INVERTIBLE MODULE CATEGORIES OVER THE REPRESENTATION CATEGORY OF THE TAFT ALGEBRA

Bojana Femić
(joint with M. Mombelli)

Facultad de Ciencias
Universidad de la República
Montevideo (Uruguay)

March 25, 2013
Tucumán (Argentina)

Lecture Outline

The setting and motivation

BrPic($\mathcal{C}$) for a finite tensor category $\mathcal{C}$(=Rep(H))

- Embedding BiGal(H, H) into BrPic(Rep(H))

Exact indecomposable Rep(T_q)-bimodule categories

Motivation

- *Extensions of tensor categories by a finite group*
  - tensor subcategory $\mathcal{C} = \mathcal{C}_e \subseteq \bigoplus_{g\in G} \mathcal{C}_g$ - G-extension of $\mathcal{C}$
  - in order to classify G-extensions of $\mathcal{C}$ one needs:
    1. a group map $g \mapsto [\mathcal{C}_g], \quad G \to \mathrm{BrPic}(\mathcal{C})$
       $\mathcal{C}_g$ is an exact invertible $\mathcal{C}$-bimodule category;
    2. a 3-cocycle and a 4-cocycle over G.

- *Mathematical Physics*

Motivation

- *Extensions of tensor categories by a finite group*
  - tensor subcategory $\mathcal{C} = \mathcal{C}_e \subseteq \bigoplus_{g\in G} \mathcal{C}_g$ - G-extension of $\mathcal{C}$
  - in order to classify G-extensions of $\mathcal{C}$ one needs:
    1. a group map $g \mapsto [\mathcal{C}_g], \quad G \to \mathrm{BrPic}(\mathcal{C})$
       $\mathcal{C}_g$ is an exact invertible $\mathcal{C}$-bimodule category;
    2. a 3-cocycle and a 4-cocycle over G.
  - in order to find extensions of $\mathcal{C} = \mathrm{Rep}(H)$ one needs to compute $\mathrm{BrPic}(\mathrm{Rep}(H))$.
- *Mathematical Physics*

Motivation

- *Extensions of tensor categories by a finite group*
  - tensor subcategory $\mathcal{C} = \mathcal{C}_e \subseteq \bigoplus_{g \in G} \mathcal{C}_g$ - G-extension of $\mathcal{C}$
  - in order to classify G-extensions of $\mathcal{C}$ one needs:
    1. a group map $g \mapsto [\mathcal{C}_g]$, $\quad G \to \mathrm{BrPic}(\mathcal{C})$
       $\mathcal{C}_g$ is an exact invertible $\mathcal{C}$-bimodule category;
    2. a 3-cocycle and a 4-cocycle over G.
  - in order to find extensions of $\mathcal{C} = \mathrm{Rep}(H)$ one needs to compute $\mathrm{BrPic}(\mathrm{Rep}(H))$.
- *Mathematical Physics*
  - the Brauer-Picard group is related to 3-dim. Topological Field Theory, see [J. Fuchs, Ch. Schweigert, A. Valentino, *Bicategories for boundary conditions and for surface defects in 3-d TFT*, **http://arxiv.org/pdf/1203.4568.pdf**]

BrPic($\mathcal{C}$) for a finite tensor category $\mathcal{C}$(=Rep(H))

The necessary ingredients

- finite tensor category $=$ k-linear abelian rigid monoidal cat. equivalent to a cat. of finite-dim. representations of a finite-dim. k-algebra s.t. the unit object is simple

The necessary ingredients

- finite tensor category $=$ k-linear abelian rigid monoidal cat. equivalent to a cat. of finite-dim. representations of a finite-dim. k-algebra s.t. the unit object is simple
- $\mathcal{M}$ a right $\mathcal{C}$-module cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a left $\mathcal{C}$-module cat.

The necessary ingredients

- finite tensor category $=$ k-linear abelian rigid monoidal cat. equivalent to a cat. of finite-dim. representations of a finite-dim. k-algebra s.t. the unit object is simple
- $\mathcal{M}$ a right $\mathcal{C}$-module cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a left $\mathcal{C}$-module cat.
- $\mathcal{M}$ a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a $(\mathcal{D}, \mathcal{C})$-bimodule cat.

