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THE BEHAVIOR OF THE SOLUTIONS OF A TWO-PHASE STEFAN PROBLEM AND THE VALUE OF AN ENERGY INTEGRAL

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§. ABSTRACT

We consider a two-phase Stefan problem, analyzing the relation between the initial data and the possibility of continuing the solution for arbitrarily large time intervals.

§. INTRODUCTION

In [1] A. Fasano and M. Primicerio analyzed a class of free-boundary problems for the heat equation in one space dimension, releasing the sign restrictions on the data and the latent heat usually required in the Stefan problem.

In [2] they considered the following two problems for a one phase Stefan problem:

a) how are the data related to the possibility of continuing the solution over arbitrarily large time intervals?

b) does any solution exist when the datum prescribed on the free boundary $x = s(t)$ does not fit the initial datum at $x = s(0)$?

In this paper we are interested in problem a) for the two-phase Stefan problem.

In 1- we give the preliminaries corresponding to the description of the problem.

In 2- we consider the case of negative initial data.

In 3- the case of positive initial data is treated.

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Finally in 4- we analyze the non-classical case withinitial data of mixed signs.

1- TWO-PHASE STEFAN PROBLEM

Let us consider Problem P which consists of finding (T, s, U, V) such that:

i) $T > 0$.

ii) $s \in C([0,T))$, $s \in C^1((0,T))$; $0 < s(t) < 1$ for $0 < t < T$.

iii) $U(x,t)$ is a function, bounded in $0 \le x \le s(t)$, $0 \le t \le T$ and continuous on the same region, except perhaps at the points $(0,0)$ and $(s(0),0)$.

$U_x(x,t)$ is a continuous function in $0 \le x \le s(t)$, $0 < t < T$.

U_{xx}, U_t are continuous functions in $0 < x < s(t)$, $0 < t < T$.

Similar conditions are imposed on the function V .

iv) The following conditions are satisfied:

(1.1) $U_{xx} - U_t = 0$ in $D_T^U = \{(x,t): 0 < x < s(t),\ 0 < t < T\}$ (liquid-phase)

(1.2) $V_{xx} - V_t = 0$ in $D_T^V = \{(x,t): s(t) < x < 1,\ 0 < t < T\}$ (solid-phase)

(1.3) $s(0) = a$

(1.4) $U(x,0) = \phi(x)$, $0 < x < a$ (1.5) $V(x,0) = \psi(x)$, $a < x < 1$

(1.6) $U_x(0,t) = 0$, $0 < t < T$ (1.7) $V_x(1,t) = 0$, $0 < t < T$

(1.8) $U(s(t),t) = 0$, $0 < t < T$ (1.9) $V(s(t),t) = 0$, $0 < t < T$.

(1.10) $V_x(s(t),t) - U_x(s(t),t) = \dot{s}(t)$ $0 < t < T$

where $0 < a < 1$ and $\phi \in C([0,a])$, $\psi \in C([a,1])$.

Defining

(1.11) $$Q = a + \int_0^a \phi(x)\, dx + \int_a^1 \psi(x)\, dx$$

It is easy to prove the following integral equation, by using Green's theorem

(1.12) $$s(t) = Q - \int_0^{s(t)} U(x,t)\, dx - \int_{s(t)}^1 V(x,t)\, dx \quad 0 < t < T .$$

We shall call case A the one in which the solution exists for all time. If there is a finite stopping time T, we speak of case B when $\liminf_{t \to T} s(t) = 0$ or $\limsup_{t \to T} s(t) = 1$, and of case C when $\liminf_{t \to T} s(t) > 0$ and $\limsup_{t \to T} s(t) < 1$, and here we have blow-up at $t = T$.

In case of finite stopping time T, it is proved in [1] that:

$$\lim_{t \to T} \min (s(t), 1 - s(t)) = 0 \quad \text{or} \quad \limsup_{t \to T} |\dot{s}(t)| = +\infty .$$

In the following section we study the non-classical cases i.e. $\phi \leq 0$ or $\psi \geq 0$.

2- OVER-COOLED LIQUID IN CONTACT WITH CLASSICAL SOLID $(\phi \leq 0, \psi \leq 0)$

Proposition 2.1: If (T, x, U, V) is a solution of Problem P, then

$$Q \leq a \quad \text{and trivially} \quad Q = a \Leftrightarrow \begin{cases} s \equiv a \\ U \equiv V \equiv 0 \end{cases}$$

moreover, if $\phi \not\equiv 0$ and $\psi \not\equiv 0$, then:

i) $U < 0$ in D_T^U and $V < 0$ in D_T^V

ii) $Q < s(t)$, $0 < t < T$

iii) s is a decreasing function in $(0,T)$, $0 < s(t) < a$, $\forall t \in (0,T)$

Proof: It follows from (1.12), (1.10) and the maximum principle.

