REPRESENTATIONS OF QUANTUM GROUPS AT ROOTS OF UNITY

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1. Introduction

At roots of unity the representation theory of quantum groups fails in semisimplicity. This is due mainly to the fact that Weyl modules are not always simple and because filtrations by Weyl modules do not always split. We will see that these problems occur in a sense that they can be managed so that suffciently much of what we would like from semisimplicity remains, and by means of the subcategory of tilting modules and an appropiate quotient of it, we can recover a suitable semisimple category of representations.

Categories of representations of quantum groups provide examples of ribbon categories, therefore, if the category is in addition semisimple, one could expect this category to be also modular, and thus provide knot invariants. In this work, for each compact, simple, simply-connected Lie group and each integer level we show that a modular tensor category can be constructed from a quotient of a certain subcategory of the category of representations of the corresponding quantum group. We give a detailed description of this category by means of a quantum version of the Racah formula for the decomposition of the tensor product and we develop the basic representation theory of quantum groups at roots of unity, including Harish-Chandra's Theorem.

Modular categories provide invariants of links and 3-manifolds. Attaching modular categories to quantum groups with this aim was first considered by Witten in [Wit89]. Witten argued that Chern-Simons theory for a compact, connected, simply-connected simple Lie group at integer level k should yield an invariant of links in a (biframed) three-manifold. He also sketched how to compute this invariant combinatorially using two-dimensional conformal field theory, and worked out the SU(2) invariant in enough detail to demonstrate that if well-defined it would have to give the Jones polynomial. A first step on Witten's program, was given in [RT91], where Reshetikhin and Turaev constructed an invariant that met all of Witten's criteria using the quantum group associated to $\mathfrak{sl}_2$ (the complexification of the Lie algebra of SU(2)) with the quantum parameter equal to a root of unity depending on the level k. Intuitively, since to each simple Lie algebra there is associated a quantum group, one could have expected that understanding the representation theory of this quantum group at roots of unity in an analogous fashion to Reshetikhin and Turaev's work on $\mathfrak{sl}_2$, one could presumably show that this representation theory formed a *modular tensor category,* and thus construct an invariant of links and three-manifolds, presumably the one Witten associated to the corresponding compact, simple Lie group.

Although it seemed that at this point an overall program was clear, it has not been fully completed. We rely on [S05] to provide the reader with a survey on the (semisimplified) category of representations of a quantum group at a root of unity.

There are, of course, more reasons why quantum roots at roots of unity became of interest. On a more algebraic side, much of the work in this field has focused on the relationship between the representation theory for algebraic groups over a field of prime characteristic p and the representation of the corresponding quantum group at a pth root of unity.

The difference between the interests of physics and mathematicians, is, according to Sawin in [S05], one of the reasons that have obstructed the completion of Witten's program. One of the main obstacles is related to the order of the root of unity, since while mathematicians became interested, as we said, in roots of order prime, the values that correspond to integer levels and hence the cases of primary interest for physicists are even roots of unity (in fact multiples of the entries of the Cartan matrix). Sawin points that it has also been a source of confusion the fact that much of the algebraic work deals only with representations with highest weights in the root lattice, while in topology and physics one is interested in all representations whose highest weights lie in the weight lattice.

In this work we prove the fundamental results on the representation theory of quantum groups needed for applications to topology and physics for all nongeneric values of the parameter and all representations.

We do not want to refer here to definitions concerning category theory and Hopf algebras, since these concepts have been largely studied during the Master Class to which this work belong. We assume the reader is familiarized with the basic concepts of these subjects, and humbly suggest to the non-specialized reader in these two areas to look through the pages of the following books:

Category theory:

(1) *Categories for the working mathematician.* Mac Lane, Saunders. [Mac71]

Hopf algebras:

(1) *Hopf algebras.* Sweedler [S69].
(2) *Quantum groups.* Kassel [Kas95].
(3) *Hopf algebras and their action on rings.* Montgomery [Mon93]

Throughout this work, when referring to Hopf algebras, we will make use of the so called *Sweedler notation* [S69] for the coproduct. Therefore, if H is a Hopf algebra with comultiplication Δ and $x \in H$, we will, for example, write:

$$\Delta(x) = x_{(1)} \otimes x_{(2)},$$

and

$$(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}.$$

We devote Section 2 we fix the notation for the concepts related to a Lie algebra $\mathfrak{g}$.

On Section 3 we define the general quantum group $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$, reviewing some elementary results from the literature.

Section 3 constructs the quantum group at roots of unity. Here, we define the affine Weyl group and use it to prove Theorem 4.15, the quantum version of Harish-Chandra's Theorem.

On Section 4 (working over $U_{\mathbf{q}}^{\mathrm{res}}(\mathfrak{g})$ and $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$), we define tilting modules as in [AP95]. We prove here that they form a tensor category and that highest weight tilting modules are irreducible in this category.

On Section 5 we introduce the so-called negligible modules and it is shown here that in fact every highest weight tilting module outside the Weyl alcove is negligible.

