Method of straight lines for a Bingham problem *

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Abstract

In this work we develop a method of straight lines for a one-dimensional Bingham problem. A Bingham fluid has viscosity properties that produce a separation into two regions, a rigid zone and a viscous zone. We propose a method of lines with the time as a discrete variable. We prove that the method is well defined, a monotone property, and a convergence theorem. Behavior of the numerical solution and numerical experiments are presented at the end of this work.

1 Introduction

We consider a fluid between two parallel plates. Using the Navier-Stokes equation for the viscous region and Newton's law for the rigid zone, we model the behavior of the system. The boundary that separates the two regions is an unknown that evolves in time. It is one of the most important unknown quantities of the problem. For weak formulations, in variational form, of free boundary problems like the Bingham problem the reader is referred to [6, 7, 11, 12, 13]. Moreover in [14] there is an extensive bibliography about these topics. In [1, 2, 9, 15] there are examples of the implementation of the method of straight lines for free boundary problems.

We recall that fluids in which the shear stress is a multiple of the shear strain called Newtonian fluids. The proportionality coefficient is the viscosity. Other fluids are known as non-Newtonian fluids. Examples of Newtonian fluids are: water, alcohol, benzene, kerosene and glycerine. Examples of non-newtonian fluids are: blood plasma, chocolate, tomato sauce, mustard, mayonnaise, toothpaste, asphalt, some greases and sewage.

Bingham fluids are non-Newtonian fluids and the relation between shear stress τ and shear strain σ is linear. That is,

$$\tau = \tau_0 + \eta\sigma. \tag{1.1}$$

where $\eta > 0$ is the viscosity and $\tau_0 > 0$ is the threshold value.

We assume that the fluid is incompressible, laminar, and with constant density ρ. Fixing the x coordinate along the direction of motion, y the perpendicular

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coordinate to the plates, and z the remaining coordinate, we make the following assumptions:

1. The pressure gradient, ∇p, is applied in only one direction, that is, $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$.

2. The fluid is laminar, that is, the velocities v and w satisfy $v = w = 0$.

3. The non-zero component of the velocity u depends only on time, t, and on the perpendicular position, y, that is, $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} = 0$.

4. There is no transport of fluid through the free boundary, $y = s(t)$. This is a condition of no deformation, that is, $u_y(s(t), t) = 0 \ \forall \ t > 0$.

5. The velocity of the fluid u at the walls of the plates is zero. This is an adherence condition.

Using the above hypotheses, we obtain a system of partial differential equation, which we call problem (P). Making a change of variables, we obtain the dimensionless system.

$$u_t - u_{yy} = f(t), \qquad s(t) < y < 1, \ t > 0, \tag{1.2}$$
$$u(1, t) = 0, \qquad t > 0, \tag{1.3}$$
$$u(y, 0) = u_0(y), \qquad s(0) = s_0, \ 0 < s_0 < y < 1, \tag{1.4}$$
$$u_y(s(t), t) = 0, \qquad t > 0, \tag{1.5}$$
$$u_t(s(t), t) = f(t) - \frac{\tau_0}{s(t)}, \qquad t > 0. \tag{1.6}$$

The problem is similar to a problem of heat transfer, where f is the pressure gradient that, according to the hypotheses, depends only on t. This system is called a free boundary problem because the function $y = s(t)$ is the boundary that separates two regions, and is part of the unknown quantities. We suppose that the pressure gradient is greater than the threshold value τ_0. This condition allows the movement between the layers of the fluid. That is,

$$f(t) > \tau_0, \quad \forall \ t > 0. \tag{1.7}$$

A more general condition can be imposed instead of a fixed boundary condition (1.3), representing that two distinct fluids are in contact. In this case we can replace (1.3) by the following equation:

$$u(1, t) = g(t), \quad t > 0. \tag{1.8}$$

It can be seen that the function g does not cause problems, because we can rewrite the system considering a new function for the pressure gradient.