We proceed to characterize cases ABC depending on the value of Q.

Proposition 2.2:

i) $B \Rightarrow Q \leq 0$; $B \Rightarrow (Q = 0 \Rightarrow \psi \equiv 0)$

ii) $A \Rightarrow 0 \leq Q \leq a$

Proof: If in case B we use Lebesgue's theorem, then

$$Q = \int_0^1 V(x,T)\, dx \text{ ; and i) follows. Part ii) follows from (1.12),}$$

since U and V tend uniformly to zero as $t \to \infty$ and also $\lim_{t \to \infty} s(t) = Q$.

Remark 2.3: If we consider case B, $Q=0$ can only occur for the one-phase problem. Then the interest lies in the fact that $\phi \not\equiv 0$, which we assume from now on.

Next we show the equivalence between case A and $0 \leq Q \leq a$ for which we need two lemmas.

Assume a hypothesis of the type (H_2') (see [2]; [7]), that is

i) $\nexists\, v \in (0,a) \,/\, \forall\, x \in (a-v, a]$, $\phi(x) \leq -1$.

ii) $\nexists\, x^* \in (0,a) \,/\, \phi(x^*) < -1\ [>1]$, $x^* \in (x_1, x_2) \subset (0,a)$,

$$\phi(x_1)\phi(x_2) < 1\ [>1],\ \phi(x_i) \geq -1\ [\leq -1],\ i=1,2 .$$

Lemma 2.4: If the hypothesis above is satisfied then:

If $Q \geq 0$ there are no points in D_T^U such that $U(x,t) = -1$ or the isotherm $U=-1$ is separated by a positive distance from $x = s(t)$ for all t $[0,T]$ such that $0 < s(t) < 1$.

Proof: It is analogous to lemma 2.5 [2].

The following lemma generalizes another result, similar to the one given in [2] and [6].

Lemma 2.5: Let M_1 and $M_2 > 0$ be such that:

$$\phi(x) \geq M_1(x-a) \quad 0 \leq x \leq a ,$$

$$\psi(x) \geq M_2(a-x) \quad a \leq x \leq 1 ,$$

and let (T, s, U, V) be a solution of P.

Let $s(T) = \lim_{t\to T} s(t) = \inf_{(0,T)} s(t) > 0$, $d \in (0, s(T))$, $d \leq 1-a$,

$M_1 d < 1$, $Z_1 \in (0,1)$, $Z_2 > 0$,

and $U(s(t)-d, t) > -Z_1 \quad 0 < t < T$,

$V(s(t)+d, t) > -Z_2 \quad 0 < t < T$.

Then there exists a constant $K > 0$ such that

$$\dot{s}(t) \geq -K ,\ 0 < t < T .$$

Proof: Let $\varepsilon > 0$. We consider two auxiliary functions

$$W_1(x,t) = \frac{-A_1}{1-e^{-bd}}\left(1-e^{b(x-s(t))}\right), \quad s(t)-d \le x \le s(t), \quad 0 \le t < T-\varepsilon$$

$$W_2(x,t) = \frac{-A_2}{1-e^{cd}}\left(1-e^{c(x-s(t))}\right), \quad s(t) \le x \le s(t)+d, \quad 0 \le t < T-\varepsilon$$

Where A_1, A_2, b and c are constant to be determined.

Using the maximun principle in the appropiate regions to compare W_1 with U and W_2 with V, we get as in [6] that:

$$(2.1) \qquad U_x(s(t),t) \le \frac{A_1 b}{1-e^{-bd}}, \quad 0 < t < T-\varepsilon \quad ,$$

with A_1 (0,1) such that $A_1 \ge \max(Z_1, M_1\, d)$ and

$$(2.2) \qquad b \ge -\sigma_\varepsilon = -\inf_{(0,T-\varepsilon)} \dot{s}(t)$$

$$V_x(s(t),t) \ge \frac{A_2 c}{1-e^{cd}}, \quad 0 < t < T-\varepsilon \quad ,$$

whith $A_2 \ge \max(Z_2, M_2\, d)$ and $c < 0$ arbitrary.

