# Simple modules of small quantum groups at dihedral groups

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Non-semisimple Tensor Categories and Their Semisimplification

# Motivation

The guiding goal is to construsct fusion categories from non-semisimple Hopf algebras.

One idea is to consider the quotient category associated to a spherical Hopf algebra.

$$\underline{\mathrm{Rep}(H)} = \mathrm{Rep}\,H \Big/ \langle M \,/\, \dim_q M = 0 \rangle$$

Andruskiewitsch, Angiono, García I, Torrecillas, V. *From Hopf algebras to tensor categories.* Conformal Field Theories and Tensor Categories, Yi-Zhu Huang et al. editors, Mathematical Lectures from Peking University, Springer.

## Theorem [Barrett, Westbury]

$\underline{\mathrm{Rep}}(H)$ is semisimple tensor category with simple objects

$$\left\{ M \in \mathrm{Rep}(H) \mid M \text{ is indecomposable and } \dim_q(M) \neq 0 \right\}$$

¿Is there some fusion subcat $\subset \underline{\mathrm{Rep}}(H)$?

¿How we can find one?

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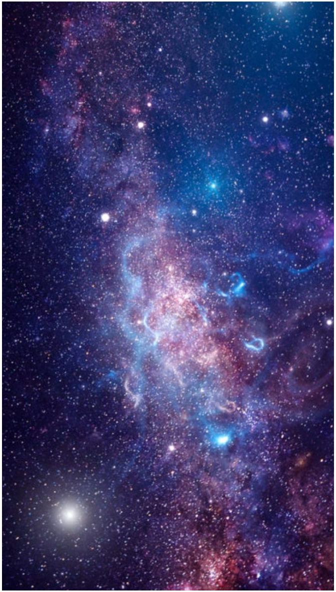
Barrett, Westbury. *Spherical Categories* Adv. Math. (1999)

- To consider sub cat generated by some object (eg $\mathcal{H}$-simple)

- To find Tilting mod or for quantum groups.

Which $H$ of the huge universe of Hopf algebras we should consider?



Something close to (small) Quantum groups at root of units

# Generalized small quantum groups

abelian group

$D(u^- \# u_0) \twoheadrightarrow u_q(\mathfrak{g}) = u^- \otimes u_0 \otimes u^+$

Nichols alg

$D(\mathcal{B}(V) \# H) \simeq \mathcal{B}(V) \otimes D(H) \otimes \mathcal{B}(\bar{V})$

f.d. Nichols alg $|_H$

# Example: Taft algebras

$$\mathcal{D} = \mathcal{D}(\mathbb{k}[x \mid x^n]\#\mathbb{Z}_n) \longrightarrow \mathfrak{u}_q(\mathfrak{sl}_2)$$

$V, W$ : $\mathcal{D}$-simples $\Rightarrow$ $V \otimes W = \bigoplus S_i \oplus P_i$

simple $\quad$ Proj

$n$ is odd $\Rightarrow$ $\mathcal{D}$ is ribbon $\Rightarrow$ spherical

$\underline{\mathrm{Rep}}\,\mathcal{D} \supseteq \langle \text{simples} \rangle =: \mathcal{S}$ fusion $\quad$ dim=0


Erdmann, Green, Snashall, Taillefer. *Representation theory of the Drinfeld doubles of a family of Hopf algebras.* J Pure Appl. Algebra (2006).

Kauffman, Radford. *A necessary and sufficient condition for a finite-dimensional Drinfel'd Double to be a ribbon Hopf algebra.* J Algebra (1993).

# Example: Fomin-Kirillov algebra

$$\mathcal{D} = \mathcal{D}(\mathcal{FK}_3 \# \mathbb{S}_3)$$ is spherical

There are 4 simples w/ $\dim_q \neq 0$

$L_1^1, L_2^1, L_3^1, L_4^{-2}$

In Rep D: $L_1^* = 1$, $L_2^* \simeq L_3$, $L_4^* = L_4$

$L_2 \otimes L_2 = B$, $L_2 \otimes L_4 = C$, $L_4 \otimes L_4 = 1 \oplus A$

$\int \langle L_1, L_2, L_3, L_4 \rangle$ is fusion?


Pogorelsky, V. *Verma and simple modules for quantum groups at non-abelian groups.* Adv. Math. (2016).

—. *On the representation theory of the Drinfeld double of the Fomin-Kirillov algebra $\mathcal{FK}_3$* Algebr. Represent. Theory (2019).

