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Method of straight lines for a Bingham problem in cylindrical pipes

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Abstract

In this work a method of straight lines for a Bingham problem in cylindrical geometry is developed. A Bingham fluid has viscosity properties that produce a separation into two regions, a rigid zone and a viscous zone. We propose a method of lines with the time as a discrete variable. We prove that the method is well defined for all times, a monotone property, qualitative behaviours of the solution, and a convergence theorem. A numerical calculation is included to illustrate the theoretical results.

Keywords: Bingham fluid; Straight lines; Non-Newtonian fluids

1. Introduction

We consider a fluid on a cylindrical pipe as shown in (1). Using the Navier-Stokes equation for the viscous region and Newton's law for the rigid zone, we model the behavior of the system. The boundary that separates the two regions is an unknown that evolves in time. It is one of the most important unknown quantities of the problem. For weak formulations in variational form of free boundary problems like the Bingham problem the reader is referred to [4,5,7–9]. Moreover, in [10] there is an extensive bibliography about these topics. In [1,2,6,12,11] there are examples of the implementation of the method of straight lines for free boundary problems.

We recall that fluids in which the shear stress is a multiple of the shear strain are called Newtonian fluids. The proportionality coefficient is the viscosity. Other fluids are known as non-Newtonian fluids. Examples of Newtonian fluids are: water, alcohol, benzene, kerosene and glycerine. Examples of non-

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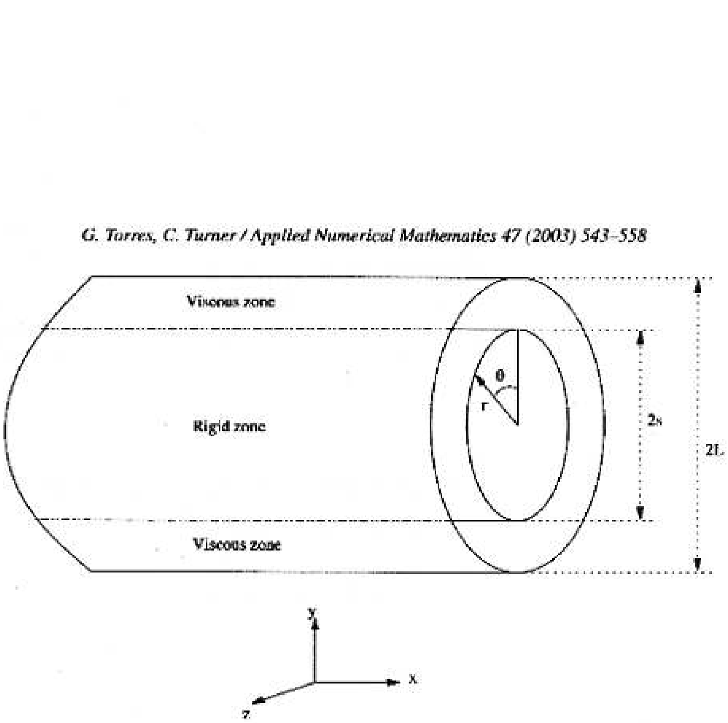

Fig. 1. Cylindrical pipe.

Newtonian fluids are: blood plasma, chocolate, tomato sauce, mustard, mayonnaise, toothpaste, asphalt, some greases and sewage.

Bingham fluids are non-Newtonian fluids and the relation between shear stress τ and shear strain σ is linear. That is,

$$\tau = \tau_0 + \eta\sigma, \tag{1}$$

where $\eta > 0$ is the viscosity and $\tau_0 > 0$ is the threshold value.

We assume that the fluid is incompressible, laminar, and with constant density ρ. Fixing the x coordinate along the direction of motion, y the perpendicular coordinate upwards, and z the remaining coordinate, we make the following assumptions:

(1) The pressure gradient, ∇p, is applied in only one direction, that is, $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$.
(2) The fluid is laminar, that is, the velocities v and w satisfy $v = w = 0$.
(3) The non-zero component of the velocity u depends only on time, t, and on the radius, r, that is, $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} = 0$.
(4) There is no transport of fluid through the free boundary, $r = s(t)$. This is a condition of no deformation, that is, $u_r(s(t), t) = 0\ \forall t > 0$.
(5) The velocity of the fluid u at the walls of the pipe is zero. This is an adherence condition.

Using the above hypotheses, we obtain a system of partial differential equation, which we call problem (P). Making a change of variables, we obtain the dimensionless system in cylindrical coordinates

$$u_t - u_{rr} - \frac{1}{r}u_r = f(t), \quad s(t) < r < 1,\ t > 0, \tag{2}$$

$$u(1, t) = 0, \quad t > 0, \tag{3}$$

$$u_r\big(s(t), t\big) = 0, \quad t > 0, \tag{4}$$

$$u_t(s(t),t) = f(t) - \frac{2\tau_0}{s(t)}, \quad t > 0, \tag{5}$$

$$u(r,0) = u_0(r), \quad s(0) = s_0, \ 0 < s_0 < r < 1. \tag{6}$$

The problem is similar to one of heat transfer, where f is the opposite of the pressure gradient that, according to the hypotheses, depends only on t. This system is called a free boundary problem because the function $r = s(t)$ is the boundary that separates two regions, and is part of the unknown quantities. We suppose that the pressure gradient is greater than the value $2\tau_0$. This will be called the *operability condition*. This condition allows the movement between the layers of the fluid. That is,

$$f(t) > 2\tau_0 \quad \forall t > 0. \tag{7}$$

We transform the problem (P) using the function $w = u_r$. The new problem (P_r) satisfies the following equations:

$$w_t - w_{rr} - \frac{1}{r}w_r + \frac{1}{r^2}w = 0, \quad s(t) < r < 1, \ t > 0, \tag{8}$$

$$w_r(1,t) + w(1,t) = -f(t), \quad t > 0, \tag{9}$$

$$w(s(t),t) = 0, \quad t > 0, \tag{10}$$

$$w_r(s(t),t) = -\frac{2\tau_0}{s(t)}, \quad t > 0, \tag{11}$$

$$w(r,0) = u_0'(r), \quad s(0) = s_0, \ 0 < s_0 < r < 1. \tag{12}$$

Notice that the original function u can be recovered from w:

$$u(r,t) = -\int_r^1 w(\xi,t)\,\mathrm{d}\xi, \quad s(t) \leqslant r \leqslant 1, \ t > 0. \tag{13}$$

