Models of Hysteresis. Edited by A. Visintin, Longman Scientific and Technical, 1993, 221 pp., $49.95

This is volume 286 in the Pitman Research Notes in Mathematics Series. It collects the proceedings of a workshop on the subject, held in Trento, Italy, in September 1991. There are 18 papers on the mathematical aspects of hysteresis, a property exhibited by phenomena such as plasticity, ferromagnetism, ferroelectricity, undercooling effects in liquid-solid (or vapour-liquid) transitions. Since all prominent mathematicians active in the field are represented, the volume offers a state-of-the-art summary of the subject. The models discussed employ mathematical topics such as partial differential equations, stochastic differential equations, control theory, stability, parabolic variational inequalities, singular perturbations, eigenvalue analysis, differential automata, semigroups, and others.

Technological Mechanics of Porous Bodies. By B. Druyanov, Oxford University Press, 1993, xii+184 pp., $57.00

This is a volume in the series Oxford Science Publications. It aims to describe and investigate the capability of some materials to acquire irreversible volumetric deformations. An example is the compacting of a porous (powder) material in a closed mould. Some theoretical questions specific to compressible bodies, such as the external friction of plastic compressible bodies, are given special consideration. A number of methods and theoretical questions are of interest not only for compressible but also for strain-hardened materials. Isothermal and non-isothermal deformation processes are considered. Chapter headings: 1. Foundation of porous body plasticity; 2. Initial and boundary-value problems, extremum theorems, and discontinuities; 3. Equations of two-dimensional flows; 4. Compacting; 5. Reduction, extrusion, and rolling; 6. Densification at evaluated temperatures; 7. Continuum theory of rigid-phase sintering.

Partial Differential Equations. By Fritz John, Springer-Verlag, 1992, x+249 pp., $32.00

This is volume 1 in the series Applied Mathematical Sciences. It is the fourth edition of the monograph first published in 1971. A considerable amount of new material has been added to this edition. There is an extensive discussion of real analytic functions of several variables in chapter 3. Chapter 6 now includes a more detailed discussion of Hilbert spaces with applications to the boundary behaviour of solutions of the Dirichlet problem in higher dimensions. To chapter 7 there has been added a proof of Widder's theorem on nonnegative solutions of the heat equation. A new chapter, chapter 8, contains H. Lewy's construction of a linear differential equation without solutions. There are also more problems, designed, in part, to extend the material discussed in the text. Chapter headings: 1. The single first-order equation; 2. Second-order equations: hyperbolic equations for functions of two independent variables; 3. Characteristic manifolds and the Cauchy problem; 4. The Laplace equation; 5. Higher-order elliptic equations with constant coefficients; 6. Parabolic equations; 7. H. Lewy's example of a linear equation without solution.

Probability. By Alan F. Karr, Springer-Verlag, 1993, xvii+282 pp., $39.00

This is a volume in the series Springer Texts in Statistics. It is a text at the introductory graduate level. On the question whether or not to include measure theory, the author concluded "that it is intellectually imperative and pedagogically sensible to introduce and use concepts and results from measure theory, but that it is not necessary (and perhaps not even desirable) to develop and prove them individually nor to treat measure theory as a subject in its own right." Hence, many proofs are downgraded or omitted, but the material is presented. Chapter headings: Prelude: Random walks; 1. Probability; 2. Random variables; 3. Independence; 4. Expectation; 5. Convergence of sequences of random variables; 6. Characteristic functions; 7. Classical limit theorems; 8. Prediction and conditional expectation; 9. Martingales.

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THE ONE-PHASE SUPERCOOLED STEFAN PROBLEM WITH A CONVECTIVE BOUNDARY CONDITION

By

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Abstract. We consider the supercooled one-phase Stefan problem with convective boundary condition at the fixed face. We analyse the relation between the heat transfer coefficient and the possibility of continuing the solution for arbitrarily large time intervals.