# Example: Dihedral groups

$V$ = simple

$$\mathcal{D} = \mathcal{D}(\mathfrak{B}(V)\#\mathbb{D}_m)$$

$\mathcal{D}$-simple → Proj → $\dim_q = 0$

$\mathcal{D}$-simple → Rigid = $D\mathbb{D}_m$-simple

$\mathcal{D}$-simple → others → $\dim_q = 0$

$\mathcal{D}$ spherical → $\dim_q = 0$

$\operatorname{Rep} \mathcal{D} \supseteq \langle \text{simples} \rangle = \langle \text{rigids} \rangle \subseteq \operatorname{Rep} D\mathbb{D}_m$

García, V. *Simple modules of small quantum groups at dihedral groups.* arXiv:2012.09323 (2020).

# Simple modules over generalized small quantum groups

$$D = D(\mathcal{B}(V) \rtimes H) = \mathcal{B}(V) \otimes \underbrace{D(H) \otimes \mathcal{B}(\overline{V})}_{D^+}$$

$\lambda \in D(H)$-simple $\rightsquigarrow$ $\lambda \in D^+$-simple $(V \cdot \lambda = 0)$

Verman $\rightsquigarrow$ $M(\lambda) = D \otimes_{D^+} \lambda$

$\rightsquigarrow$ $L(\lambda) = \operatorname{top} M(\lambda)$

$\{D(H)\text{-simple}\} \longleftrightarrow \{D\text{-simple}\}$

weight $\lambda \longleftrightarrow L(\lambda)$ highest weight

Bellamy, Thiel. *Highest weight theory for f.d. graded algebras with triangular decomposition.* Adv. Math. (2018).

¿ $\operatorname{Res}_{D(H)} L(\lambda)$?

We answer this question for $H = \mathbb{D}_m$

# Our main result

Thm We describe $\operatorname{Rep}_{\mathcal{D}(\mathbb{D}_m)} L(\lambda)$

$\forall \lambda$ $\forall V$

**Theorem [Fantino, García]**

For $m = 4t \geq 12$, the f.d. Nichols algebras over $\mathcal{D}(\mathbb{D}_m)$ are

$$\mathfrak{B}(V) \simeq \bigwedge V$$

with $V = \oplus_{(i,k)\in I} M_{i,k}$, $M_{i,k}$ is $\mathcal{D}(\mathbb{D}_m)$-simple and $\dim M_{i,k}$=2.

Fantino, García. *On pointed Hopf algebras over dihedral groups.* Pacific J. Math. (2011).

# Remarks

1. In general, the $\mathcal{D}(G)$-weights are well-known and they are no necessarily one-dimensional. For $G = \mathbb{D}_m$ we have
   1. $8$ one-dimensional weights.
   2. $(m+2)(n-1)$ two-dimensional weights.
   3. $8$ $n$-dimensional weights.

   $n = \frac{m}{2}$

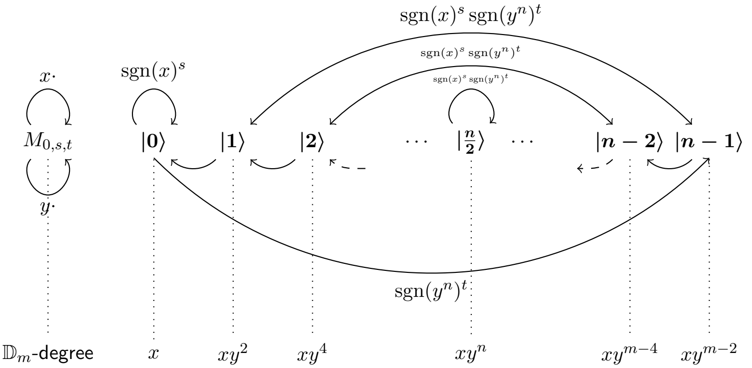
2. The tensor products of weights are no necessarily simple.

3. This is an infinity family of small quantum groups at non abelian groups.


Andruskiewitsch, Graña. *Braided Hopf algebras over non-abelian finite groups.* Bol. Acad. Nac. Cienc. (1999).

# An $n$-dimensional weight

$$\mathbb{D}_m = \langle x, y \mid x^2 = e = y^n,\ xy = y^{-1}x \rangle$$




$$D(\mathbb{D}_m) = \mathbb{D}_m \bowtie k^{\mathbb{D}_m}$$

# Notation

$H$ = f.d. Hopf alg.