In consequence, it is enough to solve the problem for w. Besides that, the implementation of the method will be easier using the problem (P_r) since the time derivative has been eliminated from the boundary conditions.

In Section 2 we prove some technical lemmas that we will use in Section 3 to show the well-posedness of the method and the convergence of the solution to the stationary solution of the problem.

2. Preliminary results

Lemma 1. *Let Z be a function that satisfies*

$$\begin{cases} Z_{rr} + \frac{1}{r}Z_r - \frac{1}{r^2}Z = 0, & s < r < 1, \\ Z_r(1) + Z(1) = -f, \\ Z(s) = 0, \\ Z'(s) = -\frac{2\tau_0}{s}, \end{cases} \tag{14}$$

where s, f and τ_0 are positive numbers. Then

$$s = \frac{2\tau_0}{f}, \qquad Z(r) = -\frac{f}{2}\left(r - \frac{s^2}{r}\right), \quad r \in [s, 1]. \tag{15}$$

Proof. We define $Y = Z_r + \frac{1}{r}Z$. Then, Y satisfies an ordinary differential equation, namely,

$$\begin{cases} Y_r = 0, & r \in (s, 1), \\ Y(1) = -f, \\ Y(s) = -\frac{2\tau_0}{s}. \end{cases} \tag{16}$$

Solving this equation the lemma follows. □

Remark 2. Notice that the previous lemma solves the stationary solution of (8)–(12). We will denote the components of the solution by s_∞ and w_∞:

$$s_\infty = \frac{2\tau_0}{f_\infty}, \qquad w_\infty(r) = -\frac{f_\infty}{2}\left(r - \frac{s_\infty^2}{r}\right), \quad r \in [s_\infty, 1]. \tag{17}$$

Lemma 3. *The ordinary differential equation*

$$x^2 y'' + xy' - (1 + x^2)y = 0, \tag{18}$$

has two independent and analytic solutions y_1 and y_2, that converge for $x > 0$.

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}, \tag{19}$$

$$y_2(x) = -\frac{1}{4} y_1(x) \ln(x) + \sum_{n=1}^{\infty} c_n x^{n-1}, \tag{20}$$

where

$$\begin{cases} a_0 = 1, \\ a_1 = 0, \\ a_n = \frac{a_{n-2}}{n(n+2)}, & n \geqslant 2. \end{cases} \tag{21}$$

$$\begin{cases} c_0 = -\frac{1}{2}, \\ c_1 = 0, \\ c_2 = -\frac{1}{16}, \\ c_{n+2} = \frac{[c_n + \frac{1}{2}(n+1)a_n]}{n(n+2)}, & n \geqslant 1. \end{cases} \tag{22}$$

Proof. See [3]. □

Lemma 4. *The ordinary differential equation*

$$x^2 y'' + xy' - (1 + x^2)y = h, \tag{23}$$

has a general solution given by

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &\quad - y_1(x) \int \frac{y_2(x)h(x)}{x^2(y_1 y_2' - y_1' y_2)(x)}\,dx + y_2(x) \int \frac{y_1(x)h(x)}{x^2(y_1 y_2' - y_1' y_2)(x)}\,dx, \end{aligned} \tag{24}$$

where $y_1(x)$ and $y_2(x)$ are the fundamental solutions of (18) *mentioned in Lemma* 3.

Proof. It is a well-known fact that the general solution for (23) is a linear combination of the fundamental solutions of the homogeneous equation plus a particular solution. The last two terms in (24) correspond to the particular solution, which is found by the method of variation of parameters. □

Lemma 5. *Let w be a function that satisfies*

$$\begin{cases} w'' + \frac{1}{r}w' - \left(\frac{1}{r^2} + q^2\right)w = g, & s < r, \\ w(s) = 0, \\ w'(s) = -\frac{2\tau_0}{s}, \end{cases} \tag{25}$$

where g is a function, and s, q, and τ_0 are positive numbers. The solution is

$$\begin{aligned} w(r) = {} & \frac{2\tau_0 y_2(sq)}{sq(y_1y_2' - y_1'y_2)(sq)} y_1(rq) - \frac{2\tau_0 y_1(sq)}{sq(y_1y_2' - y_1'y_2)(x)} y_2(rq) \\ & + y_2(rq)\int_s^r \frac{y_1(\mu q)g(\mu)}{q(y_1y_2' - y_1'y_2)(\mu q)}\,\mathrm{d}\mu - y_1(rq)\int_s^r \frac{y_2(\mu q)g(\mu)}{q(y_1y_2' - y_1'y_2)(\mu q)}\,\mathrm{d}\mu, \end{aligned} \tag{26}$$

where y_1 and y_2 are the fundamental solutions of (18).