We transform the problem (P) using the function $w = u_y$. The new problem (Py) satisfies the following equations:

$$w_t - w_{yy} = 0, \qquad s(t) < y < 1, \ t > 0, \tag{1.9}$$
$$w_y(1, t) = -f(t), \qquad t > 0, \tag{1.10}$$
$$w(y, 0) = u_0'(y), \qquad s(0) = s_0, \ 0 < s_0 < y < 1, \tag{1.11}$$
$$w(s(t), t) = 0, \qquad t > 0, \tag{1.12}$$
$$w_y(s(t), t) = -\frac{\tau_0}{s(t)}, \qquad t > 0. \tag{1.13}$$

2 Straight Lines Method

We discretize the time and choose a fixed time step $\Delta t > 0$. We define:

$$t_n = (n - 1)\Delta t, \quad n \in \mathbf{N}. \tag{2.1}$$

Denoting $w_n(y) = w(y, t_n)$, $f_n = f(t_n)$, $s_n = s(t_n)$ for $n \geq 1$, $q^2 = \frac{1}{\Delta t}$, and approximating time derivatives with the incremental quotient, the (P_y) system is transformed in the (Pd_y) system.

$$w_{n+1}''(y) - q^2 w_{n+1} = -q^2 w_n, \qquad s_{n+1} < y < 1, \quad n \geq 1, \tag{2.2}$$
$$w_{n+1}(s_{n+1}) = 0, \qquad n \geq 1, \tag{2.3}$$
$$w_{n+1}'(s_{n+1}) = -\frac{\tau_0}{s_{n+1}}, \qquad n \geq 1, \tag{2.4}$$
$$w_{n+1}'(1) = -f_{n+1}, \qquad n \geq 1, \tag{2.5}$$
$$w_1(y) = u_0'(y), \qquad 0 < s_1 < y < 1. \tag{2.6}$$

Observe that with this notation, $s_1 = s(t_1) = s(0) = s_0$ and $w_1(y) = w(y, t_1) = w(y, 0) = u_0'(y)$. It is easy to see that (2.2)-(2.4) is a second order differential equation of the form:

$$\begin{aligned} w'' - q^2 w = g, \quad s < y < 1, \\ w(s) = 0, \\ w'(s) = -\frac{\tau_0}{s}. \end{aligned} \tag{2.7}$$

Lemma 2.1 *The above system is satisfied by*

$$w(y) = -\frac{\tau_0}{qs} \sinh(q(y - s)) + \int_s^y \frac{g(\xi)}{q} \sinh(q(y - \xi)) d\xi, \quad s < y < 1. \tag{2.8}$$

Proof We transform this second order differential equation into a first order differential equation system, taking an auxiliary variable $v = w'$. In this way,

we obtain:

$$\begin{pmatrix} w' \\ v' \end{pmatrix} = A \begin{pmatrix} w \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad s < y < 1,$$

$$\begin{pmatrix} w \\ v \end{pmatrix}(s) = \begin{pmatrix} 0 \\ -\frac{\tau_0}{s} \end{pmatrix}.$$

where $A = \begin{pmatrix} 0 & 1 \\ q^2 & 0 \end{pmatrix}$. The matrix A is diagonalizable. In fact, if $P = \begin{pmatrix} 1 & 1 \\ q & -q \end{pmatrix}$ then $PAP^{-1} = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}$. Thus if $V = P^{-1} \begin{pmatrix} w \\ v \end{pmatrix}$ the we have the uncoupled system

$$V' = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix} V + \begin{pmatrix} \frac{g}{2q} \\ -\frac{g}{2q} \end{pmatrix}, \quad s < y < 1,$$

$$V(s) = \begin{pmatrix} -\frac{\tau_0}{2qs} \\ \frac{\tau_0}{2qs} \end{pmatrix}.$$