Section 5 gives an application of the technology developed in the previous sections. Specifically it gives A version of this formula for Weyl modules with highest weight in the root lattice appears in [AP95].

Finally Section 6 we consider the ribbon category associated to the set of all modules and the one associated to the set of all tilting modules. The quotient of the tilting modules by the negligible tilting modules is given and proven to be a semisimple ribbon category. We describe this category in detail, in particular giving a quantum version of the Racah formula, which expresses multiplicities of the tensor products of two Weyl modules in terms of weight multiplicities.

We end this work with some comments on modularity and further research on Section 7

2. Lie algebras

We follow Humphreys [Hum72] for all our results on Lie algebras, and for the most part, notation. The following paragraph follows [Hum72][Ch. 10].

Let $\mathfrak{g}$ be a complex simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{h}^*$ its dual vector space. We fix the following list of notation and definitions:

- Let $\Phi \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$.
- Let $\langle \cdot, \cdot \rangle$ be the unique inner product on $\mathfrak{h}^*$ (and hence on $\mathfrak{h}$) such that $\langle \alpha, \alpha \rangle = 2$ for every short root $\alpha \in \Phi$ (this convention guarantees that the inner product of two roots is an integer.
- Let $\check{\Phi} = \{\check{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle \mid \alpha \in \Phi\}$ be the dual root system to Φ.
- Let $\Lambda = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}, \, \forall \alpha \in \Phi\}$, be the weight lattice.
- Let $\Lambda_r = \mathbb{Z}\Phi \subset \Lambda$ be the root lattice and $\check{\Lambda}_r = \mathbb{Z}\check{\Phi} \subset \frac{1}{D}\Lambda$ the dual root lattice.
- Let $\mathcal{W}$, the Weyl group, be the group of isometries of $\mathfrak{h}^*$ generated by reflections about the roots $\alpha \in \Phi$. Thus in particular for each root $\alpha \in \Phi$ we have a $\sigma_\alpha \in \mathcal{W}$ defined by $\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \, \alpha$. In fact we will most often be interested in the translated action of the Weyl group, which is defined by $\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho$.
- Let $\Delta = \{\alpha_1, \dots, \alpha_N\} \subset \Phi$ be a base.
- Let $d_i = \langle \alpha_i, \alpha_i \rangle / 2$ (so $d_i = 1$ for short roots and $d_i = D$ for long).
- Let $\alpha > \beta$ mean $\alpha - \beta$ is a nonnegative linear combination of the elements of Δ.
- Let $(a_{ij}) = \langle \alpha_i, \check{\alpha_j} \rangle$ be the Cartan matrix.
- Let $\Lambda^+ = \{\lambda \in \Lambda | \langle \lambda, \alpha_i \rangle \geq 0, \forall \alpha_i \in \Delta\}$.
- Let θ be the longest root, i.e., the unique long root in $\Phi \cap \Lambda^+$,
- Let ϕ be the unique short root in the same intersection.
- Let $\rho = \sum_{\alpha > 0} \alpha/2$.

We also collect some numbers related to $\mathfrak{g}$:

- Let L be the least integer such that $L\langle \lambda, \gamma \rangle \in \mathbb{Z}$ whenever $\lambda, \gamma \in \Lambda$.
- Let D be the ratio of the square lengths of the long and short roots.
- Let $\check{h} = \langle \rho, \check{\theta} \rangle + 1$ be the *dual Coxeter number,*.
- Let $h = \langle \rho, \check{\phi} \rangle + 1$ be the *Coxeter number.*

For convenience we summarize these quantities (all information taken from [Hum72]).

	A_n	B_{2n+1}	B_{2n}	C_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8	F_4	G_2
L	$n+1$	2	1	1	2	4	3	2	1	1	1
D	1	2	2	2	1	1	1	1	1	2	3
h	$n+1$	$4n+2$	$4n$	$2n$	$4n-2$	$2n$	12	18	30	12	6
$\check{h}$	$n+1$	$4n+1$	$4n-1$	$n+1$	$4n-2$	$2n$	12	18	30	9	4

3. Quantum groups with generic parameter

We will introduce here several different (and non equivalent) versions of a quantum group related to a Lie algebra, with generic paremeter q. We will try to do it in a way such that the reader could easily keep track of each during the work. Further, we will review the most relevant facts about their representation theory, which have been covered by the Master Class and we refer the reader to look at the book [CP94] of Chari and Pressley for details.

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Given integers m, n let

$$[n]_q = (q^n - q^{-n})/(q - q^{-1}) \in \mathcal{A},$$
$$[n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q \in \mathcal{A},$$
$$\begin{bmatrix} m \\ n \end{bmatrix}_q = [m]_q!/([n]_q![m-n]_q!) \in \mathcal{A}.$$

Let $q_i = q^{d_i}$.

