On projective modules over finite quantum groups

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1. Introduction

2. Main results

3. Special features of the finite quantum groups
   - Duality of Verma modules
   - Tensor products

Introduction

- On projective modules over finite quantum groups
  arXiv:1612.09220v2
- Bellamy and Thiel
  Highest weight theory for finite-dimensional graded algebras with triangular decomposition
  arXiv:1705.08024

Theorem [BT]

The category of graded modules over a finite-dimensional algebra admitting a triangular decomposition can be endowed with the structure of a highest weight category.

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Finite quantum groups

- G = finite group.
  $\rightsquigarrow$ $\mathcal{D}(G)$ = Drinfeld double of G.
- V = Yetter-Drinfeld module over G.
  $\rightsquigarrow$ $\mathfrak{B}(V)$ = Nichols algebra of V.

Assume that $\mathfrak{B}(V)$ is finite-dimensional.

Definition

A finite quantum group is the Drinfeld double of $\mathfrak{B}(V)\#\Bbbk G$. We denote it by $\mathcal{D}$.

Facts

- $\overline{V}$ = dual object of V as $\mathcal{D}(G)$-module.
  $\rightsquigarrow$ $\mathfrak{B}(\overline{V})$ = Nichols algebra of $\overline{V}$ with the inverse braiding.

Triangular decomposition

As vector spaces, $\mathcal{D} \simeq \mathfrak{B}(V) \otimes \mathcal{D}(G) \otimes \mathfrak{B}(\overline{V})$

Graded Hopf algebra

For $n \in \mathbb{Z}$, $\mathcal{D}^n = \oplus_{n=j-i} \mathfrak{B}^i(V) \otimes \mathcal{D}(G) \otimes \mathfrak{B}^j(\overline{V})$.

Daulity

As $\mathcal{D}(G)$-modules, $\mathfrak{B}^n(\overline{V}) \simeq \mathfrak{B}^n(V)^*$.

Simmetry

$\mathcal{D}$ is a symmetric algebra.

Highest-weight data

- Λ = set of weights = simple $\mathcal{D}(G)$-modules.
- $\mathsf{M}(\lambda) = \mathcal{D} \otimes_{\mathcal{D}^{\geq 0}} \lambda$ = Verma module of $\lambda \in \Lambda$.
- $\mathsf{L}(\lambda)$ = the head of $\mathsf{M}(\lambda)$.

Theorem

1. $\mathsf{L}(\lambda)$ is simple and graded for all $\lambda \in \Lambda$.
2. Every graded simple module is isomorphic to a shift of $\mathsf{L}(\lambda)$ for some $\lambda \in \Lambda$.

Standard filtration

Theorem

Every projective module admit a graded standard filtration.

That is, given a projective module P, there is a sequence of graded subdmodules

$$0 = \mathsf{N}_0 \subset \mathsf{N}_1 \subset \cdots \subset \mathsf{N}_n = \mathsf{P}$$

such that each

$$\mathsf{N}_i / \mathsf{N}_{i-1}$$

is isomorphic to a shift of a Verma module.

Graded character

- $\mathsf{N} = \oplus_i \mathsf{N}(i) =$ graded $\mathcal{D}$-module.

$$\rightsquigarrow \quad \operatorname{ch}^\bullet \mathsf{N} = \sum_i \operatorname{ch} \mathsf{N}(i)\, t^i \in \Lambda[t, t^{-1}].$$

Theorem

The graded characters of the simple modules form a $\mathbb{Z}[t, t^{-1}]$-basis of the Grothendieck ring of the category of graded $\mathcal{D}$-modules.

Then there exist Laurent polynomials $p_{\mathsf{N},\mathsf{L}(\lambda)}$ such that

$$\operatorname{ch}^\bullet \mathsf{N} = \sum_\lambda p_{\mathsf{N},\mathsf{L}(\lambda)} \operatorname{ch} \mathsf{L}(\lambda).$$

Graded character

Theorem

The graded characters of the Verma modules form a $\mathbb{Z}[t, t^{-1}]$-basis of the Grothendieck ring of the category of graded projective $\mathcal{D}$-modules.

Then, given a graded projective $\mathcal{D}$-module P, there exist Laurent polynomials $p_{\mathsf{P},\mathsf{M}(\lambda)}$ such that

$$\operatorname{ch}^\bullet \mathsf{P} = \sum_\lambda p_{\mathsf{P},\mathsf{M}(\lambda)} \operatorname{ch} \mathsf{M}(\lambda).$$

Graded BGG Reciprocity

- $\mathsf{P}(\mu)$ = the projective cover of $\mathsf{L}(\mu)$.

Theorem

$$p_{\mathsf{P}(\mu),\mathsf{M}(\lambda)} = \overline{p_{\mathsf{M}(\lambda),\mathsf{L}(\mu)}}.$$

$$\mathrm{ch}^\bullet \mathsf{P}(\mu) = \sum_\lambda p_{\mathsf{P}(\mu),\mathsf{M}(\lambda)} \ \mathrm{ch}^\bullet \mathsf{M}(\lambda)$$

$$\mathrm{ch}^\bullet \mathsf{M}(\lambda) = \sum_\mu p_{\mathsf{M}(\lambda),\mathsf{L}(\mu)} \ \mathrm{ch}^\bullet \mathsf{L}(\mu)$$

$$p(t,t^{-1}) \in \mathbb{Z}[t,t^{-1}] \quad \rightsquigarrow \quad \overline{p} = p(t^{-1},t)$$

BGG Reciprocity

Corollary

$$[\mathsf{P}(\mu) : \mathsf{M}(\lambda)] = [\mathsf{M}(\lambda) : \mathsf{L}(\mu)],$$

i. e. the number of subquotients in a standard filtration of $\mathsf{P}(\mu)$ isomorphic to $\mathsf{M}(\lambda)$ is equal to the number of composition factors of $\mathsf{M}(\lambda)$ isomorphic to $\mathsf{L}(\mu)$.

