The Algebra – Geometry Dialectic: An Aesthetic Perspective

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“The logician cuts up, so to speak, each demonstration into a very great number of elementary operations ; when we have examined these operations one after the other and ascertained that each is correct, are we to think we have grasped the real meaning of the demonstration ? Shall we have understood it even when, by an effort of memory, we have become able to repeat this proof by reproducing all these elementary operations in just the order in which the inventor had arranged them. Evidently not; we shall not yet possess the entire reality; *that I know not what* which makes the unity of the demonstration will completely elude us.” (Poincaré, The Value of Science 21-22)

Mathematical logic allows us to account for the validity of reasoning but it does not allow to understand it, nor to create others.

Assumption: It is beauty and, more broadly, an aesthetic sensibility, which would allow us to finally understand mathematics completely and which, to repeat an expression of Poincaré, *would allow one to be a real inventor.*

Strategy

The thesis that aesthetics is the complement of mathematical logic must be argued in a functional and not evaluative perspective.

It is not a question of determining, in a quantitative way, the aesthetics of a given argument but rather of verifying whether its operational modalities include aesthetic features.

Selected tool: the theory of the aesthetics of Nelson Goodman (Languages of Art, Oxford University Press, 1968).

This theory is based on the functional analysis of symbol systems.

The analysis of syntactical and semantic properties of symbol systems allows us to supply a characterization of their aesthetic functioning, based on a symptomatology. If a work works as work of art, it is because it fills some syntactical and semantic requirements that allow it to function aesthetically.

Symptoms of the Aesthetic

- 2 syntactic requirements which are syntactic density and relative syntactic repleteness,
- 1 semantic requirement called semantic density and finally,
- 2 requirements about the reference : exemplification which is a mode of reference, and multiple complex reference, which suggests instead an itinerary of reference.

Vocabulary

- **Symbol**

Symbol is use here as a very general and colorless term. It covers letters, words, texts, pictures, diagrams, maps, models, and more, but carries no implication of the oblique or the occult. (LA xi)

- **Symbol scheme**

A symbol scheme consists of characters with modes of combining them to form others. (LA 131)

- **Symbol system**

A symbol system consists of a symbol scheme correlated with a field of reference. (LA 143)

(In a symbol system, the sense determines the validity of the characters.)

Syntactic density

Definition : a scheme is syntactically dense if it provides for infinitely many characters so ordered that between each two there is the third. (LA 132)

Relative repleteness

The relative repleteness of a system is thus the property that characterizes dense systems in which no, or few, characters can be modified without modifying the functioning of the system. In this sense, repleteness is a criterion which distinguishes syntactically dense systems that depict (the most subtle nuance of a symbol is taken into account in the interpretation and in the understanding) from those that describe (only some aspects have an constitutive and informative value).

“Compare a momentary electrocardiogram with a Hokusai drawing of Mt; Fujiyama. The black wiggly lines on white backgrounds may be exactly the same in the two cases. Yet the one is a diagram and the other a picture. What makes the difference? Obviously, some feature of the different schemes in which the two marks function as symbols. (…) But, since both schemes are dense (and assumed disjoint), what feature? The answer does not lie in what is symbolized, mountains can be diagrammed and heartbeats pictured. The difference is syntactic: the constitutive aspects of the diagrammatic as compared with the pictorial character are expressly and narrowly restricted. The only relevant features of the diagram are the ordinate and abscissa of each of the points the center of the line passes through.

The thickness of the line, its color and intensity, the absolute size of the diagram, etc., do not matter ; whether a purported duplicate of the symbol belongs to the same character of the diagrammatic scheme depends not at all upon such features. For the sketch, this is note true. Any thickening or thinning of the line, its color, its contrast whit background, its size, even the qualities of the paper – none of these is ruled out, none can be ignored. Though the pictorial and diagrammatic schemes are alike in not being articulate, some features that are constitutive in the pictorial scheme are dismissed as contingent in the diagrammatic scheme; the symbols in the pictorial scheme are relatively replete. (LA 229 - 230)

One of the differences between mathematics, as a symbol system, and an arbitrary linguistic system consists in the fact that for mathematics, the rules of combination of the symbols between them tolerate more latitude than in the linguistic systems. In linguistic systems, the rule which prevails is linear concatenation. In the case of mathematics, we combine the symbols with each other according to a variety of laws, or using operators. Operators have properties which allow several different methods of writings the same object.

Exemplification

Exemplification is possession plus reference

To have without symbolizing is merely to possess, while to symbolize without having is to refer in some other way than by exemplifying .

Semantic density

A system is semantically dense if it provides for an infinite number of characters with compliance-classes so ordered that between each two there is a third.

(Compliance-classes : it is the set of labels which with a character concords.)

The semantic density characterizes systems in which the latitude of interpretation is very broad.

It is the semantic density of mathematics, in other words, the richness of its symbolic fabric, which allows us to connect extremely different objects to each other.

Linguistic representation – Pictorial (ect.) representation

In a general way, we speak about a mathematical “figure” every time we propose a representation of a problem or of a set of data which is not purely formal, that is to say not written according to the usual logical symbolism.

This generic term, “figure”, is not satisfactory: not all mathematical "figures" have the same symbolic function nor do they have the same role within an argument. Thus, certain figures have a strictly illustrative role, while others have a demonstrative value, or still others are of use to support intuition.

The term “image” is reserved for figures which require a taking into account of syntactic density and the repleteness of the system in order to function properly. When these properties are contingent for correct functioning, then we use the term diagram.

Example of functioning as an image : Proof whitout words

The correct functioning of a *proof without words* requires taking into account the syntactic density and the repleteness of the symbol system.

Example of functioning in diagram form : representative curves of functions

The correct interpretation of a curve cannot be based on properties involving the syntactic density of the figure.

Syntactic density can be identified as the reason why certain figures cannot function as demonstrations.

In such cases, it is necessary to substitute a formal and linguistic representation for the figure (diagram).

This seesaw between geometry and algebra, to speak schematically, is based on the symptom of density: we replace a syntactically dense system by a semantically dense system.

“The infinity of the points of a curve is then made governable by the interplay of the finite number of terms of the equation” (G. G. Granger, Essai d’une philosophie du style, 54)

The symbolic system built on the pictorial symbol scheme (the graphic representation of the function) and the symbol system built on the formal symbol scheme (the equation or the formal expression of the function) are two intensions of the same extension.

The fecondity of the reasoning finds its root in this case in the dialectic between these two intensions (the geometric one versus the alebraic one).

Density is a (syntactic) obstacle to the functioning of the figure as demonstration, it becomes an advantage (semantic) in connection with the algebraic treatment of a problem.

In a Goodman-type approach, this seesaw between geometry and algebra, corresponds to the replacement of a system that does not function aesthetically by a system that does.

Must we see, in the semiological analysis of the dialectic between algebra and geometry, an element of proof of the cognitive efficacy of aesthetics so often claimed by mathematicians?

Does symptomatology allow us to restore rigor to the use the figure in mathematical argument, in a way that logic cannot take into account?