Mixed perverse sheaves on flag varieties of Coxeter groups

Pramod Achar, Simon Riche, Cristian Vay

LSU Louisiana State University · Université Clermont Auvergne · CONICET

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Objective

Introduce and investigate **"mixed perverse sheaves on flag varieties"** for general Coxeter groups using a diagrammatic approach.

Motivation

The category of "mixed perverse sheaves" on the flag variety of a semisimple algebraic group is a sort of replacement for the "mixed category $\mathcal{O}$" in arbitrary characteristics. It also plays a crucial role in the construction of a Koszul duality in different contexts.

The first definition is due to Beĭlinson-Ginzburg-Soergel [4] in the characteristic zero setting. This was adapted to the modular setting by Achar-Riche [2]. They construct a t-structure on the bounded homotopy category of "parity complexes" on the flag variety (introduced by Juteau-Mautner-Williamson [6]) and define the "mixed perverse sheaves" as the object in the heart. They also show that this turns out to be a graded highest-weight category.

On the other hand, the category of "parity complexes" is equivalent to a Elias-Williamson [5] diagrammatic category by Riche-Williamson [7].

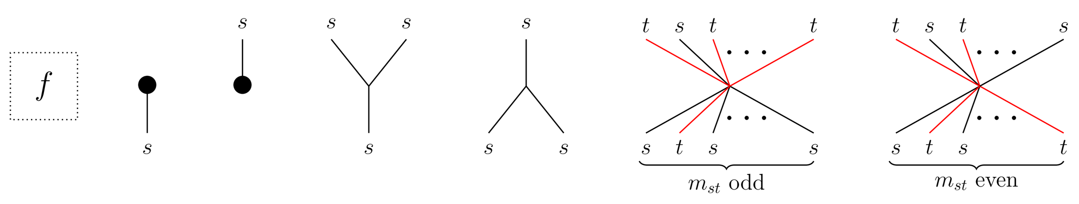
Preliminaries

(W, S) : Coxeter system with $|S| < \infty$.
$\leq$: Bruhat order on W.
$\Bbbk$: field (some results hold under mild assumptions).

A subset $I \subseteq W$ is **closed** if $x \in I \wedge y \leq x \Rightarrow y \in I$. A **locally closed** subset is the difference between two closed subsets.

The **Elias-Williamson diagrammatic category** [5] is a graded $\Bbbk$-linear monoidal category associated to (W, S) and an appropriated representation V of W. Let R be the symmetric algebra of V^*, considered as a graded ring with $\deg(V^*) = 2$.

- The objects are parametrized by the words in S.
- The generating morphisms are depicted by:



for all $f \in R$ and $s, t \in S$.
- The monoidal product is given by concatenation of words.

The Karoubian envelope $\mathscr{D}$ of this category is Krull–Schmidt. Its indecomposable objects are parametrized by W (up to grading shift).

$w \in W \rightsquigarrow B_w$: indecomposable object in $\mathscr{D}$.
I closed $\rightsquigarrow \mathscr{D}_I$: full subcategory of $\mathscr{D}$ generated by B_w, $w \in I$.
$I = I_0 \setminus I_1$ locally closed $\rightsquigarrow \mathscr{D}_I = \mathscr{D}_{I_0} /\!/ \mathscr{D}_{I_1}$
$\rightsquigarrow \overline{\mathscr{D}}_I$: category obtained from $\mathscr{D}_I$ by applying $\Bbbk \otimes_R (-)$ to Hom-spaces.

Example. The singleton $\{w\} = \{y \leq w\} \setminus \{y < w\}$ is locally closed for all $w \in W$. Then $\mathrm{End}_{\mathscr{D}_{\{w\}}}(B_w) \simeq R$ and hence

$$\mathscr{D}_{\{w\}} \cong \mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R) \quad \text{and} \quad \overline{\mathscr{D}}_{\{w\}} \cong \mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(\Bbbk).$$

[1] Achar–Makisumi–Riche–Williamson, *Free-monodromic mixed tilting sheaves on flag varieties*, arXiv:1703.05843.
[2] Achar–Riche, *Modular perverse sheaves on flag varieties II: Koszul duality and formality*, Duke Math. J. **165** (2016), 161–215.
[3] Beĭlinson–Bernstein–Deligne, *Faisceaux pervers*, in *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, Astérisque **100** (1982), 5–171.
[4] Beĭlinson–Ginzburg–Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527.
[5] Elias–Williamson, *Soergel calculus*, Represent. Theory **20** (2016), 295–374.
[6] Juteau–Mautner–Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), 1169–1212.
[7] Riche–Williamson *Tilting modules and the p-canonical basis*, preprint arXiv:1512.08296, to appear in Astérisque.