The necessary ingredients

- finite tensor category $=$ k-linear abelian rigid monoidal cat. equivalent to a cat. of finite-dim. representations of a finite-dim. k-algebra s.t. the unit object is simple
- $\mathcal{M}$ a right $\mathcal{C}$-module cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a left $\mathcal{C}$-module cat.
- $\mathcal{M}$ a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a $(\mathcal{D}, \mathcal{C})$-bimodule cat.
- a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\mathcal{M}$ is **invertible** $\leftrightarrow$ $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{op} \simeq \mathcal{C}$ and $\mathcal{M}^{op} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D}$ as bimodule cat.'s

The necessary ingredients

- finite tensor category = k-linear abelian rigid monoidal cat. equivalent to a cat. of finite-dim. representations of a finite-dim. k-algebra s.t. the unit object is simple
- $\mathcal{M}$ a right $\mathcal{C}$-module cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a left $\mathcal{C}$-module cat.
- $\mathcal{M}$ a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a $(\mathcal{D}, \mathcal{C})$-bimodule cat.
- a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\mathcal{M}$ is **invertible** $\leftrightarrow$ $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{op} \simeq \mathcal{C}$ and $\mathcal{M}^{op} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D}$ as bimodule cat.'s
- a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\mathcal{M}$ is **exact** = exact as a left $\mathcal{C} \boxtimes \mathcal{D}^{rev}$-module cat. = $\forall P \in \mathcal{C} \boxtimes \mathcal{D}^{rev} \quad \forall M \in \mathcal{M} \Rightarrow P \otimes M \in \mathcal{M}$ is projective

The necessary ingredients

- finite tensor category = k-linear abelian rigid monoidal cat. equivalent to a cat. of finite-dim. representations of a finite-dim. k-algebra s.t. the unit object is simple
- $\mathcal{M}$ a right $\mathcal{C}$-module cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a left $\mathcal{C}$-module cat.
- $\mathcal{M}$ a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\leftrightarrow$ $\mathcal{M}^{op}$ a $(\mathcal{D}, \mathcal{C})$-bimodule cat.
- a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\mathcal{M}$ is **invertible** $\leftrightarrow$ $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{op} \simeq \mathcal{C}$ and $\mathcal{M}^{op} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D}$ as bimodule cat.'s
- a $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\mathcal{M}$ is **exact** = exact as a left $\mathcal{C} \boxtimes \mathcal{D}^{rev}$-module cat. = $\forall P \in \mathcal{C} \boxtimes \mathcal{D}^{rev} \quad \forall M \in \mathcal{M} \Rightarrow P \otimes M \in \mathcal{M}$ is projective
- $\mathcal{C}$: a finite tensor cat. $\rightarrow$ BrPic($\mathcal{C}$) = the group of equiv. classes of invertible exact $\mathcal{C}$-bimodule cat.'s

How do invertible cat.'s over $\mathcal{C} = \text{Rep}(H)$ look like

Let K, L be fin.-dim. Hopf algebras

How do invertible cat.'s over $\mathcal{C} = \text{Rep}(H)$ look like

Let K, L be fin.-dim. Hopf algebras

- Rep(K)- Rep(L)-bimodule cat. = left Rep($K \otimes L^{cop}$)-module cat.

How do invertible cat.'s over $\mathcal{C} = \mathrm{Rep}(H)$ look like

Let K, L be fin.-dim. Hopf algebras

- Rep(K)- Rep(L)-bimodule cat. = left Rep($K \otimes L^{cop}$)-module cat.

- Any invertible $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\mathcal{M}$ is indecomposable $\leftrightarrow$ $\mathcal{M} \neq \mathcal{M}_1 \oplus \mathcal{M}_2$ for any non-triv. $(\mathcal{C}, \mathcal{D})$-bimodule cat.'s $\mathcal{M}_{1|2}$.