1. Introduction. In this paper we study the following problem:

Problem I: Find $\theta(y,\tau)$ the temperature and $r(\tau)$ the free boundary such that

$r(\tau)$ is Lipschitz continuous for $\tau > 0$;
$\dot r(\tau)$ is continuous for $\tau > 0$;
$\theta(y,\tau)$ is continuous for $\tau > 0$ and $0 \le y \le r(\tau)$;
$\theta_\tau(y,\tau)$, $\theta_{yy}(y,\tau)$ are continuous for $\tau > 0$ and $0 < y < r(\tau)$;
$\theta_y(y,\tau)$ is continuous for $\tau > 0$, $0 \le y \le r(\tau)$;
$r(\tau)$ and $\theta(y,\tau)$ obey the conditions:

$$\theta_\tau = \alpha\theta_{yy}, \quad 0 < y < r(\tau), \quad 0 < \tau < \tau_0,$$

$$\theta(r(\tau),\tau) = 0, \quad 0 < \tau < \tau_0,$$

$$k\theta_y(r(\tau),\tau) = -\rho\lambda\dot r(\tau), \quad 0 < \tau < \tau_0,$$

$$k\theta_y(0,\tau) = h(\theta(0,\tau) - g(\tau)), \quad \tau > 0,$$

$$\theta(y,0) = \theta_0(y), \quad 0 \le y \le b,$$

$$r(0) = b.$$

The parameters are

$\alpha = \frac{k}{\rho c}$ = material thermal diffusivity (m^2/s);
k = material thermal conductivity (KJ/s^0Cm);
ρ = material density (Kg/m^3);
λ = latent heat of melting (KJ/Kg);

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h = fluid to material surface heat transfer coefficient (KJ/s^0Cm^2);
$g(\tau)$ = ambient fluid temperature (^0C);
$\rho c = C$ = specific heat $(KJ/^0CKg)$.

The melting front at time τ is $r(\tau)$ while $\theta(y,\tau)$ is the temperature at position y and time τ.

It is known that a solution to Problem I exists [1], when $\theta_0(y) \le 0$ and $g(\tau) \le 0$. This problem is often referred to as a mathematical scheme for the freezing of a supercooled liquid (although this simple scheme for such a nonequilibrium phenomenon is far from being satisfactory) [3].

The freezing of a supercooled liquid is due to convective heat transfer from a fluid with ambient temperature $g(\tau)$ flowing across the face $x = 0$. The adimensional problem is obtained by the following transforms:

$$x = \frac{y}{b}, \qquad t = \frac{k\tau}{\rho c b^2},$$

$$z(x,t) = \frac{c}{\lambda}\theta(y,\tau), \qquad s(t) = \frac{r(\tau)}{b}.$$

Then the variables (T, s, z) satisfy the problem

Problem II:

(1.1) $z_{xx} = z_t$, in D_T;
(1.2) $s(0) = 1$;
(1.3) $z(s(t),t) = 0$, $0 < t < T$;
(1.4) $z_x(s(t),t) = -\dot{s}(t)$, $0 < t < T$;
(1.5) $z(x,0) = \varphi(x)$, $0 < x < 1$;
(1.6) $z_x(0,t) = \beta[z(0,t) - G(t)]$, $0 < t < T$,

where $\beta = \frac{h}{kb}$ is an adimensional parameter, and

$$D_T = \{(x,t) \mid 0 < x < s(t),\ 0 < t < T\},$$

$$G(t) = \frac{c}{\lambda} g\left(\frac{b^2 \rho c t}{k}\right).$$

II. The one-phase supercooled Stefan problem. In this section we consider the following hypotheses:

$$\varphi(x) \le 0,\ 0 < x < 1 \quad \text{and} \quad G(t) \le 0,\ t > 0$$

and the compatibility condition

$$\varphi'(0) = \beta[\varphi(0) - G(0)].$$

The first simple properties of the solution of (1.1)–(1.6) are summarized in the following proposition:

PROPOSITION 2.1. If (T, s, z) is a solution of Problem II, then

i) $z \le 0$ in D_T.
ii) $\dot{s}(t) < 0$, $t > 0$.
iii) If $\dot{G}(t) \le 0$, $\varphi(x) \ge G(0) = \max_{t>0} G(t)$, then $z \ge G(t)$ in D_T.
iv) If $\varphi' \ge 0$, $\dot{G} \le 0$, then $z_x \ge 0$ in D_T.
v) If $\dot{G} \ge 0$, $\varphi'' > 0$, then $z_t > 0$ in D_t.