Ambient categories

Let I be locally closed. We begin by considering the bounded homotopy categories of the diagrammatic categories associated to I.

Biequivariant category		**Right-equivariant category**
$\mathsf{BE}_I = K^{\mathrm{b}}\mathscr{D}_I$	$\xrightarrow{\mathrm{For}^{\mathsf{BE}}_{\mathsf{RE}}}$	$\mathsf{RE}_I = K^{\mathrm{b}}\overline{\mathscr{D}}_I$

The categories $\mathsf{BE} = \mathsf{BE}_W$ and $\mathsf{RE} = \mathsf{RE}_W$ were intrduced first in [1]. They show that BE admits a monoidal structure $(\star)$ extending the product of $\mathscr{D}$ and RE is a right module category over BE.

Example. $\mathsf{BE}_{\{w\}} = K^{\mathrm{b}}\,\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R) \cong D^{\mathrm{b}}\,\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(R)$.

Recollement

Let $J \subset I$ be closed and finite. We construct a recollement structure [3]:

$$\mathsf{BE}_J \;\underset{(i^I_J)^!}{\overset{(i^I_J)^*}{\leftleftarrows}}\!\xrightarrow{(i^I_J)_*}\; \mathsf{BE}_I \;\underset{(i^I_{I\setminus J})_*}{\overset{(i^I_{I\setminus J})_!}{\leftleftarrows}}\!\xrightarrow{(i^I_{I\setminus J})^*}\; \mathsf{BE}_{I\setminus J}$$

by induction on $|J|$. We can then construct pullback and pushforward functors also for J locally closed.

Example. Let $w \in W$ and $s \in S$ with $w < ws$. There exists a canonical distinguished triangle

$$B_w\langle -1\rangle \longrightarrow \left(i^{\{w,ws\}}_{\{ws\}}\right)_! B_{ws} \longrightarrow B_{ws} \xrightarrow{[1]} \qquad (\blacktriangle)$$

in $\mathsf{BE}_{\{w,ws\}}$.

Standard and costandard objects

The pushforward functors associated to singletons play an important role. Let $w \in I$.

$$\mathsf{BE}_{\{w\}} \longrightarrow \mathsf{BE}_I$$
$$B_w \xmapsto{(i^I_w)_!} \Delta^I_w \ : \textbf{standard object}.$$
$$B_w \xmapsto{(i^I_w)_*} \nabla^I_w \ : \textbf{costandard object}.$$

Example. $\Delta_e = B_\emptyset$ and Δ_s, $s \in S$, is isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow B_s \longrightarrow B_\emptyset(1) \longrightarrow 0 \longrightarrow \cdots$$

with non-zero components in degree 0 and 1.

The perverse t-structure

The recollement data allows to define a t-structure

$$\left({}^p\mathsf{BE}^{\leq 0}_I, {}^p\mathsf{BE}^{\geq 0}_I\right)$$

on BE_I by induction, starting from a suitable t-structure on $\mathsf{BE}_{\{w\}}$ for all $w \in I$ (not the natural one!).

A heart to BE

The heart of the perverse t-structure is our main object of study. We call **perverse** the object which belong to it.

$$\mathsf{P}^{\mathsf{BE}}_I = {}^p\mathsf{BE}^{\leq 0}_I \cap {}^p\mathsf{BE}^{\geq 0}_I$$