Solving directly the latter system, the lemma follows.   □

   Suppose that w_n and s_n are known. We extend w_n by zero in the interval $[0, s_n]$. In this way w_n is continuous in the interval $[0, 1]$. The solution of (2.2)-(2.4) for $s_{n+1} < y < 1$ is

$$w_{n+1}(y) = -\frac{\tau_0}{qs_{n+1}} \sinh(q(y - s_{n+1})) - \int_{s_{n+1}}^{y} qw_n(\xi) \sinh(q(y - \xi))d\xi. \quad (2.9)$$

   Up to now, the value of s_{n+1} is unknown. We can get s_{n+1} from the equation (2.5). Replacing in (2.9) we obtain:

$$-f_{n+1} = w'_{n+1}(1) = -\frac{\tau_0}{s_{n+1}} \cosh(q(1 - s_{n+1})) - \int_{s_{n+1}}^{1} q^2 w_n(\xi) \cosh(q(1 - \xi))d\xi.$$

Now we define

$$F_{n+1}(s) = f_{n+1} - \frac{\tau_0}{s} \cosh(q(1 - s)) - \int_{s}^{1} q^2 w_n(\xi) \cosh(q(1 - \xi))d\xi. \quad (2.10)$$

So, s_{n+1} has to be a root of F_{n+1} in the interval $(0, 1)$.

Proposition 2.2 *If $f_{n+1} > \tau_0$ then F_{n+1} has at least a root in the interval $(0, 1)$.*

Proof   It is clear that $F_{n+1}(1) = f_{n+1} - \tau_0 > 0$ and also that $F_{n+1}(s) \to -\infty$ when $s \to 0$. So there exists a root in the interval $(0, 1)$ because F_{n+1} is continuous.   □

Proposition 2.3 *Suppose that $f_{n+1} > 0$, $w_n \leq 0$, and that we have defined $s_{n+1} \in (0, 1)$ that satisfies (2.2)-(2.6). Then $w_{n+1} \leq 0$.*

Proof   If $y \in [0, s_{n+1}]$, then $w_{n+1}(y) = 0$. Let us think in the interval $[s_{n+1}, 1]$. It can be seen that w_{n+1} decreases locally around s_{n+1}, taking negative values, because of (2.3) and (2.4).

Suppose that w_{n+1} takes positive values in $[s_{n+1}, 1]$. So w_{n+1} has a root in $(s_{n+1}, 1]$. Let y_0 be the first root such that there is a change of sign. Let us take y_1 a point to the right of y_0 such that $w'_{n+1}(y_1) > 0$. The hypothesis says that $w'_{n+1}(1) < 0$. So, there exists a root y_2 of w'_{n+1} in $(y_1, 1)$, such that $w_{n+1}(y_2) > 0$ and $w''_{n+1}(y_2) \leq 0$.

Now $q^2 w_n(y_2) = q^2 w_{n+1}(y_2) - w''_{n+1}(y_2) > 0$, that is a contradiction. This concludes the proof.   □

Lemma 2.4 *If A is solution of*

$$A'' - q^2 A \geq 0, \quad 0 < s < y < 1,$$
$$A(s) < 0, \quad A(1) < 0, \tag{2.11}$$

then $A \leq 0$ in $[s, 1]$.

Proof   Suppose that there exists y_0 in $(s, 1)$ such that $A(y_0) > 0$. We can choose y_0 such that $A'(y_0) = 0$, $A(y_0) > 0$ and $A''(y_0) \leq 0$. This is a contradiction because $A''(y_0) - q^2 A(y_0) < 0$. This concludes the proof.   □

Proposition 2.5 *Suppose that $f_{n+1} > \tau_0$ and that $w'_n \leq 0$. Then $w'_{n+1} \leq 0$.*

Proof   If we define $A = w'_{n+1}$, $s = s_{n+1}$, and deriving (2.2)-(2.6), we conclude that A satisfies (2.11), that implies

$$A'' - q^2 A \geq 0, \quad 0 < s < y < 1,$$
$$A(s) = -\frac{\tau_0}{s} < 0, \quad A(1) = -f_{n+1} < 0. \tag{2.12}$$

By Lemma (2.4) we obtain that $w'_{n+1} \leq 0$. This concludes the proof.   □

Observation 2.6 The function $h(s) = 1 + sq \tanh(q(1 - s))$ is concave and strictly positive in $[0, 1]$ since

$$h''(s) = - 2q^2 \left[1 - \tanh^2(q(1 - s))\right]$$
$$- 2sq^3 \tanh(q(1 - s)) \left[1 - \tanh^2(q(1 - s))\right] < 0, \tag{2.13}$$

and $h(0) = 1 = h(1)$.

