GAP and Nichols algebras

Nichols algebras over the sporadic simple groups

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Notations

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Remark

Since ρ is irreducible and g belongs to the center of $C_G(g)$ then $\rho(g)$ is a scalar (by Schur lemma)

Notations

The simple Yetter-Drinfeld modules over kG are

$$M(\mathcal{C},\rho) = \mathrm{ind}_{C_G(g)}^{G} W = \oplus_{i=1}^{n} x_i \otimes_k W$$

with

$$\mathcal{C} = \{g_1 = g, g_2 = x_2 g x_2^{-1}, \ldots, g_n = x_n g x_n^{-1}\}$$

a conjugacy class of g and ρ a representation of the centralizer of g in G.

The action is

$$y \cdot (x \otimes w) = yx \otimes w,$$

And the coaction is

$$\delta(x_i \otimes w) = x_i g x_i^{-1} \otimes (x_i \otimes w)$$

Notations

The braid is

$$c((x_i \otimes w) \otimes (x_j \otimes w')) = (g_i x_j \otimes w') \otimes (x_i \otimes w)$$

Note that

$$g_i g_j = g_j g_i \Leftrightarrow x_j^{-1} g_i x_j = z \in C_g(g)$$

Nichols algebras of diagonal type

Note that if $\dim W = \deg \rho = 1$, then $\{x_i \otimes 1\}$ is a basis of $M(\mathcal{C}, \rho)$ and then we have a braid of group type:

$$c((x_i \otimes 1) \otimes (x_j \otimes 1)) = q_{ij}(x_j \otimes 1) \otimes (x_i \otimes 1)$$

where $q_{ij} = \chi(x_j^{-1} g_i x_j)$ and $\chi \in \widehat{C_g(g)}$.

Remark

Heckenberger classified Nichols algebras of diagonal type. This classification includes the classification of semisimple Lie algebras.

Some of our tools

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- Some techniques from the diagonal case:

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- Some non-diagonal techniques[1][2]:
  - Two copies of the dihedral rack $\mathcal{D}_3$; and

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Lemmas of diagonal type

The following Lemma[3] is easy to apply (it depends only of the character table of the group):

Lemma

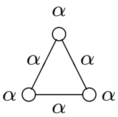
Asumme that $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$. *If* $g^{-1} \in \mathcal{C}$ *then* $\rho(g) = -1$ *and* g *has even order.*

[3] Andruskiewitsch, Zhang

Another tool from the diagonal case

Look for infinite-dimensional triangles such this one which appears in the alternating group

$$\mathbb{A}_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$$



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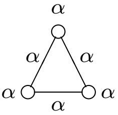

Lemma

Let $\iota : \mathbb{A}_4 \to G$ a monomorphism, σ an involution in $\mathbb{A}_4$ and $\mathcal{C}$ the conjugacy class in G of $\iota(\sigma)$. Then, $\dim \mathfrak{B}(\mathcal{C}) = \infty$.

We are interested in the classification of finite-dimensional pointed Hopf algebras over the sporadic simple groups.

We need to study Nichols algebras over these groups.

These groups are big. How can we perform the calculations?

About GAP

The GAP Group, GAP - Groups, Algorithms, and Programming
Version 4.4.10; 2007
http://www.gap-system.org

GAP is a computer software for working with group theory

Some of the authors are
Frank Celler, Steve Linton, Frank Lübeck, Werner Nickel, Martin Schönert, Thomas Breuer, Alexander Hulpke and many others.

What is ATLAS?

Atlas of finite group representations
http://brauer.maths.qmul.ac.uk/Atlas/v3

It contains information on about 716 groups

It has been prepared by
Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard Parker, Simon Norton, Simon Nikerson, Steve Linton, John Bray and Rachel Abbott.

GAP+ATLAS=AtlasRep

AtlasRep 1.3.1 (An Atlas of Group Representations)

Authors:
Robert A. Wilson, Richard A. Parker, Simon Nickerson, John N. Bray, Thomas Breuer

Example: J_1, the Janko group

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- It has 15 conjugacy classes: 1A, 2A, 3A, 5A, 5B, 6A, 7A, 7B, 10A, 10B, 11A, 15A, 15B, 19A, 19B, 19C

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- $J_1 = \langle a, b \mid a^2 = b^3 = (ab)^7 = (abab^2)^{19} = 1 \rangle$
- It has 15 conjugacy classes: 1A, 2A, 3A, 5A, 5B, 6A, 7A, 7B, 10A, 10B, 11A, 15A, 15B, 19A, 19B, 19C
- There exists a representation over $\mathbf{GL}(7, \mathbb{F}_{11})$

$$a = \begin{pmatrix} \cdot & 7 & 9 & 9 & 10 & 7 & 9 \\ 7 & \cdot & 1 & 4 & 3 & 3 & 4 \\ 9 & 1 & 2 & 8 & 3 & 6 & 2 \\ 9 & 4 & 8 & 10 & 1 & 6 & \cdot \\ 10 & 3 & 3 & 1 & 8 & 9 & 1 \\ 7 & 3 & 6 & 6 & 9 & 1 & 3 \\ 9 & 4 & 2 & \cdot & 1 & 3 & \cdot \end{pmatrix}$$

$$b = \begin{pmatrix} 5 & 7 & 1 & 8 & 5 & 8 & 2 \\ 10 & 4 & 10 & \cdot & 3 & 3 & 8 \\ 5 & 10 & 8 & 9 & 1 & 2 & 1 \\ 2 & 3 & 3 & 6 & 1 & 2 & 9 \\ 7 & 4 & 7 & 2 & 10 & 7 & 3 \\ 3 & 9 & \cdot & 4 & 2 & 3 & 5 \\ 3 & 4 & 3 & 3 & 9 & 4 & 9 \end{pmatrix}$$

How to get this information?

