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A NOTE ON THE EXISTENCE OF A WAITING TIME FOR A TWO-PHASE STEFAN PROBLEM

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Abstract. We consider a slab, represented by the interval $0 < x < x_0$, at the initial temperature $\theta_0 = \theta_0(x) \geq 0$ (or $\varphi_0 = \phi_0(x) \geq 0$) having a heat flux $q = q(t) > 0$ (or convective boundary condition with a heat transfer coefficient h) on the left face $x = 0$ and a temperature condition $b(t) > 0$ on the right face $x = x_0$ (x_0 could be also $+\infty$, i.e., a semi-infinite material). We consider the corresponding heat conduction problem and assume that the phase-change temperature is 0°C.

We prove that certain conditions on the data are necessary or sufficient in order to obtain the existence of a waiting-time at which a phase-change begins.

1. Introduction. We consider the following heat conduction problems ($0 < x_0 \leq +\infty$):

$$\begin{aligned} &\text{(i)} && \rho c\theta_t - k\theta_{xx} = 0, && 0 < x < x_0, \ t > 0; \\ &\text{(ii)} && \theta(x, 0) = \theta_0(x) > 0, && 0 \leq x \leq x_0; \\ &\text{(iii)} && k\theta_x(0, t) = q(t), && t > 0; \\ &\text{(iv)} && \theta(x_0, t) = b(t), && t > 0; \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\text{(i)} && \rho c\phi_t - k\phi_{xx} = 0, && 0 < x < x_0, \ t > 0; \\ &\text{(ii)} && \phi(x, 0) = \phi_0(x) > 0, && 0 \leq x \leq x_0; \\ &\text{(iii)} && k\phi_x(0, t) = h(D + \phi(0, t)), && t > 0; \\ &\text{(iv)} && \phi(x_0, t) = b(t), && t > 0. \end{aligned} \tag{1'}$$

where ρ is the density, k is the thermal conductivity, c is the specific heat, h is the convective heat transfer coefficient from a fluid with ambient temperature $-D < 0$ flowing across the face $x = 0$. The function $b(t)$ represents the temperature at the face $x = x_0 > 0$, and θ_0 and φ_0 are the initial temperatures for problems (1) and (1') respectively.

We take, without loss of generality, the phase-change temperature as 0°C and replace condition (1)(iv) by $\theta(+\infty, t) = \theta_0(+\infty) > 0$, $t > 0$ for the case $x_0 = +\infty$

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(idem for Problem (1')). We assume that the data satisfy the hypotheses that ensure the existence and uniqueness property of the solution of (1) and (1').

We consider the following possibilities:

(a) the heat conduction problem is defined for all $t > 0$ (waiting-time $t^* = +\infty$);

(b) there exists a time $t^* < +\infty$ such that another phase (i.e., the solid phase) appears for $t \geq t^*$ (waiting-time $0 \leq t^* < +\infty$) and then we have a two-phase Stefan problem for $t > t^*$. In this case, there exists a free boundary $x = s(t)$, which separates the liquid and solid phases with $s(t^*) = 0$.

We will separate the cases waiting-time $t^* = 0$ (i.e., there exists an instantaneous change of phase) and $0 < t^* \leq +\infty$. These possibilities depend on the data θ_0, q, b for Problem (1) and the data ϕ_0, h, b for Problem (1'). We try to clarify this dependence by finding necessary or sufficient conditions on θ_0, q, b and ϕ_0, h, b in order to have the different possibilities.

In [5, 8, 9] the one-phase Stefan problem with prescribed flux or convective boundary condition at $x = 0$ is studied.

This paper was motivated by [10, 12, 13] (see also [14]) and the term *waiting-time* was motivated by its correspondence to the term as used in the porous medium equation (see, for instance [1]).

In Sec. II we analyse problem (1) with a flux boundary condition at $x = 0$ and in Sec. III we study the problem (1') with a convective boundary condition at $x = 0$.

