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REMARKS ON A TWO-PHASE STEFAN PROBLEM WITH FLUX BOUNDARY CONDITIONS[1]

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ABSTRACT: *We consider a two-phase Stefan problem by analyzing the relation between the boundary data and the possibility of continuing the solution for arbitrarily large time intervals.*

KEY WORDS: Stefan problem.

RESUMO: *OBSERVAÇÕES SOBRE UM PROBLEMA DE STEFAN DE DUAS FASES COM CONDIÇÕES DE CONTORNO DE FLUXO. Consideramos um problema de Stefan de duas fases analizando a relação entre os dados de fronteira e a possibilidade de continuidade de solução para intervalos arbitrariamente grandes.*

PALAVRAS-CHAVE: problema de Stefan.

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1. INTRODUCTION

In this paper we study a two-phase Stefan Problem in one space dimension. The initial temperature of the material is equal to zero, in both phases. We impose a flux-boundary condition on $x = 0$ and $x = 1$, where the fluxes (g and f) are functions of time.

The classical Stefan Problem ($g \leq 0$, $f \leq 0$) is well studied in the literature (e.g. [1], [2]). Here we will treat the case ($g \geq 0$, $f \leq 0$) that corresponds to supercooling liquid. The existence and uniqueness for this problem is proved in [3], where a class of free-boundary problems for the heat equation in one space dimension was analyzed, releasing the sign restrictions on the data and the latent heat usually required in the Stefan Problem. In the next sections we relate the possibility of continuing the solution for arbitrarily large time intervals to the sign of $Q(t) = a + \int_0^t (f(\tau) - g(\tau))d\tau$. Problems of this kind also have been studied by other authors in conection with the freezing of a supercooling liquid. A one-phase Stefan Problem with initial temperature $h(x)$ and a flux equals to zero on $x = 0$ was studied in [4]. In [5], a one-phase Stefan problem with initial temperature equal to zero and the flux $g(t)$ on $x = 0$ was considered. In [6] we analyzed a two-phase Stefan problem where the liquid is initially at a temperature $\varphi(x)$ and the solid is initially at a temperature $\psi(x)$. We imposed a flux equals to zero on both sides $x = 0$, $x = 1$, and we chose the sign of the functions φ and ψ corresponding to a supercooling liquid or an overheated solid. A complete exposition of the results on the possibility of continuing the solution for arbitrarily large time intervals can be found in [7] and [8, Chap. 1].

In Section 2 we give the preliminaries corresponding to the description of the problem. In Section 3 we consider the case of a flux with a determined sign.

2. TWO-PHASE STEFAN PROBLEM

Let us consider Problem (P) which consist on finding (T, s, U, V) such that:

(i) $T > 0$.

(ii) $s \in C([0,T))$, $\quad s \in C^1((0,T))$; $\quad 0 < s(t) < 1$ for $0 < t < T$.

(iii) $U(x,t)$ is a function, bounded in $0 \leq x \leq s(t)$, $0 \leq t \leq T$, and continuous on the same region, except perhaps at the points $(0,0)$ and $(s(0),0)$.

$U_x(x,t)$ is a continuous function in $0 \leq x \leq s(t)$, $0 < t < T$.

U_{xx}, U_t are continuous functions in $0 < x < s(t)$, $0 < t < T$.

Similar conditions are imposed on the function V.

(iv) The following conditions are satisfied:

$$U_{xx} - U_t = 0 \quad \text{in} \quad D_T^U = \{(x,t) : 0 < x < s(t), \quad 0 < t < T\}, \tag{1}$$

$$V_{xx} - V_t = 0 \quad \text{in} \quad D_T^V = \{(x,t) : s(t) < x < 1, \quad 0 < t < T\}, \tag{2}$$

$$s(0) = a, \tag{3}$$

$$U(x,0)=0, \qquad 0<x<a, \tag{4}$$

$$V(x,0)=0, \qquad a<x<1, \tag{5}$$

$$U_x(0,t)=g(t), \qquad 0<t<T, \tag{6}$$

$$V_x(1,t)=f(t), \qquad 0<t<T, \tag{7}$$

$$U(s(t),t)=0, \qquad 0<t<T, \tag{8}$$

$$V(s(t),t)=0, \qquad 0<t<T, \tag{9}$$

$$V_x(s(t),t)-U_x(s(t),t)=\dot{s}(t), \quad 0<t<T, \tag{10}$$

where $0<a<1$ and the functions f and g are piecewise continuous on every interval $(0,t)$, $t>0$.

Moreover, if the solution exists, then three Cases can occur [3, Thm. 8].

(A) The problem has a solution with arbitrarily large T.

(B) There exists a constant $T_B>0$ such that $\liminf_{t\to T_B} s(t)=0$, or $\limsup_{t\to T_B} s(t)=1$.

(C) There exists a constant $T_C>0$ such that $\liminf_{t\to T_C} s(t)>0$, $\limsup_{t\to T_C} s(t)<1$ and $\limsup_{t\to T_C}|\dot{s}(t)|=\infty$.