Proof. i), ii), and iv) follow from the maximum principle.

iii) follows from the minimum principle applied to $w = z - G$, where w satisfies the following equation: $w_{xx} - w_t = \dot{G}$. Then the minimum of w is on the boundary.

v) follows from the maximum principle applied to $v = z_t = z_{xx}$.

PROPOSITION 2.2. If (T, s, z) satisfy (1.1)–(1.6) of Problem II, then the following integral representations are satisfied:

$$s(t) = 1 + \int_0^1 \varphi(x)\,dx - \int_0^t z_x(0,\tau)d\tau - \int_0^{s(t)} z(x,t)\,dx, \tag{2.1}$$

$$\frac{s^2(t)}{2} = \frac{1}{2} + \int_0^1 x\varphi(x)\,dx + \int_0^t z(0,\tau)\,d\tau - \int_0^{s(t)} xz(x,t)\,dx, \tag{2.2}$$

$$\begin{aligned} s(t)\left[1 + \frac{\beta}{2}s(t)\right] &= 1 + \frac{\beta}{2} + \int_0^1 (1+\beta x)\varphi(x)\,dx + \int_0^t \beta G(\tau)\,d\tau \\ &\quad - \int_0^{s(t)} (1+\beta x) z(x,t)\,dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \frac{\beta s^4(t)}{24} - \frac{s^3(t)}{6} &= \frac{\beta}{24} + \frac{1}{6} + \int_0^1 \left(\frac{\beta x^3}{6} + \frac{x^2}{2}\right)\varphi(x)\,dx \\ &\quad - \int_0^{s(t)} \left(\frac{\beta x^3}{6} + \frac{x^2}{2}\right) z(x,t)\,dx + \iint_{D_t} z(x,t)(\beta x + 1)\,dx\,d\tau. \end{aligned} \tag{2.4}$$

Proof. Consider Green's identity

$$\iint_{D_t} (vLu - uL^*v)\,dx\,d\tau = \oint_{\partial D_t} (vu_x - uv_x)\,d\tau + uv\,dx$$

where L denotes the heat operator and L^* its adjoint and the formulae (2.1)–(2.2) are obtained by setting $u = z(x,t)$ and $v = 1$ and x respectively. (2.3) follows from (2.1) plus β times (2.2), and (2.4) is obtained by using $v = \frac{\beta x^3}{6} + \frac{x^2}{2}$. □

REMARK 1. In the following sections we denote

$$Q(t) = 1 + \frac{\beta}{2} + \int_0^1 (1+\beta x)\varphi(x)\,dx + \int_0^t \beta G(\tau)\,d\tau. \tag{2.5}$$

If $\varphi(1) = 0$, $\varphi(x)$ is Hölder continuous for $x = 1$, and $G(t)$ is piecewise continuous on every interval $(0,t)$, $t > 0$, this problem possesses one solution for suitable T "sufficiently small" (see [1], [2], [3] where uniqueness and continuous dependence are also discussed).

Moreover, if a solution exists, then three cases can occur (see [1], Theorem 8 and [2]).

(A) The problem has a solution with arbitrarily large T.
(B) There exists a constant $T_B > 0$ such that $\lim_{t\to T_B} s(t) = 0$.
(C) There exists a constant $T_C > 0$ such that $\inf_{t\in(0,T_C)} s(t) > 0$ and $\lim_{t\to T_C} \dot{s}(t) = -\infty$.

We shall investigate the occurrence of these cases in connection with the behavior of the initial data φ, the adimensional temperature G of the external fluid, and the adimensional coefficient β.

Our next aim will be to look for some conditions on φ, G, and β giving an a priori characterization of cases (A), (B), and (C).

PROPOSITION 2.3. If $\dot{G} \leq 0$, $\varphi(x) \geq G(0)$, and the solution (T, s, z) of Problem II is case (B), then $Q(T_B) = 0$.

Proof. Setting $t \to T_B$ in (2.3) and using the boundedness of z obtained in Proposition 2.1 we conclude the thesis. □

REMARK 2. $Q(t)$ is a decreasing function of time since $\dot{Q}(t) = \beta G(t) < 0$, $\forall t$.