II. On some conduction problems with a flux boundary condition. We consider the following properties for the problem (1).

Theorem 1. If the data $q = q(t)$, $\theta_0 = \theta_0(x)$, and $b = b(t)$ verify conditions

$$
\begin{aligned}
&\text{(i)} \quad 0 < q(t) \leq q_0, \qquad 0 < t \leq t_0 \text{ with } t_0 > 0; \\
&\text{(ii)} \quad \theta_0'(x) \geq 0 \text{ and } \beta_1 \geq \theta_0(x) \geq \beta_0 > 0, \qquad 0 \leq x \leq x_0 \text{ with } \beta_0 \leq \beta_1; \\
&\text{(iii)} \quad b(t) \geq \beta_1 \text{ and } \dot{b}(t) \geq 0, \; t > 0;
\end{aligned} \tag{2}
$$

then there exists a waiting-time $t^* > 0$ for problem (1), (i.e., another phase could appear at $t \geq t^*$), where t^* verifies the following inequality:

$$
t^* \geq \operatorname{Min}(t_0, t_0^*), \quad \text{where } t_0^* = \pi k \rho c \beta_0^2 / 4 q_0^2. \tag{3}
$$

Proof. It is sufficient to prove that $\theta(x, t) \geq 0$ for $0 \leq x \leq x_0$ and $0 \leq t \leq t_0^*$. For the semi-infinite material $x > 0$, with the same thermal coefficients, we consider the following two problems:

$$
\begin{aligned}
&\rho c T_t - k T_{xx} = 0, \qquad x > 0, \; 0 < t < t_0; \\
&k T_x(0, t) = q(t), \qquad t > 0; \\
&T(x, 0) = T_0(x), \qquad x \geq 0,
\end{aligned} \tag{4}
$$

with

$$
T_0(x) = \begin{cases} \theta_0(x), & 0 \leq x \leq x_0, \\ \theta_0(x_0), & x > x_0, \end{cases} \tag{5}
$$

and

$$
\begin{aligned}
&\rho c V_t - k V_{xx} = 0, \qquad x > 0, \; t > 0; \\
&k V_x(0, t) = q_0 > 0, \qquad t > 0; \\
&V(x, 0) = \beta_0 > 0, \qquad x \geq 0,
\end{aligned} \tag{6}
$$

whose solutions are given respectively by [3, 4]

$$
T(x, t) = \int_0^{+\infty} N(x, t; \xi, 0) T_0(\xi)\, d\xi - \frac{2a}{k} \int_0^t K(x, t; 0, \tau) q(\tau)\, d\tau, \tag{7}
$$

and

$$
V(x, t) = \beta_0 - \frac{2 q_0 a}{k} \sqrt{t}\, \operatorname{ierfc}\left(\frac{x}{2a\sqrt{t}}\right), \tag{8}
$$

where

$$
\begin{gathered}
a = \left(\frac{k}{\rho c}\right)^{1/2}; \qquad K(x, t; \xi, \tau) = \frac{\exp(-(x-\xi)^2 / 4a^2(t-\tau))}{2a\sqrt{\pi(t-\tau)}}; \\
N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau); \qquad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt; \\
\operatorname{erfc}(x) = 1 - \operatorname{erf}(x); \qquad \operatorname{ierfc}(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} - x \operatorname{erfc}(x).
\end{gathered} \tag{9}
$$

By the maximum principle [6, 7] we obtain

$$
\begin{aligned}
&V(x, t) \leq T(x, t), \qquad x \geq 0, \; 0 \leq t \leq t_0, \\
&T(x, t) \leq \theta(x, t), \qquad 0 \leq x \leq x_0, \; 0 \leq t \leq t_0,
\end{aligned} \tag{10}
$$

because $T(x_0, t) \leq \beta_1 \leq b(t)$ for $0 \leq t \leq t_0$.