A first simple result is Lemma 1 below. Define

$$Q(t)=a+\int_0^t (f(\tau)-g(\tau))d\tau. \tag{11}$$

Lemma 1. *If* (T,s,U,V) *solve* (1)-(10), *then*

$$s(t)=a+\int_0^t (f(\tau)-g(\tau))d\tau-\int_0^{s(t)} U(x,t)dx-\int_{s(t)}^1 V(x,t)dx. \tag{12}$$

Proof. Consider Green's identity

$$\int\int_{D_t}(zLu-uL^*z)dxd\tau=\int_{\partial D_t}[(u_xz-uz_x)d\tau+uz\,dx],$$

where L denotes the heat operator and L^* its adjoint. Formula (12) is obtained by setting $z=1$ and $u=U$, $u=V$ respectively, and adding both equations. Other relationships of the same kind could be obtained using higher-order polynomials for z. □

In the following section we will study the non-classical case, i.e., $f\le 0$, $g\ge 0$.

3. OVER-COOLED LIQUID IN CONTACT WITH CLASSICAL SOLID

The next Propositions give an apriori estimate for the functions s, U and V.

Proposition 1. *If (T, s, U, V) is a solution of Problem (P), then:*

(i) $U \leq 0$ *in* D_T^U, $V \leq 0$ *in* D_T^V,

(ii) $U_x \geq 0$ *in* D_T^U, $V_x \leq 0$ *in* D_T^V,

(iii) $s(t)$ *is a decreasing function in* $(0, T)$,

(iv) $Q(t) \leq s(t)$, $t > 0$,

(v) $Q(t)$ *is a decreasing function in* $(0, T)$.

Proof. Since $U_x(0,t) = g(t) \geq 0$, U can not have a maximum on $x = 0$, then $U \leq 0$ in D_T^U. Using the same maximum principle for V, we get $V \leq 0$ in D_T^U. Since U and V have a maximum on $x = s(t)$ we obtain $U_x(s(t), t) \geq 0$ and $V_x(s(t), t) \leq 0$. Now, using the above estimations in (10), we conclude that $\dot{s}(t) \leq 0$, $0 < t < T$. (iv) is obtained replacing (i) in (12). (v) follows from (11). □

Remark. The temperature of the liquid is less or equal than zero so it is over-cooled, meanwhile the solid is at its usual temperature.

We proceed to characterize Cases (A) (B) (C) depending on the value of $Q(t)$.

Proposition 2. *If (T, s, U, V) is a solution of Problem (P), and $G(t) = \sup_{\tau \in (0,t)}(g(\tau))$, then $U(x, t) \geq G(t)(x - a)$ in D_T^U.*

Proof. Notice that G is differentiable a.e., because G is an increasing function. Define a function

$$W(x, t) = G(t)(x - a), \quad (x, t) \in D = \{(x, t) : 0 < x < a,\ 0 < t < T\}.$$

This function satifies the following problem in D:

$$\begin{aligned} &W_{xx} - W_t \geq 0, && \text{in} \quad D, \\ &W(x, 0) \leq 0, && 0 < x < a, \\ &W(a, t) = 0, && 0 < t < T, \\ &W_x(0, t) = G(t), && 0 < t < T. \end{aligned}$$

By comparing W with U and using the maximum principle we obtain the thesis. □

We will assume for the next propositions that f and g satisfy the following conditions:

$$g(t) \leq K e^{-b^2 t}, \qquad t > 0, \quad K > 0, \tag{13}$$

$$f(t) \geq L e^{-b^2 t}, \qquad t > 0, \quad L < 0, \tag{14}$$

where $0 < b \leq \pi/6$. Next, we have the following Propositions:

Proposition 3. *If (T, s, U, V) is a solution of Problem* (P) *for every $t > 0$ then this solution satisfies the following properties*

(i) $\lim_{t\to\infty} U(x,t) = 0$, *uniformly in x.*

(ii) $\lim_{t\to\infty} V(x,t) = 0$, *uniformly in x.*

Proof. Compare U with the function $Z(x,t) = -K(2/b)\sin(2\pi/3 + bx)e^{-b^2t}$ in D_T^U with $K > 0$ and $0 < b \le \pi/6$. Then $Z(x,t) \le U(x,t)$. And (i) follows from $\lim_{t\to\infty} Z(x,t) = 0$. We can prove (ii) for V in the same way, by using an adequate Z. □

Our next aim will be to look for some conditions on f and g giving an apriori characterization of Cases (A), (B), (C).

Proposition 4. *Case* (B) $\Longrightarrow Q(T_B) \le 0$.

