodule categories.

Proof By Corollary 6.5 there exists an object $V \in \mathcal{Q} \mathcal{M}$ such that

$$S \square_H K \simeq \text{End}_{\text{diag}(H)}(H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V) \simeq \text{Hom}_{\mathbb{k}}(V, H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V).$$

Let us explain the second isomorphism. The space $\text{Hom}_{\mathbb{k}}(V, H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V)$ is a left $H \otimes_{\mathbb{k}} H^{\text{cop}}$ -comodule via $T \mapsto T_{(-1)} \otimes T_{(0)}$ such that for all $\alpha \in (H \otimes_{\mathbb{k}} H^{\text{cop}})^*$

$$\langle \alpha, T_{(-1)} \rangle T_0(v) = \langle \alpha, T(v_{(0)})_{(-1)} \mathcal{S}^{-1}(v_{(-1)}) \rangle T(v_{(0)})_{(0)},$$

for all $v \in V$. Recall that we are identifying $\text{diag}(H)$ with the coideal subalgebra $\iota(\text{diag}(H)) \subseteq H \otimes_{\mathbb{k}} H^{\text{cop}}$. There is an isomorphism $H \otimes_{\mathbb{k}} H^{\text{cop}} \simeq Q \otimes_{\mathbb{k}} \text{diag}(H)$ of right $\text{diag}(H)$ -modules and right Q -comodules [12, Thm. 6.1].

Define $\phi : \text{End}_{\text{diag}(H)}(H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V) \rightarrow \text{Hom}_{\mathbb{k}}(V, H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V)$, $\psi : \text{Hom}_{\mathbb{k}}(V, H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V) \rightarrow \text{End}_{\text{diag}(H)}(H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V)$ by

$$\phi(T)(v) = T(1 \otimes v), \quad \psi(U)(h \otimes v) = (h \otimes 1) \cdot U(v),$$

for all $v \in V, h \in \text{diag}(H)$. One can readily prove that ϕ and ψ are one the inverse of each other and they are comodule morphisms. Thus, there are isomorphisms

$$(S \square_H K)_0 \simeq \text{Hom}_{\mathbb{k}}(V, H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V)_0 \simeq \text{Hom}_{\mathbb{k}}(V_0, \tilde{P}),$$

where $\tilde{P} = \{\sum h \otimes v \in H \otimes_{\mathbb{k}} H^{\text{cop}} \square_Q V : h \in H_0 \otimes_{\mathbb{k}} H_0\}$. Since there is an isomorphism $\tilde{P} \simeq \text{diag}(H_0) \otimes_{\mathbb{k}} V_0$ then

$$\text{Hom}_{\mathbb{k}}(V_0, \tilde{P}) \simeq \text{End}_{\text{diag}(H_0)}(\tilde{P}),$$

which implies that the bimodule categories $_{(S \square_H K)_0} \mathcal{M}, \text{diag}(H_0) \mathcal{M}$ are equivalent. By hypothesis (6.5) the bimodule categories $_{S_0 \square_{H_0} K_0} \mathcal{M}, \text{diag}(H_0) \mathcal{M}$ are equivalent. Using Theorem 6.4 we get that both categories $_{S_0} \mathcal{M}, \text{diag}(H_0) \mathcal{M}$ are invertible $\text{Rep}(H_0)$ -bimodule categories. \square

Let H be a pointed Hopf algebra such that the coradical is the group algebra of an Abelian group G . Corollary 6.6 tells us that to find invertible $\text{Rep}(H)$ -bimodule categories we have to look at those comodule algebras K such that $K_0 = \mathbb{k}_{\psi} F$ where $F \subseteq G$ is a subgroup, $\psi \in Z^2(F, \mathbb{k}^{\times})$ is a 2-cocycle such that the Morita class of the pair (F, ψ) belongs to the Brauer-Picard group of $\text{Rep}(\mathbb{k}G)$ that has been computed in [7].

Remark 6.7 In general there is an inclusion $(S \square_H K)_0 \supseteq S_0 \square_{H_0} K_0$. Equality is not true for arbitrary comodule algebras, however (6.5) seems to be fulfilled in many examples of comodule algebras over pointed Hopf algebras such that the bimodule categories are invertible.

6.4 The Brauer-Picard group of $\text{Rep}(G)$

In this section we compare the product of the Brauer-Picard group of the category of representations of a finite Abelian group G obtained in [7] and the product (6.1).