REMARK 3. If we consider the particular case where the initial temperature $\varphi(x)$ is zero and the temperature of the external fluid $G(t) = -B < 0$ is a negative constant for all time, then $Q(t)$ is a linear function of time:

$$Q(t) = 1 + \frac{\beta}{2} - t\beta B.$$

If the solution is case (B), then the stopping time is

$$T_B = \frac{1 + \frac{\beta}{2}}{\beta B} > 0.$$

PROPOSITION 2.4. If (T, s, z) is a solution of Problem II, and the initial and boundary data satisfy the following hypotheses:

i) $\varphi(x) \geq M(x-1)$, $0 \leq x \leq 1$, $0 < M < 1$,

ii) $G(t) \geq -M$

and there exists a time T_B such that $Q(T_B) = 0$, then the solution (T_B, s, z) is case (B).

Proof. First we prove that $z(x,t) \geq M(x-1)$. This easily follows from the maximum principle applied to $w = z - M(x-1)$.

We replace this inequality in (2.3) for $t = T_B$. Then $s(T_B)$ satisfies the following inequality:

$$s(T_B)\left[(1-M) + s(T_B)\left[\frac{\beta(1-M)+M}{2}\right] + \beta s^2(T_B)\frac{M}{3}\right] \leq 0.$$

The quadratic form in brackets has coefficients $1 - M > 0$ and $\frac{\beta(1-M)+M}{2} > 0$. Then $s(T_B) = 0$. □

Following [4] we obtain:

PROPOSITION 2.5. Suppose that $t_0 < T$ and $\lim_{t\to t_0} s(t) > 0$, and suppose φ satisfies the hypotheses iv) of Proposition 2.1. Moreover, $Q(t) > 0$ for all $t \leq t_0$. Then if we define a function

$$\eta(t) = \begin{cases} \max\{x \in [0, s(t)] \mid z(x,t) \leq -1\} \\ 0 \quad \text{if } z(x,t) > -1,\ x \in [0, s(t)], \end{cases}$$

it follows that

$$\lim_{t\to t_0} \eta(t) < \lim_{t\to t_0} s(t).$$

Proof. Notice first that $\lim_{t\to t_0} s(t)$ exists because of Proposition 2.1. From iv) Proposition 2.1 we have $z(x,t) \leq -1$ in $[0, \eta(t)]$ and $-1 < z(x,t) \leq 0$ in $(\eta(t), s(t)]$ for $t < t_0$. Let $\bar{s} = \limsup_{t\to t_0} \eta(t)$ and let $\{t_n\}$ be a sequence such that $t_n \to t_0$ and $\eta_n = \eta(t_n) \to \bar{s}$. Then from iii) of Proposition 2.2

$$\begin{aligned} s(t_n)\left(1 + \frac{\beta}{2}s(t_n)\right) &= Q(t_n) - \int_0^{\eta(t_n)} (1+\beta x) z(x,t)\,dx - \int_{\eta(t_n)}^{s(t_n)} (1+\beta x) z(x,t)\,dx \\ &> Q(t_n) + \int_0^{\eta(t_n)} (1+\beta x)\,dx \\ &= Q(t_n) + \eta(t_n) + \frac{\beta}{2}\eta^2(t_n). \end{aligned}$$

Performing the limit with $\eta \to \infty$,

$$\begin{aligned} \lim_{t\to t_0}\left(s(t)\left[1 + \frac{\beta}{2}s(t)\right]\right) &\geq Q(t_0) + \lim_{t\to t_0}\left(\eta(t)\left(1 + \frac{\beta}{2}\eta(t)\right)\right) \\ &> \lim_{t\to t_0}\left(\eta(t)\left[1 + \frac{\beta}{2}\eta(t)\right]\right). \end{aligned}$$

This above inequality is equivalent to

$$\lim_{t\to t_0}\left[(s(t) - \eta(t))\left[1 + \frac{\beta}{2}(s(t) + \eta(t))\right]\right] > 0.$$

Since $1 + \frac{\beta}{2}(s + \eta) > 0$, $\forall t \leq t_0$, $\lim_{t\to t_0} s(t) - \eta(t) > 0$. □

PROPOSITION 2.6. Let (T, s, z) be a solution of Problem II such that $\varphi(x) \geq M(x-1)$, $0 \leq x \leq 1$, and $S_T = \inf_{t\in(0,T)} s(t) > 0$. If there exist two constants $d \in (0, S_T)$, $z_0 \in (0,1)$ such that $Hd \leq z_0$, and

$$z(s(t) - d, t) \geq -z_0, \quad 0 \leq t \leq T,$$

then

$$\dot{s}(t) \geq \frac{\ln(1 - z_0)}{d}.$$

Proof. It is the same as the proof of Lemma 2.4 in [2]. (See also [4].) □

PROPOSITION 2.7. Let (T, s, z) be a solution of Problem II and let φ satisfy the hypotheses of Proposition 2.1 iv). Then if the solution is case (C), $Q(T_C) \leq 0$.