Proposition 2.7 *If $w_n \leq 0$ and $w'_n \leq 0$, the function F_{n+1} has at most a critical point, that is, there exists at most a point x_0 such that $F'_{n+1}(x_0) = 0$.*

Proof   From (2.10) we obtain:

$$F'_{n+1}(s) = \frac{\tau_0}{s^2}\cosh(q(1-s)) + \frac{\tau_0 q}{s}\sinh(q(1-s)) + q^2 w_n(s)\cosh(q(1-s)). \quad (2.14)$$

If F_{n+1} has no critical points, the proposition is proved. Suppose now that there exists at least s_* such that $F'_{n+1}(s_*) = 0$. Clearly $s_* \neq 0$ because $w_n \equiv 0$ in $[0, s_n]$. We multiply by $\frac{s_*^2}{\tau_0 \cosh(q(1-s_*))}$ and we get

$$h(s_*) + \frac{s_*^2 q^2 w_n(s_*)}{\tau_0} = 0. \quad (2.15)$$

So, the critical point s_* satisfies (2.15). Clearly the function above has a unique root because $s_*^2 q^2 w_n(s_*)/\tau_0$ is negative (and decreasing in $(s_n, 1)$). Since h is concave and positive, we deduce that there exists a unique s_* that holds $F'_{n+1}(s_*) = 0$. This concludes the proof. $\square$

Observation 2.8 From Proposition (2.2) and (2.7), we conclude that F_{n+1} has a unique root in $(0, 1)$ if in each step of time holds $w_n \leq 0$ and $w'_n \leq 0$.

Observation 2.9 The process of the algorithm is as follows: since $w_0 = u'_0 \leq 0$ and $w'_0 = u''_0 \leq 0$, using Proposition (2.3) and Proposition (2.5) we get that $w_1 \leq 0$ and $w'_1 \leq 0$; then we have a unique solution of $F_2(s) = 0$, and besides that, $w_2 \leq 0$ and $w'_2 \leq 0$; following inductively, we obtain the movement of the free boundary.

3   Properties of the Straight Lines Method

Proposition 3.1 *Suppose that $f_n > \tau_0$ for all n and that $u_0' \leq 0$. Then $w_n \leq 0$ for all $n \geq 1$.*

Proof   The hypothesis says that $w_1 = u'_0 \leq 0$. Let us assume that $w_n \leq 0$. By Proposition 2.3 we get $w_{n+1} \leq 0$. By induction the proof is concluded. $\square$

Proposition 3.2 *Suppose that $f_n > \tau_0$ for all n and that $u_0'' \leq 0$. Then $w'_n \leq 0$ for all n.*

Proof   We know that $w'_1 = u''_0 \leq 0$. Suppose that $w'_n \leq 0$. By Proposition 2.5 we get that $w'_{n+1} \leq 0$. This inductive step concludes the proof. $\square$

Proposition 3.3 *Suppose that $f_n > \tau_0$ for all n and that $u_0'' \leq 0$. Then $w_n < 0$ for all $y \in (s_n, 1]$ for all $n > 1$.*

Proof By Proposition 3.2 we know that $w'_n \leq 0 \ \forall \ n$. Because of $w_n(s_n) = 0$ and $w'_n(s_n) = -\frac{\tau_0}{s_n} < 0$. Then $w_n(y)$ is strictly decreasing in a neighborhood of s_n, then $w_n < 0$ in $(s_n, 1]$. This concludes the proof. $\square$