Proof: $[\mathsf{P}(\mu) : \mathsf{M}(\lambda)]$ and $[\mathsf{M}(\lambda) : \mathsf{L}(\mu)]$ are the values of $p_{\mathsf{P}(\mu),\mathsf{M}(\lambda)}$ and $\overline{p_{\mathsf{M}(\lambda),\mathsf{L}(\mu)}}$ at $t = 1$, resp. □

Remarks

- The category of $\mathcal{D}$-modules is not highest weight because $\mathcal{D}$ is symmetric and non-semisimple, then it has infinite global dimension.

- Λ does not admit a partial order $\leq$ such that $\mu \leq \lambda$ if $\mathsf{L}(\mu)$ is a composition factor of the Verma module $\mathsf{M}(\lambda)$. For instance if $\mathfrak{B}(V)$ is the Fomin-Kirillov algebra $\mathcal{FK}_3$ and $G = \mathbb{S}_3$, there are two Verma modules, $\mathsf{M}(\tau, 0)$ and $\mathsf{M}(e, \rho)$, with the same composition factors: $\mathsf{L}(\tau, 0)$, $\mathsf{L}(\sigma, -)$ and $\mathsf{L}(e, \rho)$. Then, such an order on Λ will imply that $(\tau, 0) = (e, \rho)$.

- If G is non-abelian, there are weights of dimension ≥ 1. For instance, $\dim(\tau, 0) = \dim(e, \rho) = 2$ and $\dim(\sigma, -) = 3$.

- The tensor product of weights is not necessarily a weight.

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Duality of Verma modules

- λ_V = homogeneous component of maximum degree of $\mathfrak{B}(V)$. It is a one-dimensional weight spanned by x_{top}.
  $\Longrightarrow$ $soc_{\mathcal{D}^{\leq 0}}\mathsf{M}(\lambda) = \Bbbk x_{top} \otimes \lambda$ is a weight.

Lemma

$$\mathsf{M}(\lambda)^* \simeq \mathsf{M}\left((\lambda_V \cdot \lambda)^*\right)$$

Proof: • $(\mathfrak{B}^{n_{top}}(V) \otimes \lambda)^*$ is a highest-weight of $\mathsf{M}(\lambda)^*$ isomorphic to $(\lambda_V \otimes \lambda)^*$.

- This induces a morphism $f : \mathsf{M}((\lambda_V \cdot \lambda)^*) \longrightarrow \mathsf{M}(\lambda)^*$.
- f is injective on $soc_{\mathcal{D}^{\leq 0}}(\mathsf{M}(\lambda_V \cdot \lambda)^*)$:

$$\begin{aligned}\langle x_{top} \cdot f((\lambda_V \cdot \lambda)^*), 1 \otimes \lambda\rangle &= \langle (\mathfrak{B}^{n_{top}}(V) \otimes \lambda)^*, \mathcal{S}(x_{top}) \otimes \lambda\rangle \\ &= \langle (\mathfrak{B}^{n_{top}}(V) \otimes \lambda)^*, x_{top} \otimes g_{x_{top}}^{-1} \cdot \lambda\rangle \neq 0,\end{aligned}$$

where x_{top} is G-comodule via $g_{x_{top}}$. □

Tensor products

- $\mathsf{W}(\lambda) = \mathcal{D} \otimes_{\mathcal{D}^{\leq 0}} \lambda =$ coVerma module.
- $\mathsf{Ind}(\lambda) = \mathcal{D} \otimes_{\mathcal{D}(G)} \lambda$.

If $\lambda \otimes \mu \simeq \oplus_i \lambda_i$, we set $\mathsf{Ind}(\lambda \cdot \mu) := \oplus_i \mathsf{Ind}(\lambda_i)$.

Lemma

$$\mathsf{W}(\lambda) \otimes \mathsf{M}(\mu) \simeq \mathsf{Ind}(\lambda \cdot \mu)$$

Proof: • Let $f : \mathsf{Ind}(\lambda \cdot \mu) \to \mathsf{W}(\lambda) \otimes \mathsf{M}(\mu)$ induced by $\lambda \otimes \mu \xrightarrow{\sim} (1 \otimes \lambda) \otimes (1 \otimes \mu)$.

• f is injective on $soc_{\mathcal{D}^{\leq 0}} \mathsf{Ind}(\lambda \cdot \mu)$:
If z is in the socle, then $z = x_{top} \sum_i y_i (h_i \otimes k_i)$ where $y_i \in \mathfrak{B}(\overline{V})$ and $(h_i \otimes k_i) \in \lambda \otimes \mu$. Hence

$$f(z) \in \sum g_{x_{top}}(y_i h_i) \otimes (x_{top} k_i) + \mathsf{W}(\lambda) \otimes (\oplus_{i=0}^{n_{top}-1} \mathsf{M}^{-i}(\mu)).$$

□

Tensor products

Corollary

Let P and Q be projective modules. Then

$$\mathsf{P} \otimes \mathsf{Q} \simeq \oplus_{\lambda,\mu\in\Lambda}\, p_{\mathsf{P},\mathsf{W}(\lambda)}\, p_{\mathsf{Q},\mathsf{M}(\mu)}\, \mathsf{Ind}(\lambda\cdot\mu).$$

Gracias!