Proof. Suppose $Q(T_C) > 0$. Then from Proposition 2.5 the isotherm $z = -1$ is separated from the free-boundary. Using Proposition 2.6, $\dot{s}$ has a lower bound, which contradicts the case (C). □

COROLLARY 2.8. Let (T, s, z) be a solution of Problem II and let φ, G satisfy the following hypotheses:

i) $\varphi(x) \geq M(x-1)$, $0 \leq x \leq 1$;

ii) $G(t) \geq -M$, $0 < M < 1$;

iii) $\dot{\varphi}(x) \geq 0$, $0 \leq x \leq 1$.

If the solution is case (C), then $Q(T_C) < 0$.

Proof. It follows from Propositions 2.4 and 2.7. □

PROPOSITION 2.9. Let (T, s, z) be a solution of Problem II, and let φ and G satisfy the following hypotheses:

i) $\varphi(x) \geq M(x-1)$, $M > 0$, $0 \leq x \leq 1$;

ii) $G \in L^1(0, \infty)$.

If the solution is case (A), then $Q(t) \geq 0$, $t > 0$. Moreover, if $G(t) \geq -M (M > 0)$, $\forall t > 0$, then case (A) implies that $Q(t) > 0$, $\forall t > 0$.

Proof. We suppose that the thesis is not true. Then there exists a first time T_0 such that $Q(T_0) < 0$. Since $Q(t)$ is a decreasing function, $Q(t) < 0$ for $t > T_0$. We replace this estimation in the inequality (2.3) and we obtain the following inequality:

$$-\int_0^{s(t)} (1+\beta x) z(x,t)\, dx = s(t)\left[1 + \frac{\beta}{2} s(t)\right] - Q(t) > -Q(T_0), \quad t > T_0.$$

Now we integrate the above equation with respect to time:

$$\int_{t_0}^{t} \int_0^{s(t)} (1+\beta x) z(x,\tau)\, dx\, d\tau \leq Q(T_0)(t - T_0), \quad t \geq T_0; \tag{2.6}$$

then

$$\iint_{D_t} (1+\beta x) z(x,t)\, dx\, dt \leq Q(T_0)(t - T_0), \quad t \geq T_0. \tag{2.6}$$

The following step will be to seek an inequality that contradicts (2.6).

Using Eq. (2.4) we obtain

$$\begin{aligned} \iint_{D_t} z(x,t)(\beta x + 1)\, dx\, dt &= -\int_0^1 \left(\frac{\beta x^3}{6} + \frac{x^2}{2}\right)\varphi(x)\, dx + \frac{\beta s^4(t)}{24} + \frac{s^3(t)}{6} \\ &\quad - \left(\frac{\beta}{24} + \frac{1}{6}\right) + \int_0^{s(t)} \left(\frac{\beta x^3}{6} + \frac{x^2}{2}\right) z(x,t)\, dx \qquad (2.7) \\ &\geq -\left(\frac{\beta}{24} + \frac{1}{6}\right) + \int_0^{s(t)} \left(\frac{\beta x^3}{6} - \frac{x^2}{2}\right) z(x,t)\, dx. \end{aligned}$$