Proof. Making a change of variable ($x = rq$) we obtain an equation like (23) and this equation can be solved using Lemma 4. The coefficients c_1 and c_2 in (24) are obtained using the boundary conditions and solving a linear system. □

Lemma 6. *Let Z be a function that satisfies*

$$\begin{cases} s^2Z'' + sZ' - gZ \leqslant 0, & s \in (0,1), \\ \lim_{s\to\infty} Z(s) = +\infty, \\ Z(1) > 0, \end{cases} \tag{27}$$

where g is a positive function in $(0,1)$. *Then* $Z \geqslant 0$ *in* $[0,1]$. *Besides that, if we replace the* sign $\leqslant$ *by the* sign $<$ *in the differential equation, it holds that* $Z > 0$ *in* $[0,1]$.

Proof. Suppose that there exists $\bar{s} \in [0,1]$ such that $Z(\bar{s}) < 0$. Then $\bar{s} \neq 0$ and $\bar{s} \neq 1$. Consequently, there exists $\bar{s} \in (0,1)$ a negative local minimum in $(0,1)$, that is, $Z(\bar{s}) < 0$, $Z'(\bar{s}) = 0$ and $Z''(\bar{s}) \geqslant 0$. Evaluating the differential equation we have

$$0 \geqslant \bar{s}^2 Z''(\bar{s}) + \bar{s}Z'(\bar{s}) - g(\bar{s})Z(\bar{s}) > 0, \tag{28}$$

that is a contradiction. If we consider a strict inequality in the differential equation, suppose that there exists $\bar{s} \in [0,1]$ such that $Z(\bar{s}) \leqslant 0$ and repeat the previous steps. This concludes the proof. □

Definition 7. Let y_1 and y_2 be the fundamental solutions of (18) and q a positive number. We define:

(1) $W(s) \doteq (y_1y_2' - y_1'y_2)(s)$. This is called the Wronskian of y_1 and y_2.
(2) $\alpha \doteq qy_2'(q) + y_2(q)$.
(3) $\beta \doteq -(qy_1'(q) + y_1(q))$.
(4) $A(s) \doteq \alpha y_1(sq) + \beta y_2(sq)$.

Lemma 8. *The following properties are true*:

(A) $\beta < 0$.
(B) $W(s) > 0\ \forall s \in [0, 1]$.
(C) $A(s) > 0\ \forall s \in [0, 1]$.
(D) $A'(s) < 0\ \forall s \in [0, 1]$.

Proof. (A) It is a consequence of (19) and (21).

(B) Abel's theorem [3] says that the Wronskian is a function that never vanishes. It can be proved that

$$W(x) = \frac{1}{x} + \mathrm{O}(x) + \mathrm{O}(x)\ln x. \tag{29}$$

It is clear that W is a positive function.

(C) The function A satisfies

$$\begin{cases} s^2A'' + sA' - \left(1 + s^2q^2\right)A = 0, & 0 < s < 1, \\ \lim_{s\to 0^+} A(s) = +\infty, \\ A(1) = qW(q) > 0. \end{cases} \tag{30}$$

The differential equation is obvious because A is a linear combination of fundamental solutions. The last two equations are obtained from the definition of A, the asymptotic behaviour of y_1 and y_2, and from (29). By Lemma 6 we deduce that $A \geqslant 0$ in $[0, 1]$.

Suppose now that there exists $\bar{s}$ in $[0, 1]$ such that $A(\bar{s}) = 0$. The boundary conditions tell us that $\bar{s}$ is in $(0, 1)$. We deduce that $\bar{s}$ is a local minimum, so it holds that $A(\bar{s}) = A'(\bar{s}) = 0$. The point $\bar{s}$ have to be an isolated root of A. If not, $A = 0$ in a neighborhood I of $\bar{s}$, and this cannot be because y_1 and y_2 are independent solutions. Therefore, there exists $n \geqslant 2$ such that

$$A^{(n)}(\bar{s}) \neq 0, \quad A^{(j)}(\bar{s}) = 0, \quad j = 0, \ldots, n-1. \tag{31}$$

Differentiating the first equation of (30) we obtain

$$\bar{s}^2A^{(n)}(\bar{s}) + \sum_{j=0}^{n-1}(\text{coefficient})_j A^{(j)}(\bar{s}) = 0. \tag{32}$$

We conclude that $A^{(n)}(\bar{s}) = 0$ and this is a contradiction because of (31). Finally, $A(s) > 0$ for all s in $[0, 1]$.

(D) The function $B = A'$ satisfies

$$\begin{cases} s^2B'' + 3sB' - s^2q^2B > 0, & s \in (0, 1), \\ \lim_{s\to 0^+} B(s) = -\infty, \\ B(1) = -qW(q) < 0. \end{cases} \tag{33}$$

This equation is obtained by taking the first derivative in (30) and analyzing the asymptotic behaviour of y_1' and y_2' when s tends to 0^+. By Lemma 6 applied to $-B$ and using the strict inequality in the lemma, we conclude that $B < 0$ in $[0, 1]$, that is equivalent to $A' < 0$ in $[0, 1]$. □

Definition 9. Let q be a positive number. We define

$$H(s) = -\frac{A'(s)}{A(s)}, \tag{34}$$

$$K(s) = \frac{A(s)}{qW(sq)}, \tag{35}$$

$$J(s) = sH(s). \tag{36}$$

Lemma 10. *The following properties hold*:

(A) $K > 0$.
(B) $K(1) = 1$.
(C) $\lim_{s\to 0^+} \frac{1}{s}K(s) = \infty$.
(D) $H > 0$.
(E) $H(1) = 1$.
(F) $\frac{K'}{K} - \frac{1}{s} = \frac{A'}{A}$.
(G) $J(1) = 1$.
(H) $\lim_{s\to 0^+} J(s) = 1$.
(I) J *has exactly a critical point in* $(0, 1)$ *and it is a local maximum.*
(J) $J \geqslant 1$ *in* $[0, 1]$.

Proof. (A) and (D) are a consequence of Lemma 8. (B) and (E) are deduced from (30) and (33).