Proof. Letting $t \to T_B$ in (12) we obtain the following equation.

$$Q(T_B) = \int_0^1 V(x, T_B)dx.$$

Then $Q(T_B) \le 0$ is proved according to Proposition 1. □

Proposition 5. *Let (T, s, U, V) be a solution of Problem* (P). *Denote*

$$s_T = \lim_{t\to T} s(t) > 0, \quad d \in (0, s(T)), \quad d \le 1 - a \quad \textit{and} \quad z_1 \in (0,1), \quad z_2 > 0.$$

If the solution (T, s, U, V) satisfies

$$U(s(t) - d, t) \le -z_1 \quad \textit{and} \quad V(s(t) + d, t) > -z_2 \quad \textit{in} \quad (0, T),$$

then

$$\dot{s}(t) \ge -k, \quad \textit{for some} \;\; k > 0.$$

Proof. We can use a similar technique as in [6, Lem. 2.5], because the proof is independent of $U_x(0,t)$ and $V_x(1,t)$. Let $\varepsilon > 0$. Consider the auxiliary functions

$$W_1(x,t) = \frac{-A_1}{1 - e^{-bd}}(e^{b(x - s(t))}), \qquad s(t) - d \le x \le s(t), \quad 0 \le t < T - \varepsilon,$$

$$W_2(x,t) = \frac{-A_2}{1 - e^{cd}}(1 - e^{c(x - s(t))}), \qquad s(t) \le x \le s(t) + d, \quad 0 \le t < T - \varepsilon,$$

where A_1, A_2, b and c are constants to be determined. Using the maximum principle in the appropriate regions to compare W_1 with U and W_2 with V, we get

$$\dot{s}(t) \ge \frac{-A_2}{d} + \frac{A_1\sigma_\varepsilon}{1 - e^{d\sigma_\varepsilon}},$$

with $A_1 \in (0,1)$ such that $A_1 \geq \max(z_1, d)$, $b \geq -\sigma_\epsilon = -\inf_{(0,T-\epsilon)} \dot{s}(t)$, $A_2 \geq \max(z_2, d)$ and $\epsilon < 0$ arbitrary. Therefore,

$$\frac{d\sigma_\epsilon + A_2}{dA_1\sigma_\epsilon} \leq \frac{1}{1 - e^{d\sigma_\epsilon}}.$$

Then by analyzing this as a function of σ_ϵ we obtain

$$\dot{s}(t) \geq -k \qquad 0 < t < T, \qquad \text{for some} \quad k > 0. \quad \square$$

Corollary. *If Case* (C) *occurs, the isotherm* $U = -1$ *exists and reaches the free boundary at* $t = T_C$.

Proposition 6. *If* $f - g \in L^1(0, \infty)$, *then Case* (A) $\Longrightarrow Q(t) > 0$, $t > 0$.

Proof. Letting $t \to \infty$ in (12) and using Proposition 3, we obtain the result. $\square$

Lemma 2. *Suppose* $t_0 \leq T$, *let* $\lim_{t \to t_0} s(t) > 0$, *and*

$$Q(t_0) = a + \int_0^{t_0} (f(\tau) - g(\tau))d\tau > 0. \tag{15}$$

If we define a function η *as*

$$\eta(t) = \begin{cases} \max\{x \in [0, s(t)] : U(x,t) \leq -1\} \\ 0 \quad if \quad U(x,t) > -1, \ \forall x \in [0, s(t)], \end{cases}$$

then

$$\limsup_{t \to t_0} \eta(t) < \lim_{t \to t_0} s(t). \tag{16}$$

Proof. (The proof is the same as the one in [5, Lem. 2.3]. We repeat it briefly for sake of completeness). Notice first that $\lim_{t \to t_0} s(t)$ exists because of Proposition 1 (iii). Using Proposition 1 (ii) we obtain

$$U \leq -1 \qquad \text{in} \qquad [0, \eta(t)],$$

$$-1 < U \geq 0 \qquad \text{in} \qquad (\eta(t), s(t)].$$

Define $\bar{s} = \limsup_{t \to t_0} \eta(t)$. Then there exists $\{t_n\}$ such that $t_n \to t_0$, and $\bar{s} = \lim_{n \to \infty} \eta(t_n) = \lim_{n \to \infty} \eta_n$. Replace $t = t_n$ in (12) so that $s(t_n) \geq Q(t_n) + \eta_n$. Then $\lim_{t \to t_0} s(t) \geq Q(t_0) + \bar{s} > \limsup_{t \to t_0} \eta(t)$. $\square$

Proposition 7. *Case* (C) $\Longrightarrow Q(T_C) \leq 0$.

Proof. Suppose $Q(T_C) > 0$. Then, by using Lemma 2, the free-boundary should be separated from the isoterm $U = -1$, contradicting the hypothesis. □

Proposition 8. *If* $Q(t) > 0$ *for every* $t > 0$, *then we have Case* (A).

Proof. If we had Case (B), Q should be negative or zero at some point T_B, contradicting the hypothesis. If we had Case (C), $Q(T_C) \leq 0$, from Proposition 7, contradicting the hypothesis. □

Remarks.

(i) Notice that, for $Q < 0$ it is impossible to characterize Cases (B) or (C) only by the value of Q; they also depend on the flux (g, f). We can prove that, when $Q < 0$ one can find functions (f, g) that give Case (B) and others that give Case (C). This contrasts with the one-phase Stefan Problem.

(ii) In the same way, we can obtain results for the case $g \leq 0$ and $f \geq 0$ (i.e., an overheated solid in contact with a liquid at its usual temperature).

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