Let W be the function $W = \theta_x$, which satisfies the following heat conduction problem:

$$
\begin{aligned}
&\rho c W_t - k W_{xx} = 0, \qquad 0 < x < x_0, \; t > 0; \\
&W(x, 0) = \theta_0'(x), \qquad 0 \leq x \leq x_0; \\
&W(0, t) = \frac{q(t)}{k}, \qquad W_x(x_0, t) = \frac{\rho c}{k} \dot{b}(t), \; t > 0.
\end{aligned} \tag{11}
$$

By the maximum principle we have $W = \theta_x \geq 0$ for $0 \leq x \leq x_0$, $t \geq 0$. Then, we deduce that

$$
\theta(x, t) \geq \theta(0, t) \geq V(0, t) = \beta_0 - \frac{2 q_0 a}{k} \sqrt{\frac{t}{\pi}} \geq 0 \quad \text{for } 0 \leq t \leq t_0^*, \tag{12}
$$

where t_0^* is defined by (3) proving our assertion.

Remark 1. When the data verify conditions (2), problem (1) represents a heat conduction problem for the initial phase (in our case, the liquid phase) for $t \leq t^*$.

Remark 2. We can see that t_0^* does not depend on the length of the slab $x_0 > 0$.

Corollary 2. Under the hypotheses (2)(ii),(iii), a necessary condition in order to have an instantaneous change of phase (i.e., $t^* = 0$) for problem (1) is given by

$$
q(0^+) = +\infty. \tag{13}
$$

REMARK 3. If we consider the case

$$x_0 = +\infty, \quad \theta_0(x) \geq \beta_0 > 0, \quad \forall x \geq 0,$$
$$q(t) \leq q_0(t) = \frac{\beta_0 k}{a\sqrt{\pi t}}, \quad \forall t > 0, \tag{14}$$

then problem (1) is a heat conduction problem for the liquid phase for all $t > 0$ i.e., there is not a phase-change process for any $t > 0$ because we have

$$\theta(x, t) \geq \theta q_0(x, t), \quad x \geq 0, \ t > 0, \tag{15}$$

where θq_0 is the solution of (1) with data: heat flux $q_0(t)$ at $x = 0$, $x_0 = +\infty$, and initial temperature β_0. It is given by [12]

$$\theta q_0(x, t) = \beta_0 \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) \geq 0, \quad x \geq 0, \ t > 0. \tag{16}$$

Moreover, the particular case

$$q(t) = \frac{\beta_0 k}{a\sqrt{\pi t}} = q_0(t), \quad t > 0, \tag{17}$$

shows us that condition (13) is not sufficient in order to have an instantaneous change of phase for problem (1).

REMARK 4. If $x_0 = +\infty$ and $\theta_0(x) \geq \beta_0 > 0$ for $x \geq 0$, then a necessary condition for problem (1) to have an instantaneous change of phase (i.e., the waiting-time is $t^* = 0$) is for there to exist a $t_0 > 0$ such that

$$q(t_0) > \frac{\beta_0 k}{a\sqrt{\pi t_0}}. \tag{18}$$

THEOREM 3. If the data verify the conditions

$$x_0 = +\infty; \quad 0 \leq \theta_0(x) \leq \beta_1 \quad \text{for } x \geq 0,$$
$$q(t) \geq \frac{q_0}{t^\beta}, \quad 0 < t < 1, \text{ with } q_0 > 0 \text{ and } \tfrac{1}{2} < \beta < 1, \tag{19}$$

then an instantaneous phase-change occurs, that is, the waiting-time is $t^* = 0$.