How do invertible cat.'s over $\mathcal{C} = \text{Rep}(H)$ look like

Let K, L be fin.-dim. Hopf algebras

- Rep(K)- Rep(L)-bimodule cat. = left Rep($K \otimes L^{cop}$)-module cat.

- Any invertible $(\mathcal{C}, \mathcal{D})$-bimodule cat. $\mathcal{M}$ is indecomposable $\leftrightarrow$ $\mathcal{M} \neq \mathcal{M}_1 \oplus \mathcal{M}_2$ for any non-triv. $(\mathcal{C}, \mathcal{D})$-bimodule cat.'s $\mathcal{M}_{1|2}$.

Theorem. [Andrusk.-Mombelli, 2007]

Any exact indecomposable left Rep($K \otimes L^{cop}$)-module cat. is equivalent to ${}_A\mathcal{M}^f$ - the cat. of fin.-dim. A-modules, where A is a fin.-dim. right $K \otimes L^{cop}$-simple, left $K \otimes L^{cop}$-comodule algebra with trivial coinvariants.

Some candidates for invertible Rep(H)-categories

Let A be a left **coideal subalgebra** of a fin.dim. Hopf algebra H. Then:

Some candidates for invertible Rep(H)-categories

Let A be a left **coideal subalgebra** of a fin.dim. Hopf algebra H. Then:

- [Skryabin, 2007] A is H-**simple**.

Some candidates for invertible Rep(H)-categories

Let A be a left **coideal subalgebra** of a fin.dim. Hopf algebra H. Then:

- [Skryabin, 2007] A is H-**simple**.
- [Mombelli, 2008] if σ is a 2-cocycle over H compatible with A $\Rightarrow {}_{A^\sigma}\mathcal{M}$ is an **exact indecomposable** module cat. over Rep(H).

Some candidates for invertible Rep(H)-categories

Let A be a left **coideal subalgebra** of a fin.dim. Hopf algebra H. Then:

- [Skryabin, 2007] A is H-**simple**.
- [Mombelli, 2008] if σ is a 2-cocycle over H compatible with A $\Rightarrow$ ${}_{A^\sigma}\mathcal{M}$ is an **exact indecomposable** module cat. over Rep(H).

$\Rightarrow$

• we are interested in finding all **coideal subalgebras** of $H \otimes H^{cop}$

Some candidates for invertible Rep(H)-categories

Let A be a left **coideal subalgebra** of a fin.dim. Hopf algebra H. Then:

- [Skryabin, 2007] A is H-**simple**.
- [Mombelli, 2008] if σ is a 2-cocycle over H compatible with A $\Rightarrow {}_{A^\sigma}\mathcal{M}$ is an **exact indecomposable** module cat. over Rep(H).

$\Rightarrow$

• we are interested in finding all **coideal subalgebras** of $H \otimes H^{cop}$

• and their **2-cocycle twists**.

Product of module categories

Let A be a right $L \otimes K^{cop}$-simple, left $L \otimes K^{cop}$-comodule algebra and B a right $K \otimes L^{cop}$-simple, left $K \otimes L^{cop}$-comodule algebra.

Product of module categories

Let A be a right $L \otimes K^{cop}$-simple, left $L \otimes K^{cop}$-comodule algebra and B a right $K \otimes L^{cop}$-simple, left $K \otimes L^{cop}$-comodule algebra.

Theorem. *"Product Theorem"*

If any of the following two conditions is fulfilled, then there is an equivalence of Rep(L)-bimodule categories:

$$_A\mathcal{M} \boxtimes_{\mathrm{Rep}(K)} {}_B\mathcal{M} \simeq {}_{A \square_K B}\mathcal{M}$$

Product of module categories

Let A be a right $L \otimes K^{cop}$-simple, left $L \otimes K^{cop}$-comodule algebra and B a right $K \otimes L^{cop}$-simple, left $K \otimes L^{cop}$-comodule algebra.