(C) Analyzing the asymptotic behavior of $\frac{1}{s}K(s)$ we obtain:

$$\frac{1}{s}K(s) = \alpha\frac{sq + O(s^3q^3)}{1 + O(s^2q^2) + O(s^2q^2)\ln(sq)} - \beta\frac{1 + O(s^2q^2) + O(s^2q^2)\ln(sq)}{2sq + O(s^3q^3) + O(s^3q^3)\ln(sq)}. \tag{37}$$

It is clear that (C) follows.

(F) We know that $W' + \frac{1}{s}W = 0$ (Abel's theorem). Let us differentiate K

$$K'(s) = \frac{A'(s)}{A(s)}K(s) - \frac{A(s)}{W(sq)}\left(-\frac{1}{sq}\right) = K(s)\left(\frac{A'(s)}{A(s)} + \frac{1}{s}\right). \tag{38}$$

Divide by K and the result is obtained.

(G) See (E) and the definition of J.

(H) The asymptotic behavior of sH is:

$$sH(s) = \frac{1 + O(s^2q^2) + O(s^2q^2)\ln(sq)}{1 + O(s^2q^3) + O(s^2q^3)\ln(sq)}. \tag{39}$$

It is clear that (H) follows.

(I) From (30) we have

$$-\frac{A''}{A} = \frac{1}{s}\frac{A'}{A} - \left(\frac{1}{s^2} + q^2\right) = -\frac{1}{s}H - \left(\frac{1}{s^2} + q^2\right). \tag{40}$$

Now,

$$H' = -\frac{A''}{A} + \left(\frac{A'}{A}\right)^2 = -\frac{1}{s}H + H^2 - \left(\frac{1}{s^2} + q^2\right). \tag{41}$$

Taking into account that $J' = H + sH'$ and (41) we have that

$$J^2 = sJ' + (1 + s^2q^2). \tag{42}$$

Manipulating (42) and differentiating we have,

$$s^2 J'' = -J^2 + 2sJJ' + 1 - s^2q^2. \tag{43}$$

Every critical point $\bar{s} \in (0, 1)$ of J holds $J''(\bar{s}) < 0$. Suppose that $J'(\bar{s}) = 0$ and $J''(\bar{s}) \geqslant 0$. Using (42) we see that $J^2(\bar{s}) = 1 + \bar{s}^2q^2$. By (43)

$$0 \leqslant \bar{s}^2 J''(\bar{s}) = -J^2(\bar{s}) + 1 - s^2q^2 = -2\bar{s}^2q^2 < 0. \tag{44}$$

And this is a contradiction. Besides that, if there exists a critical point (in fact at least a critical point exists because $J(0) = J(1) = 1$), then it has to be an isolated local maximum.

There exists exactly a critical point (that is a local maximum). Let us take $s_1 < s_2$ two consecutive critical point. Both s_1 and s_2 are local maximums. So it holds $J''(s_1) < 0$ and $J''(s_2) < 0$. Take δ positive and sufficiently small such that $J'(s_1 + \delta) < 0$ and $J'(s_2 - \delta) > 0$. Then, there exists $s_3 \in (s_1, s_2)$ a critical point of J. But this is a contradiction because s_1 and s_2 were consecutive.

(J) By (I) and the fact that $J(0) = J(1) = 1$ the inequality follows. □

Lemma 11. *Let Z be a function that satisfies*

$$\begin{cases} Z'' + \frac{1}{r}Z' - gZ \geqslant 0, & s < r < 1, \\ Z(1) < 0, \\ Z(s) < 0, \end{cases} \tag{45}$$

where g is a positive function and s is a positive number. Then $Z \leqslant 0$ in $[s, 1]$.

Proof. Suppose that there exists a point $\bar{s} \in [s, 1]$ such that $Z(\bar{s}) > 0$ ($\bar{s} \neq s$ and $\bar{s} \neq 1$ because of the boundary conditions). Then there exists $\bar{s} \in (s, 1)$ a positive local maximum, that is, $Z(\bar{s}) > 0$, $Z'(\bar{s}) = 0$ and $Z''(\bar{s}) \leqslant 0$. Evaluating in the differential equation we have,

$$0 \leqslant Z''(\bar{s}) + \frac{1}{\bar{s}}Z'(\bar{s}) - g(\bar{s})Z(\bar{s}) < 0, \tag{46}$$

and this is a contradiction. □

Lemma 12. *Let Z be a function that satisfies*

$$\begin{cases} Z'' + \frac{1}{r}Z' - gZ \geqslant 0, & s < r < 1, \\ Z(s) \leqslant 0, \\ Z'(1) + Z(1) \leqslant 0, \end{cases} \tag{47}$$

where g is a positive function and s is a positive number. Then $Z \leqslant 0$ in $[s, 1]$.

Proof. Suppose that there exists $\bar{s}$ in $[s, 1]$ such that $Z(\bar{s}) > 0$. It is clear that $\bar{s} \neq s$. Also it can be proved that $Z(1) > 0$. If $Z(1) \leqslant 0$, then there exists $\bar{s}$ in $(s, 1)$ such that $\bar{s}$ is a positive maximum. If we evaluate the differential equation in $\bar{s}$ we will get a contradiction. Because of the boundary condition in $s = 1$ we conclude that $Z'(1) < 0$, and this implies that the maximum of Z is in $(s, 1)$. As before, we reach a contradiction. The proof is finished. □