Proof. Let $U = U(x, t)$ be the solution of the following heat conduction problem:

$$\begin{aligned} \rho c U_t - k U_{xx} &= 0, & x > 0, \ t > 0; \\ U(x, 0) &= \beta_1, & x \geq 0; \\ k U_x(0, t) &= \frac{q_0}{t^\beta}, & t > 0, \end{aligned} \tag{20}$$

which is given by [3]

$$U(x, t) = \beta_1 - \frac{2aq_0}{k} \int_0^t \frac{K(x, t; 0, \tau)}{\tau^\beta} d\tau. \tag{21}$$

By using the maximum principle we have that $\theta(x, t) \leq U(x, t)$ for $x \geq 0$, $t > 0$. Therefore, we obtain

$$\theta(0, t) \leq U(0, t) \leq \beta_1 - \frac{aq_0}{k\sqrt{\pi}} \int_0^t \frac{d\tau}{\tau^\beta \sqrt{t - \tau}} \tag{22}$$

and, for $0 < \varepsilon = t/2 < t < 1$,

$$\int_0^t \frac{d\tau}{\tau^\beta \sqrt{t-\tau}} \geq \int_0^\varepsilon \frac{d\tau}{\varepsilon^\beta \sqrt{t-\tau}} + \int_\varepsilon^t \frac{d\tau}{\tau^\beta \sqrt{t-\varepsilon}} = C_\beta \left(\frac{2}{t}\right)^{\beta - 1/2}. \tag{23}$$

Moreover, the temperature on the fixed face $x = 0$ verifies the inequality

$$\theta(0, t) \leq \beta_1 - \frac{aq_0}{k\sqrt{\pi}} C_\beta \left(\frac{2}{t}\right)^{\beta-1/2} < 0, \tag{24}$$

for all $t < \min(1, t_\beta)$, where

$$\begin{aligned} t_\beta &= 2\left(\frac{aq_0}{k\beta_1\sqrt{\pi}} C_\beta\right)^{1/(\beta-1/2)} > 0, \\ C_\beta &= \frac{1}{1-\beta}[2^{1-\beta} - 1 + 2(1-\beta)(\sqrt{2} - 1)] > 0, \end{aligned} \tag{25}$$

that is, the thesis is achieved.

REMARK 5. If we consider the density jump under the phase of change, that is, $\rho_1 \neq \rho_2$, and the data verify the conditions

$$x_0 = +\infty: \quad 0 \leq \theta_0(x) \leq \beta_1 \quad \text{for } x \geq 0,$$
$$q(t) \geq \frac{q_0}{\sqrt{t}} \quad \text{for } t > 0 \text{ with } q_0 > \frac{\beta_1 k_2}{a_2\sqrt{\pi}}, \tag{26}$$

where $k_i, c_i, \rho_i, a_i = (k_i/\rho_i c_i)^{1/2}$ are the corresponding thermal coefficients for the phase i ($i = 2$: liquid phase, $i = 1$: solid phase), then the temperature $\theta = \theta_{q,\theta_0}$, solution of problem (1), verifies the inequality $\theta_{q,\theta_0}(x, t) \leq T_{q_0,\beta_1}(x, t)$, $x \geq 0$, $t > 0$, where T_{q_0,β_1} is the solution of (1) with initial constant temperature β_1 and a flux condition of type $q_0/\sqrt{t}$ on $x = 0$. Therefore, we obtain [2]

$$\theta_{q,\theta_0}(0, t) \leq T_{q_0,\beta_1}(0, t) = \beta_1 - \frac{q_0 a_2 \sqrt{\pi}}{k_2} < 0, \quad t > 0, \tag{27}$$

that is, the waiting-time is $t^* = 0$ (i.e., we have an instantaneous two-phase Stefan problem) for data q and θ_0. Moreover, its free boundary $x = s_{q,\theta_0}(t)$ verifies $s_{q,\theta_0}(0) = 0$ and it is characterized by $\theta_{q,\theta_0}(s_{q,\theta_0}(t), t) = 0$ for all $t > 0$.

