Symmetries on modules over Drinfeld doubles

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We are interested on the representations of the Drinfeld double

$$\mathcal{D} = \mathcal{D}(\mathfrak{B}(V)\#H)$$

of the bozonization of a Nichols algebra and a Hopf algebra.

Why?

- These are natural generalization of (small) quantum groups.
- The category of graded $\mathcal{D}$-modules is highest-weight [Bellamy-Thiel].
- Categorification of $\mathbb{Z}$-fusion datum associated with cyclic complex reflection groups [Bonnafé-Rouquier].
- These could give information about the Nichols algebra.
- To construct new examples of fusion categories.

Goals of the talk

- BGG Reciprocity
- Symmetric Hilbert Series
- Tate duality
- Braided autoequivalences

- H = finite dimensional Hopf algebra.

The Drinfeld double $\mathcal{D}(H)$ of H is

a Hopf algebra which is constructed as a kind of double crossed product between H and H^*

$$\mathcal{D}(H) = H^* \bowtie H.$$

Example: $H = \Bbbk G$ a group algebra

$\mathcal{D}(G) = \Bbbk^G \otimes \Bbbk G$ as coalgebras and

$$\delta_h \, g = g \, \delta_{g^{-1}hg}$$

for all $g, h \in G$.

- V = Yetter-Drinfeld module over $H \equiv \mathcal{D}(H)$-module.

The Nichols algebra $\mathfrak{B}(V)$ of V is

a graded braided Hopf algebra in the category of $\mathcal{D}(H)$-modules;

$$\mathfrak{B}(V) = \frac{T(V)}{\mathcal{J}}$$

where $\mathcal{J}$ is the maximal ideal which is a coideal and generated by homogeneous element of degree ≥ 2.

Example: $V_3 = \langle x_{(12)}, x_{(23)}, x_{(13)} \rangle$ over $\mathcal{D}(\mathbb{S}_3)$

$$g x_{(ij)} = \operatorname{sgn}(g)\, x_{g(ij)g^{-1}} \quad \text{and} \quad \delta_h \cdot x_{(ij)} = \delta_{(ij),h}\, x_{(ij)}$$

Example: The Fomin-Kirillov algebra $\mathcal{FK}_3 = \mathfrak{B}(V_3)$

$$\begin{aligned} x_{(12)}^2 = x_{(13)}^2 = x_{(23)}^2 &= 0 \\ x_{(12)}x_{(13)} + x_{(13)}x_{(23)} + x_{(23)}x_{(12)} &= 0 \\ x_{(13)}x_{(12)} + x_{(23)}x_{(13)} + x_{(12)}x_{(23)} &= 0 \end{aligned}$$

Properties of finite dimensional Nichols algebras

- The homogeneous component of maximum degree is one dimensional:

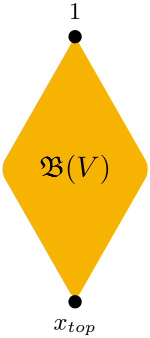
$$\mathfrak{B}^{n_{top}}(V) = \Bbbk\{x_{top}\}.$$

- $\mathfrak{B}(V)$ is Frobenius whose non-degenerate bilinear form is

$$\mathfrak{B}(V) \otimes \mathfrak{B}(V) \xrightarrow{\text{mult}} \mathfrak{B}(V) \xrightarrow{(x_{top})^*} \Bbbk$$

- The Hilbert series of $\mathfrak{B}(V)$ is symmetric

$$\dim \mathfrak{B}^i(V) = \dim \mathfrak{B}^{n_{top}-i}(V).$$



The Hilbert series h_M of a graded module M is

$$h_M = \sum_i \dim M^i \, t^i$$

Example: V_3

$$h_{\mathfrak{B}(V_3)} = 1 + 3t + 4t^2 + 3t + 1$$

$\mathfrak{B}^0(V_3)$	$\langle 1 \rangle$
$\mathfrak{B}^1(V_3)$	$\langle x_{(12)},\ x_{(23)},\ x_{(13)} \rangle$
$\mathfrak{B}^2(V_3)$	$\langle x_{(12)}x_{(13)},\ x_{(12)}x_{(23)},\ x_{(13)}x_{(23)},\ x_{(13)}x_{(12)} \rangle$
$\mathfrak{B}^3(V_3)$	$\langle x_{(12)}x_{(13)}x_{(23)},\ x_{(12)}x_{(13)}x_{(12)},\ x_{(13)}x_{(12)}x_{(23)} \rangle$
$\mathfrak{B}^4(V_3)$	$\langle x_{top} \rangle$

Properties of the Drinfeld double $\mathcal{D} = \mathcal{D}(\mathfrak{B}(V)\#H)$

The bosonization $\mathfrak{B}(V)\#H$ is

a Hopf algebra which is constructed as a kind of crossed product between H and $\mathfrak{B}(V)$

Notation

$$\mathcal{D} := \mathcal{D}(\mathfrak{B}(V)\#H)$$

Properties of $\mathcal{D}$

- Triangular decomposition: $\mathcal{D} \simeq \mathfrak{B}(V) \otimes \mathcal{D}(H) \otimes \mathfrak{B}(\overline{V})$
- Graded: $\mathcal{D}^n = \oplus_{n=j-i}\, \mathfrak{B}^i(V) \otimes \mathcal{D}(H) \otimes \mathfrak{B}^j(\overline{V})$
- Symmetric algebra

where $\overline{V}$ is the dual object of V as $\mathcal{D}(H)$-module endowed with the inverse braiding. It holds $\mathfrak{B}^n(\overline{V}) \simeq \mathfrak{B}^n(V)^*$ as $\mathcal{D}(H)$-modules.