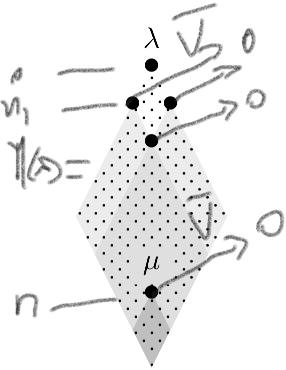
$\mathcal{B}(V)$ = fd Nichols alg $|_H$

$$D = D(\mathcal{B}(V)\# H) = \mathcal{B}(V) \otimes D(H) \otimes \underbrace{\mathcal{B}(\bar{V})}_{D^+}$$

$$M(\lambda) = M_H^V(\lambda) = D \otimes_{D^+} \lambda = \mathcal{B}(V) \otimes \lambda$$

$$L(\lambda) = L_H^V(\lambda) = \operatorname{top} M(\lambda)$$

$$S(\lambda) = \operatorname{socle} M(\lambda)$$




$\{\text{weights}\} \longleftrightarrow \{D\text{-simples}\}$

$\lambda \longleftrightarrow L(\lambda)$

$\lambda \longleftrightarrow S(\lambda)$ [N]

<u>Prop</u> $S(\lambda) = L(\mu)$ where $\mu$ is the unique highes-w of minimum degree in $M(\lambda)$

<u>Rmk</u> $S(\lambda) = D\mu = B(V)\mu$

# Example

$V =$ simple 2-dim $\quad B(V) = \Lambda(V)$

$$\text{gr} \quad B(V) \xrightarrow{\otimes \lambda} M(\lambda) \qquad V\otimes\lambda \;??\; \Delta(\lambda)\text{-mod}$$

$\lambda \,\cdot$

$0 \quad 1 \quad \cdot$

$-1 \quad V \quad \cdot \qquad \mu \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

$-2 \quad \lambda_V \quad \cdot$ 1-dim $\qquad \lambda_V\lambda \cdot$

→ the HW of min degree could be:

$-2 \rightsquigarrow S(\lambda) = L(\lambda_V \lambda) = \lambda_V\lambda$ is rigid

$-1 \rightsquigarrow S(\lambda) = L(\mu) = \mu \oplus \lambda_V\lambda$

$0 \rightsquigarrow S(\lambda) = L(\lambda) = M(\lambda)$ is proj [PV]

$$L(\mu) \underset{D(H)\text{-mod}}{=} \mu \quad \text{rigid}$$

<u>Thm</u> If $L(\mu) = \mu$ is rigid and $\lambda \otimes \mu$ is $D(H)$-s.s. $= \bigoplus \lambda_i$

$$\Rightarrow L(\lambda) \otimes L(\mu) = \bigoplus L(\lambda_i)$$

<u>Rm</u> 1) $L(\mu) \otimes L(\mu)$ is generated in deg $0$.

2) All the H.W. of $L(\lambda) \otimes L(\mu)$ are in degree $0$.

$V = W \oplus U$ on ${}^{D(H)}$-mod

$\mathcal{B}(V) \# H \simeq \mathcal{B}(Z) \# (\mathcal{B}(U) \# H)$

$Z = \mathrm{ad}_c\, \mathcal{B}(U)(W) \in {}^{\mathcal{D}(\mathcal{B}(U)\#H)}$-mod

$\Rightarrow \mathcal{D} = \mathcal{D}(\mathcal{B}(V) \# H) \simeq \mathcal{D}\left(\mathcal{B}(Z) \# \left(\mathcal{B}(U) \# H\right)\right)$

$\Rightarrow \mathcal{B}(V) \otimes \mathcal{D}(H) \otimes \mathcal{B}(\overline{V}) \simeq \mathcal{B}(Z) \otimes \mathcal{D}(\mathcal{B}(U) \# H) \otimes \mathcal{B}(\overline{Z})$

$\Rightarrow$ We can compute the $\mathcal{D}$-simples in two differents ways!

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Andruskiewitsch, Angiono. *On Nichols algebras over basic Hopf algebras.* Math. Zeitschrift (2020).