The free boundary $x = s_{q_0,\beta_1}(t)$ corresponding to the temperature T_{q_0,β_1} is given by [2]

$$s_{q_0,\beta_1} = 2\omega\sqrt{t}, \tag{28}$$

where ω is the unique solution of the equation

$$F_0(x) = x, \quad x > 0 \tag{29}$$

with

$$F_0(x) = \frac{q_0}{h\rho_1} \exp\left(\frac{-x^2}{a_1^2}\right) - \frac{k_2\beta_1}{h\rho_1 a_2\sqrt{\pi}} \frac{\exp(-x^2/a_2)}{\operatorname{erfc}(x/a_2)}, \tag{30}$$

where $h > 0$ is the latent heat. Owing to

$$\theta_{q,\theta_0}(s_{q_0,\beta_1}(t), t) \leq T_{q_0,\beta_1}(s_{q_0,\beta_1}(t), t) = 0, \quad \text{for all } t > 0, \tag{31}$$

it follows that

$$s_{q,\theta_0}(t) \geq s_{q_0,\beta_1}(t) = 2\omega\sqrt{t}, \qquad t > 0. \quad \blacksquare \tag{32}$$

From now on we consider the particular case of constant temperature $b(t) = b > 0$, $t > 0$ at $x = x_0$ and constant heat flux $q(t) = q > 0$, $t > 0$ at $x = 0$ for problem (1). The steady-state solution is given by

$$\theta_\infty(x) = \frac{q}{k}(x - x_0) + b, \tag{33}$$

and a necessary and sufficient condition in order to have a two-phase steady-state Stefan problem is given by

$$q > kb/x_0, \tag{34}$$

where k is the thermal conductivity of the liquid phase [11]. (See [13, 14] for the general steady-state case for an n-dimensional domain).

Using the fact that $\theta = \theta(x, t)$, the solution of problem (1) with data $q > 0$ and $b > 0$, converges to $\theta_\infty = \theta_\infty(x)$ when t goes to $+\infty$ [6], for any initial temperature $\theta_0 = \theta_0(x)$, we can formulate the following problem: Find the relation between the heat flux $q > 0$ on $x = 0$ and a time t_1 such that another phase appears for $t \geq t_1$, and then we can reformulate problem (1) in a two-phase Stefan problem for $t \geq t_1$.

We obtain the following result.

THEOREM 4. Suppose the initial temperature verifies the conditions $b \geq \theta_0 \geq 0$ in $[0, x_0]$ and $\theta_0(x_0) = b$. If the time $t_1 > 0$ and the constant heat flux $q > 0$ verify the inequality

$$q > \frac{bk}{x_0(1 - \exp(-\alpha\pi^2 t_1/4x_0^2))}, \qquad \alpha = \frac{k}{\rho c}, \tag{35}$$

then another phase (the solid phase) appears for $t \geq t_1$. Moreover, $\theta(0, t) < 0$ for all $t \geq t_1$ and the free boundary $x = s(t)$ begins at a point $(0, t')$ with $0 \leq t' < t_1$.

Proof. The temperature $\theta(x, t)$ is given by

$$\theta(x, t) = \theta_\infty(x) + \sum_{n=0}^{\infty} C_n \cos(\sqrt{\lambda_n}x)\exp(-\alpha t\lambda_n), \tag{36}$$

where

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{x_0^2}, \qquad n = 0, 1, 2, \dots, \tag{37}$$

$$C_n = \frac{2}{x_0}\int_0^{x_0} [\theta_0(x) - \theta_\infty(x)]\cos(\sqrt{\lambda_n}x)\,dx. \tag{38}$$

Therefore, the temperature at $x = 0$ is given by

$$\theta(0, t) = b - \frac{qx_0}{k}(1 + S(t)) + S_0(t) \tag{39}$$

with

$$S(t) = \frac{2}{x_0^2}\sum_{n=0}^{\infty}\exp(-\alpha t\lambda_n)\int_0^{x_0}(x - x_0)\cos(\sqrt{\lambda_n}x)\,dx = -\frac{2}{\pi^2}\sum_{n=0}^{+\infty}\frac{\exp(-\alpha t\lambda_n)}{(n + \frac{1}{2})^2} \tag{40}$$

and

$$S_0(t) = \frac{2}{x_0}\sum_{n=0}^{+\infty}\exp(-\alpha t\lambda_n)\int_0^{x_0}(\theta_0(x) - b)\cos(\sqrt{\lambda_n}x)\,dx. \tag{41}$$