3. Method of the straight lines

We discretize the time and choose a fixed time step $\Delta t > 0$. We define:

$$t_n = (n-1)\Delta t, \quad n \in \mathbb{N}, \tag{48}$$

$$s_n = s(t_n), \quad n \in \mathbb{N}, \tag{49}$$

$$f_n = f(t_n), \quad n \in \mathbb{N}, \tag{50}$$

$$w_n(r) = w(r, t_n), \quad n \in \mathbb{N}, \tag{51}$$

$$q = \sqrt{\frac{1}{\Delta t}}. \tag{52}$$

Approximating time derivatives with the incremental quotient, the (P_r) system is transformed into another system, called (Pd_r). For all n in $\mathbb{N}$:

$$w''_{n+1} + \frac{1}{r}w'_{n+1} - \left(\frac{1}{r^2} + q^2\right)w_{n+1} = -q^2 w_n, \quad r \in (s_{n+1}, 1), \tag{53}$$

$$w'_{n+1}(1) + w_{n+1}(1) = -f_{n+1}, \tag{54}$$

$$w_{n+1}(s_{n+1}) = 0, \tag{55}$$

$$w'_{n+1}(s_{n+1}) = -\frac{2\tau_0}{s_{n+1}}, \tag{56}$$

$$w_1 = u'_0, \qquad s_1 = s_0, \quad 0 < s_1 < r < 1. \tag{57}$$

Gathering (53), (55) and (56), and with the help of Lemma 5, it is clear that

$$\begin{aligned} w_{n+1}(r) = {} & \frac{2\tau_0 y_2(s_{n+1}q)}{s_{n+1}q W(s_{n+1}q)} y_1(rq) - \frac{2\tau_0 y_1(s_{n+1}q)}{s_{n+1}q W(s_{n+1}q)} y_2(rq) \\ & - y_2(rq) \int_{s_{n+1}}^{r} \frac{q^2 y_1(\mu q) w_n(\mu)}{q W(\mu q)} \,\mathrm{d}\mu + y_1(rq) \int_{s_{n+1}}^{r} \frac{q^2 y_2(\mu q) w_n(\mu)}{q W(\mu q)} \,\mathrm{d}\mu, \end{aligned} \tag{58}$$

and

$$\begin{aligned} w'_{n+1}(r) = {} & \frac{2\tau_0 y_2(s_{n+1}q)}{s_{n+1} W(s_{n+1}q)} y'_1(rq) - \frac{2\tau_0 y_1(s_{n+1}q)}{s_{n+1} W(s_{n+1}q)} y'_2(rq) \\ & - y'_2(rq) \int_{s_{n+1}}^{r} \frac{q^2 y_1(\mu q) w_n(\mu)}{W(\mu q)} \,\mathrm{d}\mu + y'_1(rq) \int_{s_{n+1}}^{r} \frac{q^2 y_2(\mu q) w_n(\mu)}{W(\mu q)} \,\mathrm{d}\mu. \end{aligned} \tag{59}$$

Theorem 13. *If the initial data u_0 satisfies*

$$\begin{cases} u'_0 \leqslant 0, & \text{in } [s_0, 1] \\ u''_0 + \frac{1}{r}u'_0 \leqslant 0, & \text{in } [s_0, 1] \end{cases} \tag{60}$$

and the operability condition (7) holds, then:

(1) *the system (P_r) admits a unique solution for all n in $\mathbb{N}$.*
(2) *$w_n \leqslant 0$, in $[0, 1]$, for all n in $\mathbb{N}$.*
(3) *$w'_n + \frac{1}{r}w_n \leqslant 0$, in $[s_n, 1]$, for all n in $\mathbb{N}$.*

Proof. We will proof (1), (2) and (3) at the same time by induction. We extend u_0 by zero in the interval $[0, s_1]$. In this way w_1 is continuous in the interval $[0, 1]$ because $u_0'(s_0) = 0$ (compatibility of the initial data). Now, (2) and (3) hold because of the hypothesis (60). So, the step $n = 1$ of the induction is completed.

Suppose now that we have s_n and w_n such that $s_n \in (0, 1)$, and w_n satisfies (2) and (3).

Eqs. (53), (55) and (56) have solution w_{n+1} as expressed in (58), provided that s_{n+1} is known. We introduce this expression into (54) in order to obtain an equation for s_{n+1}. After some algebraic operations, and using the definition of K, we have that,

$$0 = f_{n+1} - \frac{2\tau_0}{s_{n+1}} K(s_{n+1}) - \int_{s_{n+1}}^{1} q^2 w_n(\mu) K(\mu)\, d\mu. \tag{61}$$

Therefore s_{n+1} has to be a root of a function F_{n+1} defined by

$$F_{n+1}(s) = f_{n+1} - \frac{2\tau_0}{s} K(s) - \int_{s}^{1} q^2 w_n(\mu) K(\mu)\, d\mu. \tag{62}$$

There exists at least a root of F_{n+1} in $(0, 1)$. It is clear that F_{n+1} is continuous in $(0, 1)$. Besides that, $F_{n+1}(1) = f_{n+1} - 2\tau_0 > 0$, because of the operability condition. By Lemma 10 and by the extension by zero of w_n in $[0, s_n]$, we deduce that $\lim_{s \to 0^+} F_{n+1}(s) = -\infty$. Therefore, there exists a root in $(0, 1)$.

F_{n+1} has at most a critical point in $(0, 1)$. It can be proved that

$$F_{n+1}'(s) = \frac{2\tau_0}{s^2} K(s) \left[J(s) + \frac{s^2 q^2}{2\tau_0} w_n(s) \right]. \tag{63}$$

We define

$$B_n(s) = \frac{s^2 q^2}{2\tau_0} w_n(s), \tag{64}$$

$$G_n(s) = J(s) + B_n(s). \tag{65}$$

Notice that B_n is a negative and decreasing function, because

$$B_n'(s) = \frac{s^2 q^2}{2\tau_0} \left(w_n'(s) + \frac{1}{s} w_n(s) \right) + \frac{s q^2}{2\tau_0} w_n(s) \leqslant 0. \tag{66}$$

If $F_{n+1}(s) \neq 0$ for all $s \in (0, 1)$, then F_{n+1} has no critical points. Suppose now that there is a root $\bar{s}$ of F_{n+1}'. Using (A) of Lemma 10, we deduce that $\bar{s}$ is a root of G_n. The critical points of F_{n+1} are roots of G_n. But by Lemma 10 the function J has only a critical point that is a maximum, and the fact that B_n is a negative and decreasing function implies that G_n has at most a unique root. Therefore, there exists at most a critical point of F_{n+1}.