Proposition 3.4 *The stationary solution of Problem* (Py) *(2.2)-(2.6) is*

$$s_\infty = \frac{\tau_0}{f_\infty}, \quad w_\infty(y) = \begin{cases} -f_\infty(y - s_\infty) & \text{if } y \in [s_\infty, 1] \\ 0 & \text{if } y \in [0, s_\infty) \end{cases} \tag{3.1}$$

where $s_\infty = \lim_{n \to \infty} s_n$, $w_\infty(y) = \lim_{n \to \infty} w_n(y)$ *and* $f_\infty = \lim_{n \to \infty} f_n$, *if the limits exist.*

Proof If we take $\lim_{n \to \infty}$ in (2.2)-(2.6) we can get the system

$$\begin{aligned} w''_\infty &= 0, \quad \text{in } [s_\infty, 1], \\ w_\infty(s_\infty) &= 0, \\ w'_\infty(s_\infty) &= -\frac{\tau_0}{s_\infty}, \\ w'_\infty(1) &= -f_\infty. \end{aligned} \tag{3.2}$$

Since w'_∞ is constant, then $w'_\infty(s_\infty) = w'_\infty(1)$. That implies $s_\infty = \frac{\tau_0}{f_\infty}$. Since w_∞ is a straight line with slope $-f_\infty$ and root s_∞, then we have $w_\infty(y) = -f_\infty(y - s_\infty)$ in the interval $[s_\infty, 1]$. If $y \in [0, s_\infty)$, then there exists $N \in \mathbf{N}$ such that $y \in [0, s_n]$ for all $n \geq N$. This said $w_n(y) = 0$ for all $n \geq N$. This is equivalent to $w_\infty(y) = 0$ in $[0, s_\infty]$. This concludes the proof. $\square$

Lemma 3.5 *If* W *is a solution of*

$$\begin{aligned} W'' - q^2 W &\geq 0, \quad 0 < s < y < 1, \\ W(s) &< 0, \quad W'(1) < 0, \end{aligned} \tag{3.3}$$

then $W \leq 0$ *on* $[s, 1]$.

Proof Suppose that W takes positive values. Since $W(s) < 0$ there exists $y_0 \in (s, 1]$ such that $W(y_0) = 0$. Now we can choose $y_1 \in (y_0, 1)$ such that $W(y_1) > 0$ and $W'(y_1) > 0$. Since $W'(1) < 0$ there exists $y_2 \in (y_1, 1)$ such that $W'(y_2) = 0$. We can choose y_2 such that $W(y_2) > 0$ and $W''(y_2) \leq 0$. This is a contradiction because $W''(y_2) \geq q^2 W(y_2) > 0$. This concludes the proof. $\square$

Lemma 3.6 *If* V *is solution of*

$$\begin{aligned} V''(y) &\leq 0, \quad 0 < s < y < 1, \\ V(s) &\geq 0, \quad V'(1) \geq 0, \end{aligned} \tag{3.4}$$

then $V \geq 0$ *in* $[s, 1]$.

Proof Suppose that V assumes negative values in $(s, 1]$. There exists $y_0 \in (s, 1]$ such that $V'(y_0) < 0$. Since $V'' \leq 0$ we obtain $V'(1) < 0$, and this is a contradiction. The proof is finished. □

Theorem 3.7 *Suppose $f_{n+1} > 0$ for all n and $u_0'' \leq 0$. Then:*

(A) *If $f_{n+1} > f_n$ and $w_n \leq w_{n-1}$ then $s_{n+1} < s_n$ and $w_{n+1} \leq w_n$. Moreover, if $\{f_{n+1}\}$ is a strictly increasing sequence convergent to f_∞ then $s_{n+1} > \frac{\tau_0}{f_{n+1}}$, $s_\infty < s_{n+1}$ and $w_\infty \leq w_{n+1}$.*