Representation theory of $\mathcal{D}$

- $\Lambda =$ the set of simple $\mathcal{D}(H)$-modules.

Theorem [Bellamy-Thiel, V]

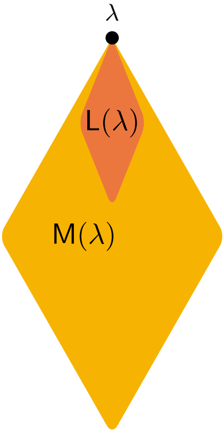
If H is semisimple, then the category of graded $\mathcal{D}$-modules is a highest weight category whose set of weights is $\Lambda \times \mathbb{Z}$.

The standard (Verma) modules are:

$$\mathsf{M}(\lambda[n]) = \mathcal{D} \otimes_{\mathcal{D}^{\geq 0}} \lambda[n].$$

The simple modules are:

$$\mathsf{L}(\lambda[n]) = top\left(\mathsf{M}(\lambda[n])\right).$$

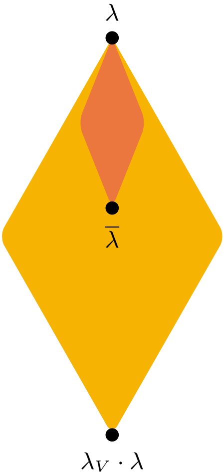


- $\lambda_V = \mathfrak{B}^{n_{top}}(V) \in \Lambda$.

Lemma

$$\mathsf{M}(\lambda)^* \simeq \mathsf{M}((\lambda_V \lambda)^*)$$

$$\mathsf{L}(\lambda)^* \simeq \mathsf{L}(\overline{\lambda}^*)$$



Problem

Describe $\operatorname{ch}^\bullet \mathsf{L}(\lambda)$ for all $\lambda \in \Lambda$.

- $\mathsf{N} = \oplus_i \mathsf{N}(i)$ a graded $\mathcal{D}(H)$-module.

$$\rightsquigarrow \quad \operatorname{ch}^\bullet \mathsf{N} = \sum_i \operatorname{ch} \mathsf{N}(i)\, t^i \in \Lambda[t, t^{-1}].$$

Diagonal case

- $H = \Bbbk\Gamma$ a finite abelian group
- $\mathfrak{B}(V)$ = Nichols algebra of diagonal type with finite root system

Theorem [Yamane]

λ "typical", it holds a Weyl-Kac-type formula

$$\operatorname{ch}^{\bullet} \mathsf{L}(\lambda) = \sum_{\dot{\omega} \in \dot{W}^{\lambda}} \operatorname{sgn}(\dot{\omega}) \operatorname{ch}^{\bullet} \mathsf{M}(\dot{\omega} \cdot \lambda).$$

- $\mathsf{P}(\lambda)$ = projective cover of $\mathsf{L}(\lambda)$.

Theorem [Holmes-Nakano, B-T, V]

$\mathsf{P}(\lambda)$ admits a standard filtration, *i.e.*

$$\exists \quad 0 = \mathsf{N}_0 \subset \mathsf{N}_1 \subset \cdots \subset \mathsf{N}_n = \mathsf{P}(\lambda) \quad \text{s.t.}$$
$$\forall i \quad \mathsf{N}_i/\mathsf{N}_{i-1} \simeq \mathsf{M}(\lambda_i) \quad \text{for some } \lambda_i \in \Lambda$$

BGG Reciprocity [B-T, V]

$$[\mathsf{P}(\lambda) : \mathsf{M}(\mu)] = [\mathsf{M}(\mu) : \mathsf{L}(\lambda)]$$

Theorem [Holmes-Nakano, B-T, V]

$\mathsf{P}(\lambda)$ admits a graded standard filtration.

- $p_{\mathsf{P}(\lambda),\mathsf{M}(\mu)}$ and $p_{\mathsf{M}(\mu),\mathsf{L}(\lambda)} \in \mathbb{Z}[t, t^{-1}]$ s.t.

$$\mathrm{ch}^\bullet \mathsf{P}(\lambda) = \sum_\mu p_{\mathsf{P}(\lambda),\mathsf{M}(\mu)} \, \mathrm{ch}^\bullet \mathsf{M}(\mu) \quad \text{and}$$

$$\mathrm{ch}^\bullet \mathsf{M}(\mu) = \sum_\lambda p_{\mathsf{M}(\mu),\mathsf{L}(\lambda)} \, \mathrm{ch}^\bullet \mathsf{L}(\lambda)$$

Graded BGG Reciprocity [B-T, V]

$$p_{\mathsf{P}(\mu),\mathsf{M}(\lambda)} = \overline{p_{\mathsf{M}(\lambda),\mathsf{L}(\mu)}}$$

where $\overline{p(t, t^{-1})} = p(t^{-1}, t)$.