We get that $S_0(t) = v(0, t) \leq 0$ for all $t \geq 0$ because of the maximum principle, where the function $v = v(x, t) \leq 0$ is the solution of the problem

$$\begin{aligned} &\rho c v_t - k v_{xx} = 0, && 0 < x < x_0, \ t > 0; \\ &v_x(0, t) = 0, && t > 0; \\ &v(x_0, t) = 0, && t > 0; \\ &v(x, 0) = \theta_0(x) - b \leq 0, && 0 \leq x \leq x_0. \end{aligned} \tag{42}$$

By some manipulations, it follows that

$$0 < |S(t)| = -S(t) \leq \exp\left[-\frac{\alpha\pi^2 t}{4x_0^2}\right] < 1 \quad \text{for all } t \geq t_1. \tag{43}$$

Therefore, if q and t_1 verify (35), we obtain

$$\theta(0, t) \leq b - \frac{qx_0}{k}(1 + S(t)) < 0 \quad \text{for all } t \geq t_1, \tag{44}$$

i.e., the thesis is achieved.

REMARK 6. If $\theta_0(x) = b$ in $[0, x_0]$, we deduce $S_0(t) = 0$ for $t > 0$. We remark that inequality (35) was obtained for this particular initial temperature because $S_0(x) \leq 0$ in $(0, +\infty)$ for $\theta_0(x) \leq b$ in $[0, x_0]$.

COROLLARY 5. If we consider the t, q plane and define the following set

$$Q = \{(t, q) \mid q > f(t), \ t > 0\}, \qquad f(t) = \frac{bk}{x_0[1 - \exp(-\alpha\pi^2 t/4x_0^2)]}, \tag{45}$$

then we have a two-phase problem for all $(t, q) \in Q$.

III. On some conduction problems with a convective boundary condition. Now we consider the same kind of techniques used in Sec. 2 for problem (1′) corresponding to a heat conduction problem with a convective boundary condition at $x = 0$.

THEOREM 6. If the data $\phi_0 = \phi_0(x)$, $b = b(t)$, and D verify the conditions

$$\begin{aligned} &\text{(i)} \quad \phi_0'(x) \geq 0 \quad \text{and} \quad \beta_1 \geq \phi_0(x) \geq \beta_0 > 0, \qquad 0 \leq x \leq x_0; \\ &\text{(ii)} \quad b(t) \geq \beta_1 \quad \text{and} \quad \dot{b} \geq 0, \qquad t > 0; \\ &\text{(iii)} \quad D > 0, \end{aligned} \tag{46}$$

then there exists a waiting-time $t^* > 0$ for problem (1′), where t^* verifies the inequality

$$t^* \geq t_1^*, \quad \text{where } t_1^* = \frac{kc\rho}{h^2}\left(F^{-1}\left(1 + \frac{\beta_0}{D}\right)\right)^2. \tag{47}$$

where F^{-1} is the inverse function of

$$F(x) = \frac{\exp(-x^2)}{\operatorname{erfc}(x)}, \qquad x > 0. \tag{48}$$

Proof. By using the maximum principle we get that

$$\begin{aligned}&\phi_x(x,t)\ge 0, \quad 0\le x\le x_0,\ t\ge 0,\\ &\phi(x,t)\ge z(x,t), \quad 0\le x\le x_0,\ t\ge 0.\end{aligned} \tag{49}$$

where $z = z(x,t)$ is the solution of the following heat conduction problem:

$$\begin{aligned}&\text{(i)}\quad \rho c z_t - k z_{xx} = 0, \quad x>0,\ t>0;\\ &\text{(ii)}\quad z(x,0)=\beta_0, \quad x>0;\\ &\text{(iii)}\quad k z_x(0,t) = h(D+z(0,t)), \quad t>0.\end{aligned} \tag{50}$$

which is given by $(a^2 = k/\rho c)$:

$$z(x,t) = (\beta_0 + D)\left[\operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right) + \exp\left(\frac{hx}{k}+\eta^2\right)\right]\operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}+\eta\right), \quad x\ge 0,\ t\ge 0. \tag{51}$$

where

$$\eta = \frac{ha\sqrt{t}}{k}. \tag{52}$$

Taking into account (47) and (49) we get

$$\phi(x,t)\ge\phi(0,t)\ge z(0,t) = -D + (\beta_0 + D)\exp(\eta^2)\operatorname{erfc}(\eta)\ge 0, \quad t\le t_1^*, \tag{53}$$

because the function $F(x)$ verifies the conditions

$$F(0)=1, \quad F(+\infty)=+\infty, \quad F'>0 \text{ in } R^+. \tag{54}$$

REMARK 7. We can see that t_1^* does not depend on the length of the slab $x_0 > 0$. ■

From now on we consider the particular case of constant temperature $b(t) = b > 0$, $t > 0$ at $x = x_0$ for problem (1′). The corresponding steady-state solution is given by

$$\phi_\infty(x) = b - \frac{h(D+b)}{k+hx_0}(x_0 - x), \quad 0\le x\le x_0, \tag{55}$$

and a necessary and sufficient condition, in order to have a two-phase steady-state Stefan problem is given by

$$h > \frac{kb}{Dx_0}. \tag{56}$$

We consider the following problem related to problem (1′): Find the relation between the heat transfer coefficient h and a time t_2 such that another phase appears for $t\ge t_2$, and then we can reformulate problem (1′) in a two-phase Stefan problem for $t\ge t_2$. We obtain the following result.

THEOREM 7. Suppose the initial temperature verifies the conditions $b\ge\theta_0\ge 0$ in $[0, x_0]$ and $\theta_0(x_0) = b$. If the time $t_2 > 0$ and the constant heat transfer coefficient $h > 0$ verify the inequality

$$h > g(t_2), \tag{57}$$

then another phase (the solid phase) appears for $t\ge t_2$, where the function $g = g(t)$ is defined implicitly by the equation

$$\psi(t, g(t)) = 0, \quad t>0, \tag{58}$$

with

$$\psi(t,h) = -D + \frac{k(D+b)}{k+hx_0} + \frac{2k(D+b)}{hx_0}\gamma(t), \quad t>0,\ h>0, \tag{59}$$

$$\gamma(t) = \sum_{n=1}^{\infty}\exp\left[-\frac{(2n-1)^2\pi^2a^2t}{x_0^2}\right], \quad t>0. \tag{60}$$

Proof. The solution of problem (1′) is given by

$$\phi(x,t) = \phi_\infty(x) + \sum_{n=1}^{\infty} B_n\exp(-\mu_n^2a^2t)\left[\operatorname{Sin}(\mu_n x) + \frac{k\mu_n}{h}\operatorname{Cos}(\mu_n x)\right], \quad 0\le x\le x_0,\ t>0. \tag{61}$$

where

$$B_n = \frac{2}{x_0}\int_0^{x_0}[\phi_0(x) - \phi_\infty(x)]\left[\operatorname{Sin}(\mu_n x) + \frac{k\mu_n}{h}\operatorname{Cos}(\mu_n x)\right]dx, \tag{62}$$

and $\mu_n = \omega_n/x_0$, where ω_n is the nth root of the eigenvalue equation

$$tg(\omega) = -\frac{k}{hx_0}\omega, \quad \omega>0. \tag{63}$$

Moreover, we get that

$$(2n-1)\tfrac{\pi}{2} < \omega_n < n\pi, \quad n\in N. \tag{64}$$

After some manipulation, we deduce that the temperature at $x = 0$ is bounded by

$$\phi(0,t)\le\psi(t,h), \quad t>0, \tag{65}$$

where the function ψ has been defined before.