From (2.3) and the hypotheses i) and ii)

$$\begin{aligned} \int_0^{s(t)} (1+\beta x) z(x,t)\, dx\, dt &= -s(t)\left[1 + \frac{\beta}{2} s(t)\right] + \frac{\beta}{2} + \int_0^1 (1+\beta x)\varphi(x)\, dx \\ &\quad + \beta \int_0^t G(\tau)\, d\tau \\ &\geq -1 - \frac{\beta}{2} + 1 + \frac{\beta}{2} + M\left[\frac{1}{2} + \frac{\beta}{3} - \frac{\beta}{2} - 1\right] - \beta \|G\|_{1,t} \\ &\geq -M\left[\frac{1}{2} + \frac{\beta}{6}\right] - \beta\|G\|_{1,t} = -C, \quad C > 0,\ C \text{ constant}, \end{aligned}$$

where

$$\|G\|_{1,t} = -\int_0^t G(\tau)\, d\tau.$$

Since $0 < x < s(t) < 1$ and $z \leq 0$ in D_t:

$$\int_0^{s(t)} z(x,t)(1+\beta x)x^2\, dx > \int_0^{s(t)} (1+\beta x) z(x,t)\, dx \geq -C.$$

Then

$$\int_0^{s(t)} \left(\frac{\beta x^3}{6} + \frac{x^2}{2}\right) z(x,t)\, dx \geq -\frac{C}{2} - \frac{C}{6} = -D, \quad D > 0. \tag{2.8}$$

We replace (2.8) in (2.7):

$$\iint_{D_t} (1+\beta x) z(x,t)\, dx\, dt \geq -\left(\frac{\beta}{24} + \frac{1}{6}\right) - D, \quad t > 0.$$

This last inequality is in contradiction with (2.6). Then $Q(t) \geq 0$, $\forall t > 0$. Moreover, if $G(t) \geq -M$ $(M > 0)$, $\forall t > 0$, the case (A) and Proposition 2.4 imply that $Q(t) > 0$, $\forall t > 0$. □

III. Asymptotic behavior of the solution.

PROPOSITION 3.1. Let (T, s, z) be a solution of Problem II of case (A) under the hypotheses of Proposition 2.9 and (iii) of Proposition 2.1. Moreover, we assume that the limit of $G(t)$ when $t \to \infty$ exists. If we denote $Q_\infty = \lim_{t\to\infty} Q(t)$ and $s_\infty = \lim_{t\to\infty} s(t)$, then s_∞ is given by

$$s_\infty = \frac{-1 + \sqrt{1 + 2\beta Q_\infty}}{\beta}. \tag{3.1}$$

Proof. The existence of the limit of $G(t)$ when $t \to \infty$ and $G \in L^1(0,\infty)$ assure that $\lim_{t\to\infty} G(t) = 0$.

We denote by z_∞ the limit of z when t tends to infinity. The existence of $\lim_{t\to\infty} z(x,t)$ is due to Proposition 2.1 and [6, Chapter 6]. The function z_∞ satisfies: $z''_\infty = 0$ in $(0, s_\infty)$, $z_\infty(s_\infty) = 0$, $z'_\infty(0) = \beta z_\infty(0)$; then $z_\infty(x) = 0$, $0 < x < s_\infty$.

Taking the limit when $t \to \infty$ in (2.3), we have

$$s_\infty\left[1 + \beta\frac{s_\infty}{2}\right] - Q_\infty = 0.$$

That means that $s_\infty \in (0,1)$ is the root of the above equation, that is, (3.1).

Moreover, we have $s_\infty < 1$ since

$$s_\infty < 1 \Leftrightarrow 1 + 2\beta Q_\infty < (1+\beta)^2 \Leftrightarrow 2Q_\infty - 2 - \beta < 0.$$

By taking the limit when $t \to \infty$ in (2.3) the last inequality always holds due to the following expression:

$$2Q_\infty - 2 - \beta = 2\int_0^1 (1 - \beta x)\varphi(x)\, dx - 2\beta\|G\|_1 < 0$$

where $\|G\|_1 = -\int_0^\infty G(\tau)\, d\tau$. □

REMARK 4. We notice that

$$s_\infty = 0 \Leftrightarrow Q_\infty = 0 \Leftrightarrow 1 + \frac{\beta}{2} + \int_0^1 (1+\beta x)\varphi(x)\, dx - \beta\|G\|_1 = 0.$$

PROPOSITION 3.2. i) For any $t > 0$ the free boundary of Problem II obeys the condition

$$s(t) \geq 1 + \int_0^1 \varphi(x)\,dx + \beta t \inf_{0 \leq \tau \leq t} G(\tau). \tag{3.2}$$

ii) If $\dot{G} \leq 0$ and $\varphi(x) \geq G(0)$, then

$$s^2(t) \geq 1 + 2\int_0^1 \varphi(x)\,dx + 2t \inf_{0 \leq \tau \leq t} G(\tau).$$

Proof. i) Using the integral representation (2.1) and i) of Proposition 2.1 we obtain (3.2). ii) It follows from the integral representation (2.2) and iii) of Proposition 2.1.