Theorem. *"Product Theorem"*

If any of the following two conditions is fulfilled, then there is an equivalence of Rep(L)-bimodule categories:

$$_A\mathcal{M} \boxtimes_{\mathrm{Rep}(K)} {}_B\mathcal{M} \simeq {}_{A\square_K B}\mathcal{M}$$

1. [Mombelli, 2008]
   - $A \otimes B$ is free as a left $A\square_K B$-module;
   - the module cat. $_{A\square_K B}\mathcal{M}$ is exact;
   - $_A\mathcal{M}$ is an invertible (L, K)- and $_B\mathcal{M}$ an invertible (K, L)-bimodule category.

Product of module categories

Let A be a right $L \otimes K^{cop}$-simple, left $L \otimes K^{cop}$-comodule algebra and B a right $K \otimes L^{cop}$-simple, left $K \otimes L^{cop}$-comodule algebra.

Theorem. *"Product Theorem"*

If any of the following two conditions is fulfilled, then there is an equivalence of Rep(L)-bimodule categories:

$$_A\mathcal{M} \boxtimes_{\mathrm{Rep}(K)} {}_B\mathcal{M} \simeq {}_{A\square_K B}\mathcal{M}$$

1. [Mombelli, 2008]
   - $A \otimes B$ is free as a left $A\square_K B$-module;
   - the module cat. $_{A\square_K B}\mathcal{M}$ is exact;
   - $_A\mathcal{M}$ is an invertible (L, K)- and $_B\mathcal{M}$ an invertible (K, L)-bimodule category.
2. [Femić, Mombelli]
   A is a Hopf-Galois extension, as a left L-comodule algebra.

Embedding BiGal(H, H) into BrPic(Rep(H))

Lemma.

If A be an (H, H)-biGalois object then $[{}_A\mathcal{M}] \in \mathrm{BrPic}(\mathrm{Rep}(H))$.

Embedding BiGal(H, H) into BrPic(Rep(H))

Lemma.

If A be an (H, H)-biGalois object then $[{}_A\mathcal{M}] \in \mathrm{BrPic}(\mathrm{Rep}(H))$.

Illustration: ${}_A\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{A^{op}}\mathcal{M} \simeq {}_{A\square_H A^{op}}\mathcal{M} \simeq {}_H\mathcal{M}$

Embedding BiGal(H, H) into BrPic(Rep(H))

Lemma.

If A be an (H, H)-biGalois object then $[{}_A\mathcal{M}] \in \mathrm{BrPic}(\mathrm{Rep}(H))$.

Illustration: $\quad {}_A\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{A^{op}}\mathcal{M} \simeq {}_{A\square_H A^{op}}\mathcal{M} \simeq {}_H\mathcal{M}$

Proposition.

There is a group embedding $\mathrm{BiGal}(H, H) \hookrightarrow \mathrm{BrPic}(\mathrm{Rep}(H))$ given by $[A] \mapsto [{}_A\mathcal{M}]$.

Embedding BiGal(H, H) into BrPic(Rep(H))

Lemma.

If A be an (H, H)-biGalois object then $[{}_A\mathcal{M}] \in \mathrm{BrPic}(\mathrm{Rep}(H))$.

Illustration: ${}_A\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{A^{op}}\mathcal{M} \simeq {}_{A\square_H A^{op}}\mathcal{M} \simeq {}_H\mathcal{M}$

Proposition.

There is a group embedding $\mathrm{BiGal}(H, H) \hookrightarrow \mathrm{BrPic}(\mathrm{Rep}(H))$ given by $[A] \mapsto [{}_A\mathcal{M}]$.

Used: $\Psi : \mathrm{BrPic}(\mathrm{Comod}(H)) \to \mathrm{BrPic}(\mathrm{Rep}(H^{op}))$

$$[\mathcal{N}] \mapsto [Vec^{op} \boxtimes_{\mathrm{Comod}(H)} \mathcal{N} \boxtimes_{\mathrm{Comod}(H)} Vec]$$

$$\Psi([\mathrm{Comod}(H)^{A\square_H -}]) = [\mathcal{M}_A].$$

Exact indecomposable Rep(T_q)-bimodule categories

Left coideal subalgebras of $T_q \otimes T_{q^{-1}}$

It is $T_q \otimes T_q^{cop} \cong T_q \otimes T_{q^{-1}}$. Set $H = T_q \otimes T_{q^{-1}}$.