Then:

$$(2.3) \qquad V_x(s(t),t) \ge \lim_{c \to 0} \frac{A_2 c}{1-e^{cd}} = -\frac{A_2}{d}$$

By (1.10) and (2.1), (2.2), (2.3)

$$\dot{s}(t) = V_x(s(t),t) - U_x(s(t),t) \ge \frac{-A_2}{d} + \frac{A_1 \sigma_\varepsilon}{1-e^{d \cdot \sigma_\varepsilon}} \quad .$$

Therefore:

$$\sigma_\varepsilon \ge \frac{-A_2}{d} + \frac{A_1 \sigma_\varepsilon}{1-e^{d\sigma_\varepsilon}}$$

as $\sigma_\varepsilon < 0$ then:

$$(2.4) \qquad \frac{d\sigma_\varepsilon + A_2}{d A_1 \sigma_\varepsilon} \leq \frac{1}{1-e^{d\sigma_\varepsilon}}$$

The function $f(x) = \dfrac{x+A_2}{A_1 x} - \dfrac{1}{1-e^x}$ $\quad -\infty < x < 0$, is monotone.

And:

$$\lim_{x \to -\infty} f(x) = \frac{1}{A_1} - 1 > 0, \quad \lim_{x \to 0} f(x) = -\infty .$$

Therefore there exists a unique $\alpha < 0$ such that $f(\alpha) = 0$ (it depends only on A_1 and A_2) .

By (2.4) $\sigma_\varepsilon \geq \dfrac{\alpha}{d}$.

We have obtained a lower bound for σ_ε, independent of ε, then, it follows:

$$\dot{s}(t) \geq -K \quad , \ 0 < t < T, \ \text{with } K > 0 .$$

From these two lemmas, we deduce:

Proposition 2.6: Under the hypotheses of lemmas 2.4 and 2.5

$$C \Rightarrow Q < 0$$

Proof: If $Q \geq 0$, from lemma 2.4, the isotherm $U = -1$ either does not exist or is separated from the free boundary. This allows us to apply Lemma 2.5, so that $\dot{s}$ is bounded from below.

Remark 2.7: If case C occurs, the isotherm $U = -1$ exists and reaches the free boundary. On the other hand, if $\phi > -1$, C can not occur.

Proposition 2.8: Under the hypotheses of lemmas 2.4 and 2.5 and $\psi \neq 0$ we have:

$$0 \leq Q \leq a \rightarrow A$$

Proof: From Proposition 2.2, since $\psi \neq 0$, we have $B \Rightarrow Q < 0$.

From the previous proposition we have $C \Rightarrow Q < 0$.

Therefore $Q \geq 0 \Rightarrow A$.

Note that for $Q<0$, it is impossible to characterize cases B or C only by the value of Q; they depend on the initial configuration (ϕ,ψ) too; this is in contrast to the one-phase problem.

Lemma 2.9: If there exist $M_1>0$, $M_2>0$ such that

$$\phi(x)\geq M_1(x-1)\ ,\ |0,a|$$

$$\psi(x)\geq -M_2\,x\ ,\ |a,1|$$

and if $Q<\frac{-M_2}{2}$ then C occurs.

Proof: From the maximum principle

$$U(x,t)\geq M_1(x-1)\quad\text{in}\quad D_T^U\ ,$$

$$V(x,t)\geq -M_2\,x\quad\text{in}\quad D_T^V\ .$$

From the integral equation (1.12) it follows

$$0\leq Q-\left(\frac{M_1+M_2}{2}\right)s^2(t)+(M_1-1)\,s(t)+\frac{M_2}{2}\ ,$$

with which B cannot occur if $Q<-\frac{M_2}{2}$.

Proposition 2.10: If $Q<0$ then there exists a pair (ϕ,ψ) such that C occurs.

Proof: Given $Q<0$ we choose $M_2\in(0,1)$ such that $Q<-\frac{M_2}{2}$

Now it will be enough to choose

$$a\in(0,1)\ ,\ \phi\in C(|0,a|)\ \text{and}\ \psi\in C(|a,1|)\ \text{such that}\ \phi\leq 0\ ,\ \psi\leq 0\ ,$$

$$Q=a+\int_0^a\phi(x)\,dx+\int_a^1\psi(x)\,dx\ \text{ and }\ \psi(x)\geq -M_2\,x\ \text{ in }\ |a,1|\ .$$

A possible choice is $a=\frac{1}{2}$,

$$\psi(x) = \begin{cases} -M_2(x-1/2) & \frac{1}{2} \le x \le \frac{3}{4} \\ M_2(x-1) & \frac{3}{4} \le x \le 1 \end{cases} \quad \text{and} \quad \phi(x) = M(x-1/2) \quad \text{in} \quad [0,1/2] \;,$$

with $M > 0$ such that $Q = \frac{1}{2} + \int_0^{1/2} M(x - \frac{1}{2})\, dx + \int_{1/2}^{3/4} -M_2(x - \frac{1}{2})\, dx +$

$+ \int_{3/4}^{1} M_2(x-1)\, dx$.