IV. The oxygen-consumption problem. As in [4] we are interested in the dependence on the temperature $G(t)$ of the external fluid at the fixed face $x = 0$. If, in Problem II, we perform the classical transformation

$$u(x,t) = \int_x^{s(t)} \left\{ \int_\gamma^{s(t)} [1 + z(\alpha,t)]\,d\alpha \right\} d\gamma$$

then we obtain the following oxygen-consumption Problem III:

$u_{xx} - u_t = 1, \quad \text{in } D_t;$
$s(0) = 1;$
$u(s(t),t) = u_x(s(t),t) = 0, \quad t > 0;$
$u(x,0) = H(x), \quad 0 \leq x \leq 1;$
$u_x(0,t) - H'(0) = \beta[u(0,t) - H(0) + \|G\|_{1,t}], \quad t > 0,$

where

$$H(x) = \int_x^1 \int_\gamma^1 (1 + \varphi(\alpha))\,d\alpha\,d\gamma.$$

From now on, in this section, we consider the following hypotheses for φ:

$$-1 < \varphi(x) \leq 0, \quad 0 \leq x \leq 1.$$

Then

$$H(x) > 0,\ 0 \leq x \leq 1; \quad H'(x) < 0,\ 0 \leq x \leq 1; \quad H''(x) > 0,\ 0 \leq x \leq 1.$$

PROPOSITION 4.1. Let (T, s, u) be a solution of Problem III with $-1 < \varphi \leq 0$ in $[0,1]$ and iii) of Proposition 2.1. Then $u(x,t) < H(x)$, $x \in (0, s(t))$, $t > 0$.

Proof. We apply the maximum principle to $W(x,t) = u(x,t) - H(x)$, which satisfies the following problem: $W_{xx} - W_t = 1 - H''(x) = -\varphi(x) > 0$, $W(x,0) = 0$, $W(s(t),t) = -H(s(t)) < 0$, and $W_x(0,t) = \beta[W(0,t) + \|G\|_{1,t}]$.

We suppose there exists a $T_0 > 0$ such that $W(0,T_0) > 0$. The point $(0,T_0)$ will be a maximum for W. Then the maximum principle implies $W_x(0,T_0) < 0$, which is a contradiction to

$$W_x(0,T_0) = \beta[W(0,T_0) + \|G\|_{1,T_0}] > 0.$$

We conclude that there does not exist such T_0. Then $W(x,t) = u(x,t) - H(x) < 0$, $\forall x \in (0,s(t)), t > 0$. ❑

COROLLARY 4.2. Let (T, s, u) be a solution of Problem III. If $G(t) > -1$, $t > 0$, then $u(x,t) \geq 0$ in D_t.

Proof. Using Proposition 2.1 iii) we obtain the following inequality:

$$u(x,t) = \int_x^{s(t)} \int_\gamma^{s(t)} 1 + z(x,t)\,d\alpha\,d\gamma \geq \int_x^{s(t)} \int_\gamma^{s(t)} 1 + G(t)\,d\alpha\,d\gamma$$
$$= (1 + G(t)) \left(\frac{s^2(t)}{2} - s(t)x + \frac{x^2}{2} \right) \geq 0. \quad \square$$

We now consider some properties related to the qualitative behavior of the free-boundary.

PROPOSITION 4.3. Let (T, s, u) be a solution of Problem III. Then s and u satisfy the following integral representations:

i) $\int_0^t s(\tau)\,d\tau = \int_0^1 H(x)\,dx - \int_0^{s(t)} u(x,t)\,dx - \int_0^t \beta[u(0,\tau) - H(0) + \|G\|_\tau] + H'(0)]\,d\tau;$
ii) $\int_0^t s^2(\tau)\,d\tau = \int_0^1 xH(x)\,dx - \int_0^{s(t)} xu(x,t)\,dx + \int_0^t u(0,\tau)\,d\tau;$
iii) $\int_0^t s(\tau)[1 + \beta s(\tau)]\,d\tau = \int_0^1 H(x)[1 + \beta x]\,dx - \int_0^{s(t)} u(x,t)[1 + \beta x]\,dx + \int_0^t [\|G\|_\tau + H'(0) - H(0)]\,d\tau.$

Proof. i) and ii) follow by applying the Green's formula used in Proposition 2.2 and iii) is obtained as a combination of i) and ii). ❑

We now address the question of how the solution to Problem III depends upon $G(t)$.