Left coideal subalgebras of $T_q \otimes T_{q^{-1}}$

It is $T_q \otimes T_q^{cop} \cong T_q \otimes T_{q^{-1}}$. Set $H = T_q \otimes T_{q^{-1}}$.

- We classified all **homogeneous** left coideal subalgebras of H

Left coideal subalgebras of $T_q \otimes T_{q^{-1}}$

It is $T_q \otimes T_q^{cop} \cong T_q \otimes T_{q^{-1}}$. Set $H = T_q \otimes T_{q^{-1}}$.

- We classified all **homogeneous** left coideal subalgebras of H
- We found their 2-cocycle twists

Left coideal subalgebras of $T_q \otimes T_{q^{-1}}$

It is $T_q \otimes T_q^{cop} \cong T_q \otimes T_{q^{-1}}$. Set $H = T_q \otimes T_{q^{-1}}$.

- We classified all **homogeneous** left coideal subalgebras of H
- We found their 2-cocycle twists
- We computed the **"liftings"** of the latter

Left coideal subalgebras of $T_q \otimes T_{q^{-1}}$

It is $T_q \otimes T_q^{cop} \cong T_q \otimes T_{q^{-1}}$. Set $H = T_q \otimes T_{q^{-1}}$.

- We classified all **homogeneous** left coideal subalgebras of H
- We found their 2-cocycle twists
- We computed the **"liftings"** of the latter
- All the latter 5 (families) are proved to be H-simple, left H-comodule algebras with trivial coinvariants

Left coideal subalgebras of $T_q \otimes T_{q^{-1}}$

It is $T_q \otimes T_q^{cop} \cong T_q \otimes T_{q^{-1}}$. Set $H = T_q \otimes T_{q^{-1}}$.

- We classified all **homogeneous** left coideal subalgebras of H
- We found their 2-cocycle twists
- We computed the **"liftings"** of the latter
- All the latter 5 (families) are proved to be H-simple, left H-comodule algebras with trivial coinvariants
- Any **exact indecomposable** Rep(T_q)-**bimodule category** turns out to be equivalent to ${}_A\mathcal{M}$, where A is one of the above 5 (families of) algebras.

Left coideal subalgebras of $T_q \otimes T_{q^{-1}}$

It is $T_q \otimes T_q^{cop} \cong T_q \otimes T_{q^{-1}}$. Set $H = T_q \otimes T_{q^{-1}}$.

- We classified all **homogeneous** left coideal subalgebras of H
- We found their 2-cocycle twists
- We computed the **"liftings"** of the latter
- All the latter 5 (families) are proved to be H-simple, left H-comodule algebras with trivial coinvariants
- Any **exact indecomposable** Rep(T_q)-**bimodule category** turns out to be equivalent to ${}_A\mathcal{M}$, where A is one of the above 5 (families of) algebras.

• It remains to see which of the above 5 (families of) bimodule categories are invertible.

Deciding which bimodule categories are invertible

1. The family ${}_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.

Deciding which bimodule categories are invertible

1. The family $_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.

$\Rightarrow {}_{L_1}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{L_2}\mathcal{M} \simeq {}_{L_1 \square_{T_q} L_2}\mathcal{M} \simeq {}_{L_3}\mathcal{M}$

and we computed $({}_L\mathcal{M})^{-1} \simeq {}_{L'}\mathcal{M}$.

Deciding which bimodule categories are invertible

1. The family ${}_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.
$\Rightarrow {}_{L_1}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{L_2}\mathcal{M} \simeq {}_{L_1 \square_{T_q} L_2}\mathcal{M} \simeq {}_{L_3}\mathcal{M}$
and we computed $({}_L\mathcal{M})^{-1} \simeq {}_{L'}\mathcal{M}$.
These parameters/invertible categories contribute to the subgroup $(k^\times \ltimes k^+) \times \mathbb{Z}_2$ in BrPic(Rep(T_q)).