Proposition 2.11: For any $Q < 0$ there exist pairs (ϕ,ψ) such that B occurs.

Proof: We want to find (ϕ,ψ) such that

$$Q = a + \int_0^a \phi(x)\, dx + \int_a^1 \psi(x)\, dx \;,\; \phi \le 0 \;,\; \psi \le 0$$

and $\phi > -1$ in $[0,a]$, excluding case C (see Remark 2.7).

For that, it is enough to take

$$a = \frac{1}{2} \;,\; \phi(x) = x - a,\; \psi(x) = -H(x-a) \quad \text{with} \quad H = 3 - 8Q \;.$$

3. CLASSICAL LIQUID IN CONTACT WITH OVER-HEATED SOLID ($\phi \ge 0$, $\psi \ge 0$)

Due to its similarity with ($\phi \le 0$, $\psi \le 0$) we shall only state the important conclusions, the proofs are similar to the proceeding ones.

The isotherm $V = 1$ plays in the over-heated solid the same role as the isotherm $U = -1$ in the case of the over-cooled liquid.

Proposition 3.1: Let (T, s, U, V) be a solution of P. Then

$$Q \ge a \quad \text{and} \quad Q = a \Leftrightarrow \begin{cases} s \equiv a \\ U\, V = 0 \end{cases}$$

Moreover, if $\phi \not\equiv 0$ and $\psi \not\equiv 0$ it follows that

i) $U > 0$ in D_T^U and $V > 0$ in D_T^V

ii) $Q > s(t)$ if $0 < t < T$

iii) s is an increasing function in $[0,T)$.

By using the analogues of lemmas 2.5 and 2.6, we obtain under the corresponding assumptions

Proposition 3.2: If $\phi \not\equiv 0$, then:

i) $B \Rightarrow Q > 1$

ii) $A \Leftrightarrow a \leq Q \leq 1$

iii) $C \Rightarrow Q > 1$.

Proposition 3.3: Let $Q > 1$ be given, then there exists (ϕ,ψ) such that B occurs and another one such that C occurs.

4. OVER-COOLED LIQUID IN CONTACT WITH OVER-HEATED SOLID ($\phi \leq 0$, $\psi \geq 0$)

The possible kinds of behaviour are:

i) The solution exists for all positive time (A).

ii) A finite stopping time (T) exists.

a) If ($\liminf_{t \to T} s(t) = 0$ or $\limsup_{t \to T} s(t) = 1$) , we call these B_ℓ and B_r respectively.

b) If ($\liminf_{t \to T} s(t) > 0$ and $\limsup_{t \to T} s(t) < 1$) , we call it C .

The energy integral Q may take any real value. The isotherms $U = -1$ and $V = 1$ play a major role in the respective phases.

Lemma 4.1: Let $M_1 > 0$, $M_2 > 0$ be such that $\phi(x) \geq M_1(x-a)$ $0 \leq x \leq a$
$\psi(x) \leq M_2(x-a)$ $a \leq x \leq 1$

$$0 < s^-(T) = \inf_{(0,T)} s(t) \leq \sup_{(0,T)} s(t) = s^+(T) < 1$$

$$0 < d < \min\,(s^-(T)\ ,\ 1 - s^+(T))\ ,\ 0 < Zo < 1$$

$$U(s(t) - d,t)\ >\ -Zo \quad \text{in} \quad (0,T)$$

$$V(s(t) + d,t)\ <\ Zo \quad \text{in} \quad (0,T)$$

Then there exists a constant $K > 0$ such that $|\dot{s}| < K$ in $(0,T)$.

Proof: Analogous to the one of Lemma 2.5.

Remark 4.2: $C \Rightarrow$ the isotherm $U = -1$ or $V = 1$ reaches the free boundary $x = s(t)$.