Main properties of the standard and costandard objects

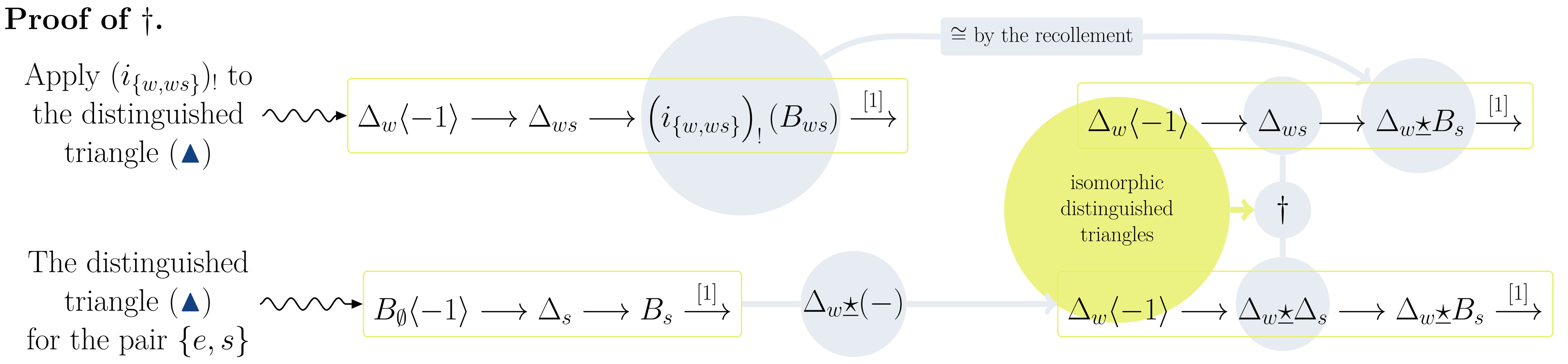
- The standard and costandard objects are perverse.
- They characterize the perverse t-structure as follows

$$ {}^p\mathsf{BE}^{\leq 0}_I = \langle \Delta^I_w\langle m\rangle[n] : w \in W,\ m \in \mathbb{Z},\ n \in \mathbb{Z}_{\geq 0}\rangle_{\mathrm{ext}},$$
$$ {}^p\mathsf{BE}^{\geq 0}_I = \langle \nabla^I_w\langle m\rangle[n] : w \in W,\ m \in \mathbb{Z},\ n \in \mathbb{Z}_{\leq 0}\rangle_{\mathrm{ext}}.$$

- Let $w \in W$ and $s \in S$ be such that $w < ws$. The following isomorphisms hold in BE:

$$\Delta_w \overset{\dagger}{\cong} \Delta_w \underline{\star} \Delta_s, \qquad \nabla_w \cong \nabla_w \underline{\star} \nabla_s,$$
$$\Delta_w \underline{\star} \nabla_{w^{-1}} \cong B_\varnothing \cong \nabla_{w^{-1}} \underline{\star} \Delta_w.$$

Proof of †.



Main properties of the heart of the right-equivariant categories

All the above also hold (*mutatis mutandis*) for the right-equivariant categories (and the functor $\mathrm{For}^{\mathsf{BE}}_{\mathsf{RE}}$ is t-exact). And even more! In particular, the heart P^{RE} of RE is our replacement for the **"mixed category $\mathcal{O}$"**.

- P^{RE} is a **graded highest weight category**. The standard and costandard objects $\overline{\Delta}_w$ and $\overline{\nabla}_w$ are defined as in BE for all $w \in W$. The irreducible objects are

$$L_w := \mathrm{im}(\overline{\Delta}_w \to \overline{\nabla}_w) \quad \forall w \in W.$$

In the definition of L_w we use that $\mathrm{Hom}_{\mathsf{RE}}(\overline{\Delta}_w, \overline{\nabla}_w) \simeq \Bbbk$.

- For all $w \in W$,

$$soc(\overline{\Delta}_w) \simeq L_e\langle -\ell(w)\rangle \quad \text{and} \quad top(\overline{\nabla}_w) \simeq L_e\langle \ell(w)\rangle.$$

- If $w \leq y$, then there exists an injective morphism

$$\overline{\Delta}_w \hookrightarrow \overline{\Delta}_y\langle \ell(y) - \ell(w)\rangle.$$

Any other morphism between standard objects is a shift or scalar multiple of it.

Tilt: additive full subcategory of P^{RE} generated by the tilting objects.

- The natural functors

$$K^{\mathrm{b}}(\mathrm{Tilt}) \xrightarrow{\sim} D^{\mathrm{b}}(\mathsf{P}^{\mathsf{RE}}) \xrightarrow{\sim} \mathsf{RE}.$$

are equivalences of triangulated categories
- If W is finite, there exist a Ringel duality and

$$\mathcal{T}_{w_0} \simeq \mathcal{P}_e\langle \ell(w_0)\rangle.$$

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