Let G be a finite Abelian group. The group $O(G \oplus \widehat{G})$ consists of group isomorphisms $\alpha : G \oplus \widehat{G} \rightarrow G \oplus \widehat{G}$ such that $\langle \alpha_2(g, \chi), \alpha_1(g, \chi) \rangle = \langle \chi, g \rangle$ for all $g \in G, \chi \in \widehat{G}$. Here $\alpha(g, \chi) = (\alpha_1(g, \chi), \alpha_2(g, \chi))$.

Theorem 6.8 [7, Corollary 1.2] *There is an isomorphism of groups*

$$\text{BrPic}(\text{Rep}(G)) \simeq O(G \oplus \widehat{G}).$$

Let $\alpha \in O(G \oplus \widehat{G})$ and define $U_{\alpha} \subseteq G \times G$ the subgroup

$$L_{\alpha} = \{(\alpha_1(g, \chi), g) : g \in G, \chi \in \widehat{G}\}.$$

and the 2-cocycle $\psi_{\alpha} : L_{\alpha} \times L_{\alpha} \rightarrow \mathbb{k}^{\times}$ defined by

$$\psi_{\alpha}((\alpha_1(g, \chi), g), (\alpha_1(h, \xi), h)) = \langle \alpha_2(g, \chi)^{-1}, \alpha_1(h, \xi) \rangle \langle \chi, h \rangle.$$

It was proved in [7] that the bimodule categories $_{\mathbb{k}_{\psi_{\alpha}} L_{\alpha}} \mathcal{M}$ are invertible.

Proposition 6.9 *There is an equivalence of $\text{Rep}(\mathbb{k}G)$ -bimodule categories*

$${}_{\mathbb{k}\psi_\alpha L_\alpha} \square_{\mathbb{k}G} {}_{\mathbb{k}\psi_\beta L_\beta} \mathcal{M} \simeq {}_{\mathbb{k}\psi_\alpha L_\alpha} \mathcal{M}.$$

Proof It follows directly from Theorem 6.4. □

Remark 6.10 The product in $\text{BrPic}(\text{Rep}(G))$ for a non-Abelian group G remains as an open problem. As pointed out by the referee to describe the elements and the product in $\text{BrPic}(\text{Rep}(G))$ one might have to use the description given in [3, Corollary 3.6.3].

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References

1. Andruskiewitsch, N., Mombelli, M.: On module categories over finite-dimensional Hopf algebras. *J. Algebra* **314**, 383–418 (2007)
2. Caenepeel, S., Crivei, S., Marcus, A., Takeuchi, M.: Morita equivalences induced by bimodules over Hopf-Galois extensions. *J. Algebra* **314**, 267–302 (2007)
3. Davydov, A.: Modular invariants for group-theoretical categories I. *J. Algebra* **323**, 1321–1348 (2010)
4. Deligne, P.: Catégories tannakiennes. In: *The Grothendieck Festschrift, Vol. II. Progr. Math.*, vol. 87, pp. 111–195. Birkhäuser, Boston (1990)
5. Doi, Y.: Unifying Hopf modules. *J. Algebra* **153**, 373–385 (1992)
6. Etingof, P., Ostrik, V.: Finite tensor categories. *Mosc. Math. J.* **4**(3), 627–654 (2004)
7. Etingof, P., Nikshych, D., Ostrik, V.: Fusion categories and homotopy theory. *Quantum Topol.* **1**(3), 209–273 (2010)
8. Gelaki, S., Naidu, D., Nikshych, D.: Centers of graded fusion categories. *Algebra Number Theory* **3**(8), 959–990 (2009)
9. Greenough, J.: Monoidal 2-structure of bimodule categories. *J. Algebra* **324**, 1818–1859 (2010)
10. Kitaev, A., Kong, L.: Models for gapped boundaries and domain walls. [arXiv:1106.3276](https://arxiv.org/abs/1106.3276)
11. MacLane, S.: *Categories for the Working Mathematician*. Graduate Texts in Mathematics, vol. 5. Springer, Berlin (1971). 2nd ed., 1998
12. Skryabin, S.: Projectivity and freeness over comodule algebras. *Trans. Am. Math. Soc.* **359**(6), 2597–2623 (2007)