Lemma 4.3: Let (T, s, U, V) be a solution of P and (ϕ,ψ) be such that:

i) a) $\phi \leq 0$.

b) $\exists\, \sigma \in (0,a)\ /\ \forall\, x \in (a-\sigma,a]\ ,\quad \phi(x) \leq -1$

c) $\exists\, x^* \in (0,a)\ /\ \phi(x^*) < -1\ |> -1|,\ x^* \in (x_1,x_2) \subset (0,a)\ ,$

$\phi(x_1)\phi(x_2) < 1\ |> 1|,\ \phi(x_i) \geq -1\ |\leq -1|,\ i = 1,2$.

ii) a) $\psi \geq 0$.

b) $\exists\, \sigma \in [0,1-a)\ /\ \forall\, x\ \ [a,a+\sigma],\ \psi(x) \geq 1$.

c) $\exists\, x^* \in (a,1)\ /\ \psi(x^*) > 1\ |< 1|,\ \ x^* \in (x_1,x_2) \subset (a,1),\ \psi(x_1)\psi(x_2) < 1\ |>1|$,

$\psi(x_i) \leq 1\ |\geq 1|,\ \ i = 1,2$

Then

i) $Q \leq 0 \Rightarrow U(0,t) < -1$ if $s(t) > 0$.

ii) $Q \geq 1 \Rightarrow V(1,t) > 1$ if $s(t) < 1$

Proof: Let us consider the following four possible situations

a) $-1 \leq \phi \leq 0\ ,\ 0 \leq \psi \leq 1$. Then $0 < Q < 1$, so there is nothing to prove.

b) $\phi \leq 0\ ,\ \phi \ngeq -1\ ,\ 0 \leq \psi \leq 1$. Then $Q < 1$.

Let us assume $Q \le 0$ and prove i).

$$s(t) - Q = -\int_0^{s(t)} U(x,t)\,dx - \int_{s(t)}^{1} V(x,t)\,dx < -\int_0^{s(t)} U(x,t)\,dx \;,$$

therefore $-1 > \underset{(0,s(t))}{\text{average}}\, U$, $0 \le t < T$, From these inequalities it follows that the isotherm $U = -1$ is seperated from $x = 0$, $\forall t > 0$, then $U(0,t) < -1$.

c) $-1 \le \phi \le 0$, $0 \le \psi$, $\psi \not\le 1$. Then $Q > 0$, i.e. we have to prove ii). Let us assume $Q \ge 1$. From the integral equation we get: $\underset{(s(t),1)}{\text{average}}\, V > 1$; therefore, the isothem $V = 1$ is separated from $x = 1, \forall t > 0$, with which $V(1,t) > 1$.

d) $\phi \le 0$, $\phi \not\geq -1$, $\phi \not\geq 0$, $\psi \not\le 1$. For this case we do not obtain a priori bounds for Q, so we have to prove i) and ii).

Let us assume $Q \le 0$, following the procedure b) it follows $U(0,t) < -1$.

Let us assume $Q \ge 1$, following the procedure c) it follows $V(1,t) > 1$.

Proposition 4.4: Under the hypotheses of Lemma 4.1 and Lemma 4.3, $\phi \not\equiv 0$ and $\psi \not\equiv 0$. Then

i) $A \Rightarrow 0 < Q < 1$

ii) $B_\ell \Rightarrow Q > 0$

iii) $B_r \Rightarrow Q < 1$

Proof:

i) $A \Rightarrow \lim_{t\to\infty} U(x,t) = \lim_{t\to\infty} V(x,t) = 0$ uniformly in x . Therefore:

$\lim_{t\to\infty} s(t) = Q$, then $0 \le Q \le 1$.

From Lemma 4.3 it follows:

$Q = 0 \Rightarrow U(0,t) < -1 \Rightarrow$ not A

$Q = 1 \Rightarrow V(1,t) > 1 \Rightarrow$ not A .

Then $0 < Q < 1$.

ii) $B_\ell \Rightarrow Q = \int_0^1 V(x,T)\,dx > 0$

iii) $B_r \Rightarrow Q = \int_0^1 U(x,T)\,dx + 1 < 1$.

Remark 4.5:

1) if $Q \leq 0$, then C or B_r may occur.

2) If $Q \geq 1$, then C or B_ℓ may occur.

Lemma 4.6: Let $M_1 > 0$ and $M_2 > 0$ be such that

$$M_1 (x - 1) \leq \phi (x) \leq 0 \quad \text{in } [0,a]$$

$$0 \leq \phi (x) \leq M_2\, x \; , \quad \text{in } [a,1] \; .$$

Then

i) $Q > \frac{M_2}{2} \Rightarrow B_\ell$ does not occur.

ii) $Q < 1 - \frac{M_1}{2} \Rightarrow B_r$ does not occur.

The proofs are analogous to the ones in Lemma 2.9.

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