F_{n+1} has a unique root in $(0, 1)$. If F_{n+1} does not have a critical point, then F_{n+1} is an increasing function, and the root of F_{n+1} is unique. If F_{n+1} has a critical point $\bar{s}$, then F_{n+1} is increasing in $[0, \bar{s}]$ and decreasing in $[\bar{s}, 1]$. As $F_{n+1}(1) > 0$, the root is in the interval $[0, \bar{s}]$, and there F_{n+1} is increasing. Finally, we deduce that there is only one root of F_{n+1}.

Now, we define s_{n+1} as the unique root of F_{n+1}, and $s_{n+1} \in (0, 1)$.

$w_{n+1} \leqslant 0$ *in* $[0,1]$. We extend continuously w_{n+1} by zero in $[0, s_{n+1}]$. The function w_{n+1} satisfies (53)–(55), and using Lemma 12 we prove that w_{n+1} is a non-positive function in the interval $[0,1]$.

$w'_{n+1} + \frac{1}{r} w_{n+1} \leqslant 0$ *in* $[s_{n+1}, 1]$. We define $Z_{n+1} = w'_{n+1} + \frac{1}{r} w_{n+1}$. With the help of (53) we deduce that

$$Z'_{n+1} = -q^2(w_n - w_{n+1}). \tag{67}$$

Differentiating (53) and using (67) and the inductive hypothesis, it is clear that

$$Z''_{n+1} + \frac{1}{r} Z'_{n+1} - q^2 Z_{n+1} = -q^2 Z_n \geqslant 0. \tag{68}$$

Finally, Z_{n+1} satisfies the following system:

$$\begin{cases} Z''_{n+1} + \frac{1}{r} Z'_{n+1} - q^2 Z_{n+1} \geqslant 0, & r \in (s_{n+1}, 1), \\ Z_{n+1}(1) < 0, \\ Z_{n+1}(s_{n+1}) < 0. \end{cases} \tag{69}$$

By Lemma 11 it is seen that $Z_{n+1} \leqslant 0$ in $[s_{n+1}, 1]$.

This last step completes the inductive step, and the proof is finished. □

Theorem 14. *Suppose that* (60) *and the operability condition hold.*

(1) *If* $f_{n+1} \geqslant f_n$, $w_n \leqslant w_{n-1}$ *in* $[0,1]$ *and* $s_n \leqslant s_{n-1}$, *then* $w_{n+1} \leqslant w_n$ *in* $[0,1]$ *and* $s_{n+1} \leqslant s_n$. *Besides that, if* $\{f_n\}$ *is a non-decreasing sequence that converges to* f_∞, *then* $s_{n+1} \geqslant \frac{2\tau_0}{f_{n+1}}$, $s_\infty \leqslant s_{n+1}$ *and* $w_\infty \leqslant w_{n+1}$ *in* $[0,1]$.

(2) *If* $f_{n+1} \leqslant f_n$, $w_n \geqslant w_{n-1}$ *in* $[0,1]$ *and* $s_n \geqslant s_{n-1}$, *then* $w_{n+1} \geqslant w_n$ *in* $[0,1]$ *and* $s_{n+1} \geqslant s_n$. *Besides that, if* $\{f_n\}$ *is a non-increasing sequence that converges to* f_∞, *then* $s_{n+1} \leqslant \frac{2\tau_0}{f_{n-1}}$, $s_\infty \geqslant s_{n+1}$ *and* $w_\infty \geqslant w_{n+1}$ *in* $[0,1]$.

Proof. (1) It is easy to see that for all $s \in [0,1]$:

$$F_{n+1}(s) - F_n(s) = (f_{n+1} - f_n) - \int_s^1 q^2\big(w_n(\mu) - w_{n-1}(\mu)\big) K(\mu)\, \mathrm{d}\mu \geqslant 0. \tag{70}$$

This implies that $s_{n+1} \leqslant s_n$.

Let us define $Z = w_{n+1} - w_n$. The function Z satisfies

$$\begin{cases} Z'' + \frac{1}{r} Z' - \left(\frac{1}{r^2} + q^2\right) Z \geqslant 0, & r \in (s_n, 1), \\ Z(s_n) \leqslant 0, \\ Z'(1) + Z(1) \leqslant 0. \end{cases} \tag{71}$$

By Lemma 12 we obtained that $Z \leqslant 0$ in $[s_n, 1]$, that is equivalent to $w_{n+1} \leqslant w_n$ in $[s_n, 1]$.

Besides that, if $s \in [0, s_{n+1}]$, then $(w_{n+1} - w_n)(s) = 0$. And if $s \in [s_{n+1}, s_n]$ then $(w_{n+1} - w_n)(s) = w_{n+1}(s) \leqslant 0$. Therefore, $w_{n+1} \leqslant w_n$ in $[0,1]$.

Rewriting (53) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(w'_{n+1} + \frac{1}{r} w_{n+1}\right) = -q^2(w_n - w_{n+1}) \leqslant 0. \tag{72}$$

Integrating between s_{n+1} and 1, we have

$$-f_{n+1}+\frac{2\tau_0}{s_{n+1}}\leqslant 0 \Longrightarrow s_{n+1}\geqslant\frac{2\tau_0}{f_{n+1}}. \tag{73}$$

Now, $f_{n+1}\leqslant f_\infty$ implies that $\frac{2\tau_0}{f_{n+1}}\geqslant\frac{2\tau_0}{f_\infty}$. Using the above result, we have $s_{n+1}\geqslant s_\infty$.