Deciding which bimodule categories are invertible

1. The family ${}_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.

$\Rightarrow {}_{L_1}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{L_2}\mathcal{M} \simeq {}_{L_1 \square_{T_q} L_2}\mathcal{M} \simeq {}_{L_3}\mathcal{M}$

and we computed $({}_L\mathcal{M})^{-1} \simeq {}_{L'}\mathcal{M}$.

These parameters/invertible categories contribute to the subgroup $(k^\times \ltimes k^+) \times \mathbb{Z}_2$ in BrPic(Rep(T_q)).

2. The family ${}_{k_\psi G}\mathcal{M}$. We know that $({}_{k_\psi G}\mathcal{M})^{-1} = {}_{(k_\psi G)^{op}}\mathcal{M}$.

Deciding which bimodule categories are invertible

1. The family ${}_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.
$\Rightarrow {}_{L_1}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{L_2}\mathcal{M} \simeq {}_{L_1 \square_{T_q} L_2}\mathcal{M} \simeq {}_{L_3}\mathcal{M}$
and we computed $({}_L\mathcal{M})^{-1} \simeq {}_{L'}\mathcal{M}$.
These parameters/invertible categories contribute to the subgroup $(k^\times \ltimes k^+) \times \mathbb{Z}_2$ in BrPic(Rep(T_q)).

2. The family ${}_{k_\psi G}\mathcal{M}$. We know that $({}_{k_\psi G}\mathcal{M})^{-1} = {}_{(k_\psi G)^{op}}\mathcal{M}$.
Suppose that both categories are invertible.

Deciding which bimodule categories are invertible

1. The family ${}_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.
$\Rightarrow {}_{L_1}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{L_2}\mathcal{M} \simeq {}_{L_1 \square_{T_q} L_2}\mathcal{M} \simeq {}_{L_3}\mathcal{M}$
and we computed $({}_L\mathcal{M})^{-1} \simeq {}_{L'}\mathcal{M}$.
These parameters/invertible categories contribute to the subgroup $(k^\times \ltimes k^+) \times \mathbb{Z}_2$ in BrPic(Rep(T_q)).

2. The family ${}_{k_\psi G}\mathcal{M}$. We know that $({}_{k_\psi G}\mathcal{M})^{-1} = {}_{(k_\psi G)^{op}}\mathcal{M}$.
Suppose that both categories are invertible.
Then they fulfill the conditions of the *"Product Theorem"*.

Deciding which bimodule categories are invertible

1. The family ${}_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.
$\Rightarrow {}_{L_1}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{L_2}\mathcal{M} \simeq {}_{L_1 \square_{T_q} L_2}\mathcal{M} \simeq {}_{L_3}\mathcal{M}$
and we computed $({}_L\mathcal{M})^{-1} \simeq {}_{L'}\mathcal{M}$.
These parameters/invertible categories contribute to the subgroup $(k^\times \ltimes k^+) \times \mathbb{Z}_2$ in BrPic(Rep(T_q)).

2. The family ${}_{k_\psi G}\mathcal{M}$. We know that $({}_{k_\psi G}\mathcal{M})^{-1} = {}_{(k_\psi G)^{op}}\mathcal{M}$.
Suppose that both categories are invertible.
Then they fulfill the conditions of the *"Product Theorem"*.
Hence: ${}_{k_\psi G}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{(k_\psi G)^{op}}\mathcal{M} \simeq {}_{k_\psi G \square_{T_q} (k_\psi G)^{op}}\mathcal{M} \simeq {}_{T_q}\mathcal{M}$.

Deciding which bimodule categories are invertible

1. The family ${}_L\mathcal{M}$. For some parameters the algebras L are left Hopf-Galois extensions over T_q.
$\Rightarrow {}_{L_1}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{L_2}\mathcal{M} \simeq {}_{L_1 \Box_{T_q} L_2}\mathcal{M} \simeq {}_{L_3}\mathcal{M}$
and we computed $({}_L\mathcal{M})^{-1} \simeq {}_{L'}\mathcal{M}$.
These parameters/invertible categories contribute to the subgroup $(k^\times \ltimes k^+) \times \mathbb{Z}_2$ in BrPic(Rep(T_q)).