(B) *If $f_{n+1} < f_n$ and $w_n \geq w_{n-1}$ then $s_{n+1} > s_n$ and $w_{n+1} \geq w_n$. Moreover, if $\{f_{n+1}\}$ is a strictly decreasing sequence convergent to f_∞ then $s_{n+1} < \frac{\tau_0}{f_{n+1}}$, $s_\infty > s_{n+1}$ and $w_\infty \geq w_{n+1}$.*

Proof From the expression of F_n in (2.10) we have

$$F_{n+1}(s) - F_n(s) = -\int_s^1 q^2 \left(w_n(\xi) - w_{n-1}(\xi)\right) \cosh\left(q(1 - \xi)\right) d\xi + (f_{n+1} - f_n).$$
(3.5)

For part (A), since $f_{n+1} > f_n$ and $w_n \leq w_{n-1}$, we see from (3.5) that $F_{n+1}(s) - F_n(s) > 0$ for all s. From this we deduce that $s_{n+1} < s_n$.

Let $W = w_{n+1} - w_n$. Then from (2.2)-(2.6), we see that on $[s_n, 1]$, W satisfies

$$W'' - q^2 W = -q^2(w_n - w_{n-1}) \geq 0,$$
$$W(s_n) = w_{n+1}(s_n) < 0,$$
$$W'(1) = -(f_{n+1} - f_n) < 0.$$

Therefore, on the interval $[s_n, 1]$, W satisfies

$$W'' - q^2 W \geq 0, \quad W(s_n) < 0, \quad W'(1) < 0. \tag{3.6}$$

and out this interval, W satisfies

$$W(y) = \begin{cases} 0, & \text{if } y \in [0, s_{n+1}], \\ w_{n+1}(y), & \text{if } y \in [s_{n+1}, s_n], \end{cases}$$

Knowing that the w_{n+1} are negative functions on (s_{n+1}, s_n) (Proposition 3.3), and using the Lemma 3.5, we observe that $W \leq 0$ on $[s_{n+1}, 1]$ finally $W \leq 0$ on $[0, 1]$, and this is equivalent to $w_{n+1} \leq w_n$.

When we integrate the equation (2.2) from s_{n+1} to 1, using (2.4),(2.5) we obtain:

$$\int_{s_{n+1}}^1 w_{n+1}''(\xi) d\xi = \int_{s_{n+1}}^1 q^2 \left(w_{n+1}(\xi) - w_n(\xi)\right) d\xi,$$

$$w_{n+1}'(1) - w_{n+1}'(s_{n+1}) = q^2 \int_{s_{n+1}}^{s_n} w_{n+1}(\xi) d\xi + \int_{s_n}^1 q^2 \left(w_{n+1} - w_n\right)(\xi) d\xi < 0,$$

$$-f_{n+1} + \frac{\tau_0}{s_{n+1}} < 0.$$

From this we deduce that $s_{n+1} > \tau_0/f_{n+1}$. Since $f_{n+1} < f_\infty$ and $s_{n+1} > \tau_0/f_{n+1} > \tau_0/f_\infty = s_\infty$, it followsthat $s_{n+1} > s_\infty$.

Let $V_{n+1} = w_{n+1} - w_\infty$ be. Using (2.2) and the Proposition (3.4), the following equation holds on $(s_{n+1}, 1)$,

$$V''_{n+1} - q^2 V_{n+1} = -q^2 V_n. \tag{3.7}$$

Then is clear that

$$V''_{n+1} = q^2(w_{n+1} - w_n) \le 0, \quad \text{in } (s_{n+1}, 1),$$
$$V_{n+1}(s_{n+1}) = w_{n+1}(s_{n+1}) - w_\infty(s_{n+1}) > 0,$$
$$V'_{n+1}(s_{n+1}) = -\frac{\tau_0}{s_{n+1}} + \frac{\tau_0}{s_\infty} > 0,$$
$$V'_{n+1}(1) = -f_{n+1} + f_\infty > 0.$$