We define $Z=w_{n+1}-w_\infty$. If $s\in[0,s_\infty]$ then $(w_{n+1}-w_\infty)(s)=0$, and if $s\in[s_\infty,s_{n+1}]$, then $(w_{n+1}-w_\infty)(s)=-w_\infty(s)\geqslant 0$. In the interval $[s_{n+1},1]$ the function Z satisfies:

$$\begin{cases} Z''+\frac{1}{r}Z'-\frac{1}{r^2}Z\leqslant 0, & r\in[s_{n+1},1],\\ Z'(1)+Z(1)\geqslant 0,\\ Z(s_{n+1})\geqslant 0.\end{cases} \tag{74}$$

By Lemma 12 it holds that $Z\geqslant 0$ in $[s_{n+1},1]$, that is equivalent to $w_{n+1}\geqslant w_\infty$ in $[s_{n+1},1]$. This concludes the proof.

(2) The proof of this part is similar to (1) and we omit it. □

Theorem 15. *Suppose that the initial data satisfies* (60) *and the operability condition holds. If* $\lim_{n\to\infty}s_n=s_*$ *and* $\lim_{n\to\infty}w_n=w_*$, *then* $s_*=s_\infty$ *and* $w_*=w_\infty$.

Proof. Taking limit to (58) when n tends to infinity we have

$$\begin{aligned} w_*(r)=&\frac{2\tau_0 y_2(s_*q)}{s_*qW(s_*q)}y_1(rq)-\frac{2\tau_0 y_1(s_*q)}{s_*qW(s_*q)}y_2(rq)\\ &-y_2(rq)\int_{s_*}^{r}\frac{qy_1(\mu q)w_*(\mu)}{W(\mu q)}\,d\mu+y_1(rq)\int_{s_*}^{r}\frac{qy_2(\mu q)w_*(\mu)}{W(\mu q)}\,d\mu.\end{aligned} \tag{75}$$

Differentiating (75) we get

$$\begin{aligned} w_*'(r)=&\frac{2\tau_0 y_2(s_*q)}{s_*W(s_*q)}y_1'(rq)-\frac{2\tau_0 y_1(s_*q)}{s_*W(s_*q)}y_2'(rq)\\ &-y_2(rq)\int_{s_*}^{r}\frac{qy_1(\mu q)w_*(\mu)}{W(\mu q)}\,d\mu+y_1(rq)\int_{s_*}^{r}\frac{qy_2(\mu q)w_*(\mu)}{W(\mu q)}\,d\mu.\end{aligned} \tag{76}$$

Differentiating one more time, it can be proved that $w_*''+\frac{1}{r}w_*'-\frac{1}{r^2}w_*=0$. Also it is obvious that $w_*(s_*)=0$ and $w_*'(s_*)=-\frac{2\tau_0}{s_*}$.

Taking limit to (59) when n tends to infinity we obtain (76). Therefore, $\lim_{n\to\infty}w_{n+1}'=w_*'$. Now, $w_*'(1)+w_*(1)=\lim_{n\to\infty}(w_{n+1}'(1)+w_{n+1}(1))=\lim_{n\to\infty}-f_{n+1}=-f_\infty$. So, gathering all the equations we have:

$$\begin{cases} w_*''+\frac{1}{r}w_*'-\frac{1}{r^2}w_*=0, & r\in(s_*,1),\\ w_*(s_*)=0,\\ w_*'(s_*)=-\frac{2\tau_0}{s_*},\\ w_*'(1)+w_*(1)=-f_\infty.\end{cases} \tag{77}$$

By Lemma 1 this system has a unique solution:

$$s_* = \frac{2\tau_0}{f_\infty} \Longrightarrow s_* = s_\infty.$$

$$w_*(r) = -\frac{f_\infty}{2}\left(r - \frac{s_\infty^2}{r}\right) = w_\infty(r) \quad \forall r \in [s_\infty, 1]. \tag{78}$$

This concludes the proof. □

Remark 16. The method works even with functions f that are not necessarily constant. Besides that, the expected behaviour is seen in the numerical experiments, namely, that the asymptotic behaviour of the free boundary is $\frac{2\tau_0}{f(t)}$, and that the physical property $w \leqslant 0$ holds.

4. Numerical experiments

The algorithm for the following results was programmed in Fortran. First we compute the root of F_{n+1}, and then we compute w_{n+1} from (58). The functions w_n are stored as splines functions and the integrals are computed by the Simpson's Rule. The numerical experiments are shown below and prove that the algorithm reproduces the physical behavior of the solution. In the following figures there are examples of several cases.

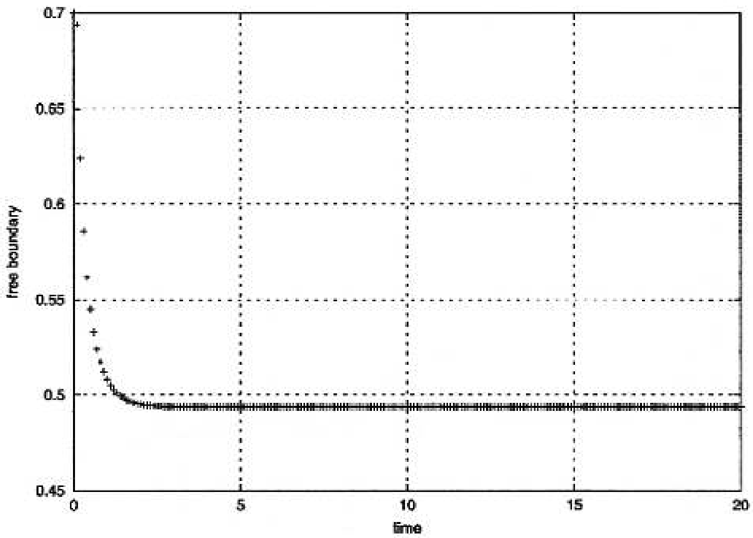

Fig. 2. Solution with $s_0 = 0.7$, $\tau_0 = 1$, $\Delta t = 0.1$, $f(t) = 4$.