1) $\lambda \rightsquigarrow L^{\vee}_{H}(\lambda)$

2) $\lambda \rightsquigarrow L^{U}_{H}(\lambda) \rightsquigarrow L^{Z}_{B(1)\#H}(\lambda) = \mathrm{top}\, M^{Z}_{B(1)\#H}(L^{U}_{H}(\lambda))$

Thm $\quad L^{\vee}_{H}(\lambda) \simeq L^{Z}_{B(1)\#H}\left(L^{U}_{H}(\lambda)\right)$

Rmk $\quad W$ is simple

$\Rightarrow \quad Z = L^{U}_{H}(W)$.

[AA]

# Example

$W =$ simple 2-dim $\quad c_{W,U} = -\text{flip}$

$B(W) = \lambda(W) \qquad \Rightarrow Z = L^0_{ff}(W)$

$1 \quad \bullet \longrightarrow M^Z(L^0_{ff}(W))$

$\cancel{B(U) \otimes Z}$

$W \quad \bullet \qquad\qquad \bullet$

$\lambda_W \quad \bullet \qquad\qquad \bullet$

**Thm** if $V = M_{i,k}$ Then

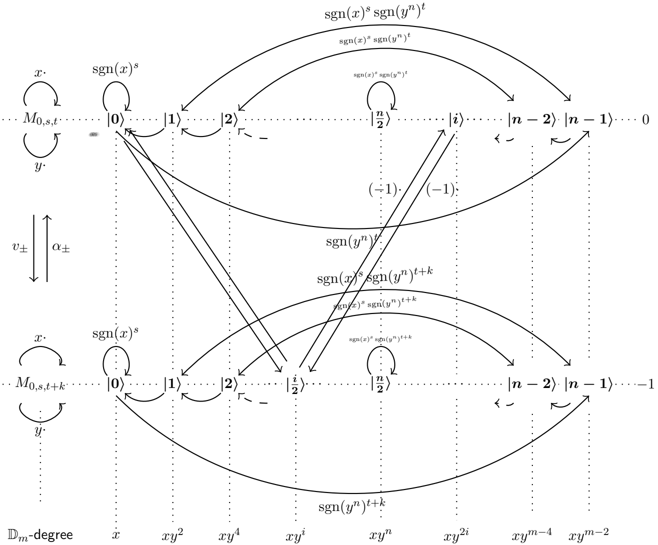
- $\dim \lambda = 1$ or $2$ $\Rightarrow$ $L(\lambda) = \lambda$ (rigid) or $= B(\gamma)\otimes\lambda$ (proj)

- $\dim \lambda = n$ $\Rightarrow$ $L(\lambda) \underset{D(\mathbb{D}_n)}{=} \lambda \oplus \bar{\lambda}$ with $\dim \bar{\lambda} = n$

**Thm** $D$ is spherical $\Leftrightarrow$ iff $k$ is odd

in that case $\dim_q L(\lambda) \neq 0 \Leftrightarrow L(\lambda)$ rigid

# Example



$\mathrm{sgn}(x)^s\,\mathrm{sgn}(y^n)^t$
$\mathrm{sgn}(x)^s$
$x\cdot$
$y\cdot$
$\cdots M_{0,s,t}\cdots$
$|\mathbf{0}\rangle$ $|\mathbf{1}\rangle$ $|\mathbf{2}\rangle$ $|\frac{\boldsymbol{n}}{\mathbf{2}}\rangle$ $|\boldsymbol{i}\rangle$ $|\boldsymbol{n}-\mathbf{2}\rangle$ $|\boldsymbol{n}-\mathbf{1}\rangle$ $0$
$(-1)\cdot$
$(-1)\cdot$
$\mathrm{sgn}(y^n)^t$
$v_\pm$
$\alpha_\pm$
$\mathrm{sgn}(x)^s\,\mathrm{sgn}(y^n)^{t+k}$
$\cdots M_{0,s,t+k}\cdots$
$|\mathbf{0}\rangle$ $|\mathbf{1}\rangle$ $|\mathbf{2}\rangle$ $|\frac{\boldsymbol{i}}{\mathbf{2}}\rangle$ $|\frac{\boldsymbol{n}}{\mathbf{2}}\rangle$ $|\boldsymbol{n}-\mathbf{2}\rangle$ $|\boldsymbol{n}-\mathbf{1}\rangle$ $-1$
$\mathrm{sgn}(y^n)^{t+k}$
$\mathbb{D}_m$-degree
$x$ $xy^2$ $xy^4$ $xy^i$ $xy^n$ $xy^{2i}$ $xy^{m-4}$ $xy^{m-2}$