Therefore, on the interval $[0, s_\infty]$, V satisfies

$$V''_{n+1} \le 0, \quad V_{n+1}(s_{n+1}) > 0, \quad V'_{n+1}(s_{n+1}) > 0, \quad V'_{n+1}(1) > 0. \tag{3.8}$$

and out of this interval, V satisfies

$$V_{n+1}(y) = \begin{cases} 0, & \text{if } y \in [0, s_\infty], \\ -w_\infty(y), & \text{if } y \in [s_\infty, s_{n+1}]. \end{cases}$$

Because of Lemma 3.6 we get that $V_{n+1} \ge 0$ on $[0, 1]$, and this is equivalent to $w_{n+1} \ge w_\infty$

The proof of part (B) is similar to (A) and we omit it.   □

Theorem 3.8 *Suppose that* $\lim_{n\to\infty} f_n = f_\infty$, $f_n > 0 \; \forall \; n$, $u'_0 \le 0$, $u''_0 \le 0$, $\lim_{n\to\infty} s_n = s^*$ *and* $\lim_{n\to\infty} w_n = w^*$. *Then* $s^* = s_\infty$ *and* $w^* = w_\infty$.

Proof   From Theorem 3.7 we can take $\lim_{n\to\infty}$ in (2.9) and we obtain:

$$w^* = -\frac{\tau_0}{qs^*} \sinh(q(y - s^*)) - \int_{s^*}^y qw^*(\xi) \sinh(q(y - \xi))d\xi. \tag{3.9}$$

Computing the derivatives of the function w^* we get that

$$w^{*''}(y) = 0, \quad w^{*'}(s^*) = -\frac{\tau_0}{s^*}, \quad w^*(s^*) = 0. \tag{3.10}$$

On the other hand, if we differentiate (2.9) for $s_{n+1} < y < 1$, we have

$$w'_{n+1}(y) = -\frac{\tau_0}{s_{n+1}} \cosh(q(y - s_{n+1})) - \int_{s_{n+1}}^y q^2 w_n(\xi) \cosh(q(y - \xi))d\xi. \tag{3.11}$$

Taking $\lim_{n\to\infty}$ we get:

$$\lim_{n\to\infty} w'_{n+1}(y) = -\frac{\tau_0}{s^*}\cosh(q(y-s^*)) - \int_{s^*}^y q^2 w^*(\xi)\cosh(q(y-\xi))d\xi. \quad (3.12)$$

From (3.9) and (3.16) we obtain that $\lim_{n\to\infty} w'_n(y) = w^{*'}(y)$. Then $w^{*'}(1) = \lim_{n\to\infty} w'_{n+1}(1) = -\lim_{n\to\infty} f_{n+1} = -f_\infty$. Since $w^{*'}$ is constant, we deduce that $w^{*'}(1) = w^{*'}(s^*)$. Then

$$s^* = \frac{\tau_0}{f_\infty}, \quad w^* = -\frac{\tau_0}{s^*}(y-s^*).$$

A comparison with Proposition 3.4 completes this proof. $\qquad\square$

Observation 3.9 Under the hypotheses of Theorem 3.7, the solutions s_n and w_n converge to the stationary solutions, s_∞, w_∞.
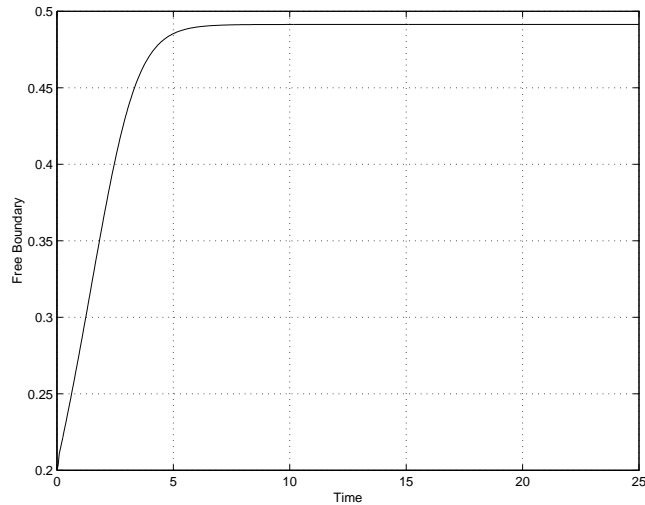