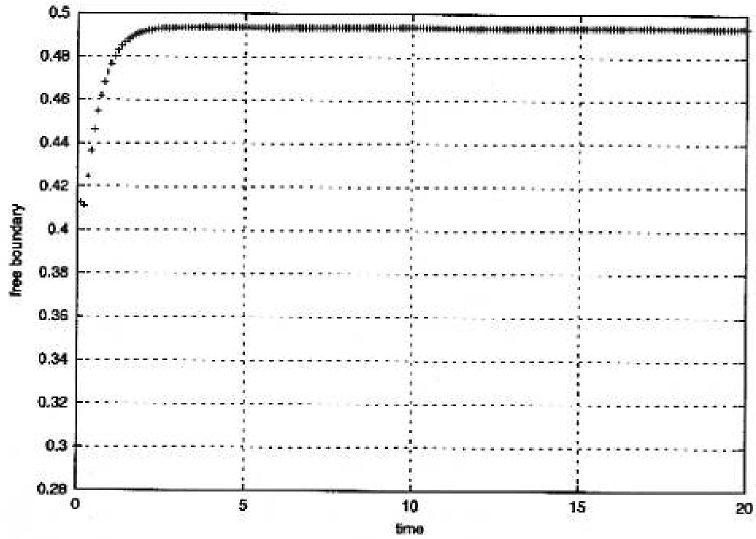

Fig. 3. Solution with $s_0 = 0.3$, $\tau_0 = 1$, $\Delta t = 0.1$, $f(t) = 4$.

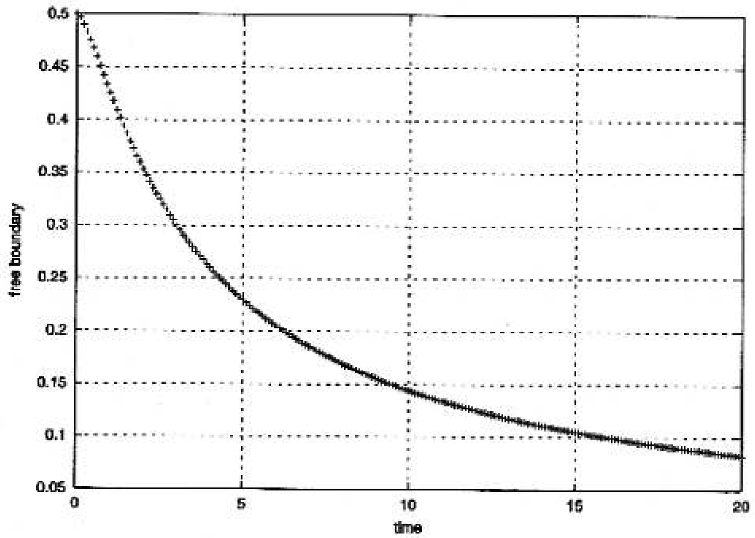

Fig. 4. Solution with $s_0 = 0.5$, $\tau_0 = 1$, $\Delta t = 0.1$, $f(t) = 4 + t$.

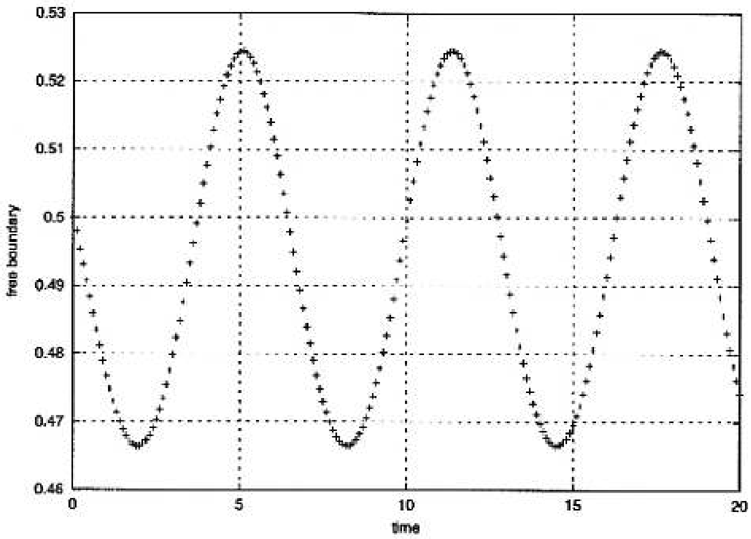

Fig. 5. Solution with $s_0 = 0.5$, $\tau_0 = 1$, $\Delta t = 0.1$, $f(t) = 4 + \frac{\sin(t)}{4}$.

5. Concluding remarks

For the discrete solution of (8)–(12) we have reproduced the physical behaviour of the fluid. That is $w_n(r) \leqslant 0$, $\{s_n\}_n$ is monotone if $\{f_n\}_n$ is monotone, the stationary solution for the discrete problem (which agrees with the stationary solution for the continuous solution) is established, and the discrete solution converges to the stationary solution. Moreover, the algorithm is well defined for all f_n that satisfy $f_n > 2\tau_0$, and can be used in real problems.

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