We notice that the function $g = g(t)$ is well defined since the functions $\gamma = \gamma(t)$ and $\psi = \psi(t,h)$ satisfy the properties

$$\gamma(0^+) = +\infty, \quad \gamma(+\infty) = 0, \quad \gamma'(t) < 0,\ \forall t>0, \tag{66}$$

$$\begin{aligned}&\text{(a)}\quad \psi(t,0^+) = +\infty, \quad \psi(t,+\infty) = -D<0, \quad t>0;\\ &\text{(b)}\quad \frac{\partial\psi}{\partial h}(t,h)<0, \quad \frac{\partial\psi}{\partial t}(t,h)<0, \quad t>0,\ h>0.\end{aligned} \tag{67}$$

Therefore, the function $g = g(t)$ satisfies the conditions

$$g(0^+) = +\infty, \quad g(+\infty) = \frac{kb}{Dx_0}, \quad g'(t)<0,\ \forall t>0. \tag{68}$$

By using the inequality (65) we get the thesis.

Corollary 8. We consider in the plane t, h the following set:

$$R_2 = \{(t, h) \mid h > g(t),\ t > 0\}; \tag{69}$$

then we have a two-phase problem for all $(t, h) \in R_2$.

Remark 8. If the initial temperature is given by $\phi_0(x) = b > 0$ in $[0, x_0]$, then we have a heat conduction problem for the initial phase for all $(t, h) \in R_1$, where

$$R_1 = \left\{(t, h) \mid 0 < h < \operatorname{Max}\left(\frac{kb}{Dx_0},\ F^{-1}\left(1 + \frac{b}{D}\right)\sqrt{\frac{k\rho c}{t}}\right),\ t > 0\right\}. \tag{70}$$

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FINITE-ELEMENT CONVERGENCE FOR CONTACT PROBLEMS IN PLANE LINEAR ELASTOSTATICS

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Dedicated to Professor E. Meister on the occasion of his sixtieth birthday

Abstract. This paper presents a convergence analysis for the finite-element approximation of unilateral problems in plane linear elastostatics. We consider in particular the deformation of a body unilaterally supported by a frictionless rigid foundation, solely subjected to body forces and surface tractions without being fixed along some part of its boundary, and establish convergence of piecewise polynomial finite-element approximations for mechanically definite problems without imposing any regularity assumption. Moreover we study the discretization of the contact problem with given friction along the rigid foundation.

1. Introduction. This paper presents a convergence analysis for the finite-element approximation of a class of unilateral problems in linear elastostatics, which were initiated by Signorini [17] over fifty years ago. In particular, we address the most interesting case from the view of applications and the most delicate case from the view of mathematics where the linear-elastic body which is supported by a rigid foundation is only subjected to body forces and surface tractions, but is not fixed along some part of its boundary. As shown by Fichera [4] and Stampacchia [18], the existence theory of these semicoercive Signorini problems hinges in the frictionless case upon the mechanically illustrative condition that the applied forces should form an obtuse angle with the "directions of escape," i.e., the rigid body motions defined by the geometry of the foundation. Here we prove that this condition, which renders the semicoercive Signorini problem mechanically definite, is also sufficient for the convergence of finite-element approximations with respect to the energy norm. For this convergence result no regularity assumption concerning the solution of the continuous problem is needed. Furthermore, by adapting and extending the discretization theory of Glowinski [5, 6], the finite-element approximation is not restricted to piecewise linear trial functions, but trial functions of higher polynomial degree are also included. Finally, as a first modest step towards the more realistic but difficult analysis of friction phenomena, we turn to Signorini problems with given friction along the rigid foundations and in addition investigate the discretization of the friction

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