To define the J_1 group and display the generators:

```
gap>LoadPackage("atlasrep");;
gap>grp:=AtlasGroup("J1",Dimension,7,Characteristic,11);;
gap>Display(grp.1);
gap>Display(grp.2);
```

And to see that all the conjugacy classes of J_1 are real:

```
gap>RealClasses(CharacterTable(grp));
```

Other calculations related to J_1

We want to study, for example, the conjugacy class of involutions. For that purpose:

```
gap>ct:=CharacterTable(grp);
gap>cc:=ConjugacyClasses(grp);
gap>ClassNames(ct);
gap>c:=cc[ct.2a];
gap>g:=Representative(c);
gap>Display(g);
```

And the centralizer in J_1 of the representative g of the class 2A and it's character table:

```
gap>z:=Centralizer(grp,g);
gap>z_cc:=ConjugacyClasses(z);
gap>z_ir:=Irr(CharacterTable(z));
```

All conjugacy classes are real. So, the classes with representatives of odd order give infinite-dimensional Nichols algebras for every representation:

$$3A, 5A, 5B, 7A, 7B, 11A, 15A, 15B, 19A, 19B, 19C.$$

Then, we have the following

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Then, we have the following

Proposition

If $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$ then $\mathcal{C}$ is one of the conjugacy classes 2A, 6A, 10A, 10B and $\rho(g) = -1$.

The class 2A

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In a group G the number of solutions of the equation

$$abc = 1, \quad (a \in \mathcal{A},\, b \in \mathcal{B},\, c \in \mathcal{C})$$

is equal to

$$\xi(\mathcal{A},\mathcal{B},\mathcal{C}) = \sum_{\chi} \frac{\chi(\mathcal{A})\chi(\mathcal{B})\chi(\mathcal{C})}{\chi(1)}$$

(here $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are conjugacy classes)

To kill the conjugacy class 2A in J_1 just note that

$$\xi(2A, 3A, 3A) = 30 > 0$$

This is easy to calculate because it is a loop of order:

$$\text{number of conjugacy classes} = 15$$

And note that the naive approach contains a loop of order:

$$\#(2A) \cdot \#(3A) = 120 \cdot 30 = 3600$$

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Remark

For the Monster group:
$194 < 208654189522007807935270771114742592307 2 \cdot 10^{10}$

The class 10A

For the real class 10A we use the $\mathcal{D}_3$ tecnique:

```
gap>10a:=cc[ct.10a];
gap>g2:=Representative(10a);
gap>z:=Centralizer(grp,g2);
gap>for g1 in c do
>if not g1 in z and g1^2 in z then
>if (g2*g1*g2*Inverse(g1))*Inverse(g2)=g1 then
>Print("Ok!\n");
>break;
>fi;
>fi;
>od;
gap>Display(g1);
gap>Display(g2);
gap>Display(g1*g2*Inverse(g1));
```

The class 10A

$$g_1 = \begin{pmatrix} 6 & 9 & 8 & 3 & 5 & 10 & 4 \\ 9 & 10 & 2 & 7 & 2 & 4 & 0 \\ 2 & 3 & 9 & 1 & 3 & 5 & 2 \\ 3 & 8 & 10 & 3 & 10 & 2 & 1 \\ 2 & 1 & 0 & 4 & 0 & 8 & 2 \\ 1 & 3 & 7 & 0 & 7 & 8 & 4 \\ 8 & 1 & 0 & 4 & 1 & 5 & 2 \end{pmatrix},$$

The class 10A

$$g_2 = \begin{pmatrix} 3 & 2 & 9 & 10 & 5 & 2 & 3 \\ 4 & 2 & 5 & 5 & 1 & 5 & 2 \\ 5 & 6 & 10 & 2 & 5 & 4 & 2 \\ 5 & 9 & 10 & 3 & 2 & 2 & 3 \\ 9 & 1 & 8 & 3 & 4 & 5 & 5 \\ 1 & 6 & 9 & 2 & 2 & 6 & 9 \\ 8 & 9 & 10 & 9 & 5 & 10 & 10 \end{pmatrix}$$

The class 10A

$$g_3 = \begin{pmatrix} 4 & 9 & 0 & 1 & 8 & 0 & 9 \\ 0 & 10 & 7 & 3 & 8 & 4 & 7 \\ 4 & 3 & 1 & 1 & 5 & 0 & 9 \\ 8 & 6 & 6 & 2 & 1 & 8 & 4 \\ 4 & 9 & 9 & 1 & 7 & 2 & 0 \\ 1 & 9 & 3 & 3 & 5 & 2 & 9 \\ 8 & 3 & 1 & 3 & 9 & 10 & 1 \end{pmatrix}$$

Conclusion

The same technique could be used for killing the conjugacy class 10B.

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Theorem

If $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$ then $\mathcal{C} = 6A$ and ρ is the only representation such that $\rho(g) = -1$.