Example: $\mathcal{D}(\Bbbk[x \mid x^n]\#\Bbbk\mathbb{Z}_n)$ [Chen]

$1 + t + t^2 + \cdots + t^i$ with $i \leq n$

- $\Lambda_{\mathcal{D}(\mathbb{S}_3)} =$

$$= \{\varepsilon, (e,-), (e,-), (e,\rho), (\sigma,+), (\sigma,-), (\tau,0), (\tau,1), (\tau,2)\}$$

Example: $\mathcal{D}(\mathcal{FK}_3\#\Bbbk\mathbb{S}_3)$ [Pogorelsky-V]

- $h_\varepsilon = 1$
- $h_{(e,\rho)} = 2 + 3t + 2t^2$
- $h_{(\tau,0)} = 2 + 3t + 2t^2$
- $h_{(\sigma,-)} = 3 + 4t + 3t^2$
- $h_\lambda = h_{\mathfrak{B}(V_3)} \cdot \dim\lambda$

Example= $\mathcal{D}(\mathfrak{ufo}(7)\#\Bbbk\Gamma)$ [Andruskiewistch-Angiono-Mejía-Renz]

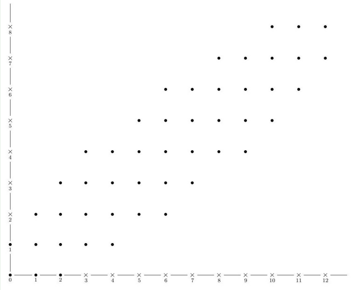
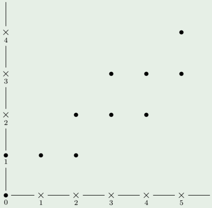
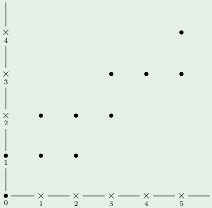


Figure B.1:

Case 11, $\lambda_1 = 1$, $\lambda_2 = \zeta$



Case 12, $\lambda_1 = 1$, $\lambda_2 = \zeta^4$



Question

Are the Hilbert series of simple modules symmetric?

Ulrich Thiel posed the same question for *restricted rational Cherednik algebras*

$$\overline{\mathbf{H}}_c = \left(\Bbbk[V]/\Bbbk[V]^G\right) \otimes \Bbbk G \otimes \left(\Bbbk[V^*]/\Bbbk[V^*]^G\right).$$

Example

Yes, for all the exceptional complex reflection groups and generic parameters c.

Counterexample

For special parameters c there are simple modules whose Hilbert series is not symmetric.

Theorem [Linckelmann]

For any symmetric algebra A and finitely generated A-modules U and V, the Tate duality holds

$$\left(\widehat{\operatorname{Ext}}_A^{-n}(U,V)\right)^* \simeq \widehat{\operatorname{Ext}}_A^{n-1}(V,U).$$

In particular, it applies for $A = \mathcal{D}$.

Problem

Compute the Brauer–Picard group of a fusion category $\mathcal{A}$:

$$\mathrm{BrPic}(\mathcal{A}) = \{\text{semisimple invertible } \mathcal{A}\text{-bimodule categories}\}$$

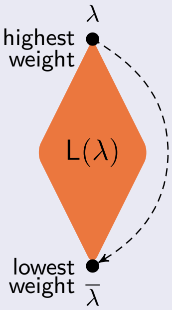
Or equivalently, the group of braided autoequivalences of $\mathcal{Z}(\mathcal{A})$.

$$\mathrm{BrPic}(\mathcal{A}) \simeq \mathrm{Aut}^{br}(\mathcal{Z}(\mathcal{A}))$$

Remark

$$\mathcal{A} = H - mod \Longrightarrow \mathcal{Z}(\mathcal{A}) = \mathcal{D}(H) - mod$$

Consider the bijection



Example: $\mathcal{D}(\mathcal{FK}_3\#\mathbb{S}_3)$

this corresponds to the unique non-trivial braided autoequivalence of the category of $\mathcal{D}(\mathbb{S}_3)$-modules:

$$\overline{(e,\rho)} = (\tau, 0),$$
$$\overline{(\tau,0)} = (e,\rho) \quad \text{and}$$
$$\overline{\lambda} = \lambda \quad \text{for the other weights.}$$

[Lentner-Priel, Nikshych-Riepel].

Question

Does this bijection induce a braided autoequivalence in the category of $\mathcal{D}(H)$-modules?