PROPOSITION 4.4. The solution (T, s, u) of Problem III depends monotonically on G. In particular, if (T_i, s_i, u_i), $i = 1, 2$, are the solutions for G_1 and G_2, respectively, and if $G_1(t) < G_2(t)$, then $s_1(t) \leq s_2(t)$ and $u_1(x,t) \leq u_2(x,t)$ however they are both defined.

Proof. This is seen by considering the difference

$$v(x,t) = u_2(x,t) - u_1(x,t)$$

at the points where they are both defined.

Let $t^* = \sup\{t > 0 \mid u_2(0,t) > u_1(0,t)\}$ and let $t^{**} = \sup\{t > 0 \mid s_2(t) > s_1(t)\}$. Let us suppose that both t^* and t^{**} are finite. By definition, v satisfies the following problem:

$$v_{xx} = v_t, \quad x \in (0, s_1(t)), \quad t \in (0, t^{**});$$
$$v(x,0) = 0;$$
$$v(s_1(t),t) = u_2(s_1(t),t) > 0;$$
$$v_x(0,t) = \beta[v(0,t) + (\|G_2\|_{1,t} - \|G_1\|_{1,t})].$$

Claim 1: $t^* \neq t^{**}$.

In order to prove that t^* and t^{**} are different from each other, let us suppose that they are equal. Then

a) $s_1(t^*) = s_2(t^*)$;
b) $\dot{s}_1(t^*) > \dot{s}_2(t^*)$;
c) $v(s_1(t^*), t^*) = u_2(s_1(t^*), t^*) = u_2(s_2(t^*), t^*) = 0.$

Moreover, $u_2(0,t) > u_1(0,t)$ for $t < t^*$. Then

$$v(0,t) > 0, \quad t < t^*$$

and

$$v(s_1(t),t) = u_2(s_1(t),t) > 0.$$

Since v has the minimum value zero at $(s_1(t^*),t^*)$, the minimum principle to v in $D^1_{t^*}$, we get $v_x(s_1(t^*),t^*) < 0$, which is a contradiction by (a) to

$$v_x(s_1(t^*),t^*) = u_{2x}(s_1(t^*),t^*) = u_{2x}(s_2(t^*),t^*) = 0.$$

Then $t^* \neq t^{**}$.

Claim 2: $t^* < t^{**}$ is impossible.

On $[0,t^*]$, $s_1(t) < s_2(t)$, whence $v(s_1(t),t) > 0$. By definition, $v(0,t) > 0$ for $t < t^*$ and $v(0,t^*) = 0$. That implies $v(0,t^*)$ is a minimum value up to time t^*, whence $v_x(0,t^*) > 0$, which contradicts

$$v_x(0,t^*) = \beta[v(0,t^*) + (\|G_2\|_{1,t^*} - \|G_1\|_{1,t^*})] = \beta[\|G_2\|_{1,t^*} - \|G_1\|_{1,t^*}] < 0.$$

Claim 3: $t^{**} < t^*$ is impossible since:

Let $t^{**} < t^*$. Since $v(0,t) > 0$, $v(s_1(t),t) = u_2(s_1(t),t) > 0$, for $t < t^{**}$, the point $(s_1(t^{**}),t^{**})$ is a minimum point for v because $v(s_1(t^{**}),t^{**}) = u_2(s_1(t^{**}),t^{**}) = u_1(s_1(t^{**}),t^{**}) = 0$.

By the corner minimum principle,

$$v_x(s_1(t^{**}),t^{**}) < 0,$$

which contradicts

$$v_x(s_1(t^{**}),t^{**}) = u_{2x}(s_2(t^{**}),t^{**}) = 0.$$

Thus the proposition is proved. □

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