Figure 1: Solution with $s_0 = 0.2$, $\tau_0 = 1$, $\Delta t = 0.05$, $f(t) = 2$

Numerical Results

The algorithm for the following results was programmed in Fortran. First we compute the root of F_{n+1}, and then we compute w_{n+1} from (2.9). The functions w_n are stored as splines functions and the integrals are computed by the Simpson's Rule. The numerical experiments are shown in Figures 1, 2, and 3. They show that the algorithm reproduces the theoretical behavior of the solution (see [3]).
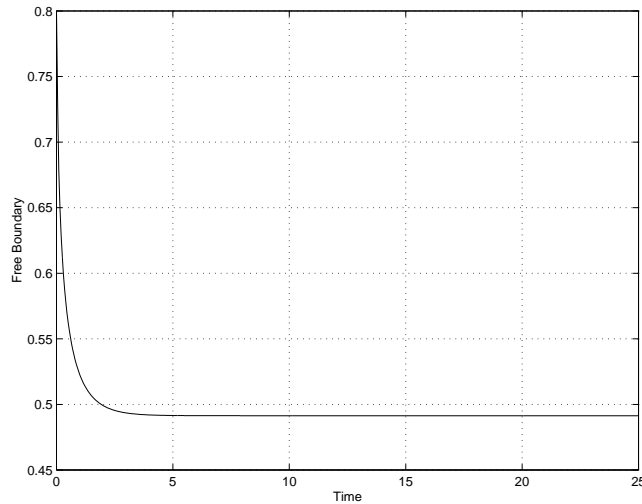
Figure 2: Solution with $s_0 = 0.8$, $\tau_0 = 1$, $\Delta t = 0.05$, $f(t) = 2$.

Concluding Remarks

For the discrete solution of $(2.2) - (2.6)$, we have obtained the same properties that the continuous solution of Problem (Py) satisfies (see [3]). That is $w_n(y) < 0$, $w_n'(y) < 0$, $\{s_n\}_n$ is monotone if $\{f_n\}_n$ is monotone, the stationary solution for the discrete problem (which agrees with the stationary solution for the continuous problem) is established, and the discrete solution converges to the stationary solution. Moreover the algorithm is well defined for all f_n that satisfy $f_n > \tau_0$. This condition is the corresponds to the motion between the layers of the fluid.

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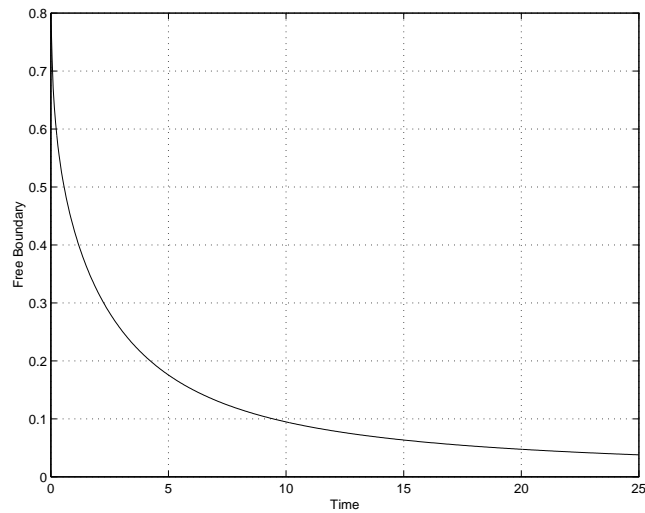[4] CRANK J., *Free and Moving Boundary Problems*, Claredon Press, Oxford (1984).

Figure 3: Solution with $s_0 = 0.8$, $\tau_0 = 1$, $\Delta t = 0.05$, $f(t) = 2 + t$

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