2. The family ${}_{k_\psi G}\mathcal{M}$. We know that $({}_{k_\psi G}\mathcal{M})^{-1} = {}_{(k_\psi G)^{op}}\mathcal{M}$.
Suppose that both categories are invertible.
Then they fulfill the conditions of the *"Product Theorem"*.
Hence: ${}_{k_\psi G}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{(k_\psi G)^{op}}\mathcal{M} \simeq {}_{k_\psi G \Box_{T_q} (k_\psi G)^{op}}\mathcal{M} \simeq {}_{T_q}\mathcal{M}$.
However, $k_\psi G \Box_{T_q} (k_\psi G)^{op}$ is semisimple and T_q is not. ↯

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.

Suppose that both categories are invertible.

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.

Suppose that both categories are invertible.

Then they fulfill the conditions of the *"Product Theorem"*.

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.

Suppose that both categories are invertible.

Then they fulfill the conditions of the *"Product Theorem"*.

Hence: ${}_{\mathcal{K}_{01}}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{\mathcal{K}_{10}}\mathcal{M} \simeq {}_{\mathcal{K}_{01}\square_{T_q}\mathcal{K}_{10}}\mathcal{M} \simeq {}_{T_q}\mathcal{M}$.

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.

Suppose that both categories are invertible.

Then they fulfill the conditions of the *"Product Theorem"*.

Hence: ${}_{\mathcal{K}_{01}}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{\mathcal{K}_{10}}\mathcal{M} \simeq {}_{\mathcal{K}_{01}\square_{T_q}\mathcal{K}_{10}}\mathcal{M} \simeq {}_{T_q}\mathcal{M}$. ↯

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.

Suppose that both categories are invertible.

Then they fulfill the conditions of the *"Product Theorem"*.

Hence: ${}_{\mathcal{K}_{01}}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{\mathcal{K}_{10}}\mathcal{M} \simeq {}_{\mathcal{K}_{01}\square_{T_q}\mathcal{K}_{10}}\mathcal{M} \simeq {}_{T_q}\mathcal{M}$. $\lightning$

5. The family ${}_{\mathcal{K}_{11}}\mathcal{M}$ - is still to be investigated.

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.
Suppose that both categories are invertible.
Then they fulfill the conditions of the *"Product Theorem"*.
Hence: ${}_{\mathcal{K}_{01}}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{\mathcal{K}_{10}}\mathcal{M} \simeq {}_{\mathcal{K}_{01}\square_{T_q}\mathcal{K}_{10}}\mathcal{M} \simeq {}_{T_q}\mathcal{M}$. ↯

5. The family ${}_{\mathcal{K}_{11}}\mathcal{M}$ - is still to be investigated.
There are candidates for some parameters, however it is difficult to compute the product of two (bi)module categories.

Deciding which bimodule categories are invertible

3./4. The families ${}_{\mathcal{K}_{01}}\mathcal{M}$ and ${}_{\mathcal{K}_{10}}\mathcal{M}$.

We know that $({}_{\mathcal{K}_{01}}\mathcal{M})^{-1} = {}_{\mathcal{K}_{10}}\mathcal{M}$.

Suppose that both categories are invertible.

Then they fulfill the conditions of the *"Product Theorem"*.

Hence: ${}_{\mathcal{K}_{01}}\mathcal{M} \boxtimes_{\mathrm{Rep}(T_q)} {}_{\mathcal{K}_{10}}\mathcal{M} \simeq {}_{\mathcal{K}_{01}\square_{T_q}\mathcal{K}_{10}}\mathcal{M} \simeq {}_{T_q}\mathcal{M}$. ↯

5. The family ${}_{\mathcal{K}_{11}}\mathcal{M}$ - is still to be investigated.

There are candidates for some parameters, however it is difficult to compute the product of two (bi)module categories.

THAT IS ALL...