Definition 3.1. *We define the Hopf algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$ with generators E_i, F_i, K_i and relations:*

$$K_iK_j = K_jK_i \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$K_iE_jK_i^{-1} = q^{\langle\alpha_i,\alpha_j\rangle}E_j, \qquad K_iF_jK_i^{-1} = q^{-\langle\alpha_i,\alpha_j\rangle}F_j,$$
$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (E_i)^{1-a_{ij}-r}E_j(E_i)^r = 0 \qquad \textit{if } i \neq j,$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (F_i)^{1-a_{ij}-r}F_j(F_i)^r = 0 \qquad \textit{if } i \neq j.$$

It is a Hopf algebra with comultiplication:

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

antipode:

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i,$$

and counit:

$$\epsilon(K_i) = 1, \qquad \epsilon(E_i) = \epsilon(F_i) = 0.$$

Proposition 3.2. [CP94] *For each $\lambda \in \Lambda^+$ there is a unique, irreducible, highest weight λ $U_q(\mathfrak{g})$-module of type I, the Weyl module W_q^λ which is a direct sum of its weight spaces, and the dimensions of whose weight spaces is the same as that of the classical Weyl module W^λ (a weight λ vector in a type I module is a vector v such that $K_iv = q^{\langle\alpha_i,\lambda\rangle}v$ for all i). The tensor product of two Weyl modules is isomorphic to a direct sum of Weyl modules with multiplicities the same as those in the classical case.*

Definition 3.3. *Define $E_i^{(l)} = E_i^l/[l]_{q_i}!$, and likewise for F_i. We define the $\mathcal{A}$-subalgebra $U_{\mathcal{A}}^{res}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ to be generated by the elements $E_i^{(r)}$, $F_i^{(r)}$, $K_i^{\pm 1}$, for $1 \leq i \leq N$ and $r \geq 1$.*

$U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ is an integral form of $U_q(\mathfrak{g})$ in the sense that $U_q(\mathfrak{g}) = U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{Q}(q)$, and $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ is a free $\mathcal{A}$-algebra.

There exist $E_{\beta_1}, F_{\beta_1}, E_{\beta_2}, F_{\beta_2}, \ldots, E_{\beta_N}, F_{\beta_N}, \in U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$, where $\beta_1, \ldots, \beta_N$ is an enumeration of the positive roots, such that each E_{β_i}, F_{β_i} satisfies $K_j E_{\beta_i} K_j^{-1} = q^{\langle \alpha_j, \beta_i \rangle} E_{\beta_i}$ and the set consisting of all $(E_{\beta_N})^{(l_N)} \cdots (E_{\beta_1})^{(l_1)}$ forms a basis for the subalgebra $U_q^+(\mathfrak{g})$ generated by $\{1, E_i^{(k)}\}$ (and likewise for F, with $U_q^-(\mathfrak{g})$ defined correspondingly).

Proposition 3.4. [CP94] *If v is a highest weight vector of a module W^λ, then $W_{\mathcal{A}}^\lambda = U_{\mathcal{A}}^{res}(\mathfrak{g}) \cdot v$ is a $U_{\mathcal{A}}^{res}(\mathfrak{g})$-submodule of W^λ and is a direct sum of its intersections with the weight spaces of W^λ, each of which is a free $\mathcal{A}$-module of finite rank.*

These two versions of a quantum groups are not ribbon Hopf algebras. Below we give an integral form of the quantum group which is a ribbon Hopf algebra (technically a topological Hopf algebra, in the sense that the comultiplication maps to a completed tensor product).

Let $\mathcal{A}' = \mathbb{Z}[s, s^{-1}]$ and define a monomorphism $\mathcal{A} \to \mathcal{A}'$ by $q \mapsto s^L$ (henceforth we will treat this monomorphism as the identity and write $q = s^L$).

Definition 3.5. *We define the $\mathcal{A}'$ Hopf algebra $U_{\mathcal{A}'}^{res}(\mathfrak{g}) = U_{\mathcal{A}}^{res}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathcal{A}'$ and the $U_{\mathcal{A}'}^{res}(\mathfrak{g})$-module $W_{\mathcal{A}'}^\lambda = W_{\mathcal{A}}^\lambda \otimes_{\mathcal{A}} \mathcal{A}'$.*

The collection of all set-theoretic functions from the addtive group Λ to $\mathcal{A}'$, $\mathrm{Map}(\Lambda, \mathcal{A}')$, is naturally an algebra over $\mathcal{A}'$ with pointwise multiplication. A topological basis for this algebra is given by $\{\delta_\lambda\}_{\lambda \in \Lambda}$, where $\delta_\lambda(\gamma) = \delta_{\lambda,\gamma}$. By topological basis we mean the elements are linearly independent and span a dense subspace of $\mathrm{Map}(\Lambda, \mathcal{A}')$ in the topology of pointwise convergence.

It is a (topological) Hopf algebra when given the comultiplication $\Delta(f)(\mu, \mu') = f(\mu + \mu')$, the counit $\epsilon(f) = f(0)$, and the antipode $S(f)(\mu) = f(-\mu)$ for $f \in \mathrm{Map}(\Lambda, \mathcal{A}')$ and $\mu, \mu' \in \Lambda$. Here

$$\Delta\colon \mathrm{Map}(\Lambda, \mathcal{A}') \to \mathrm{Map}(\Lambda \times \Lambda, \mathcal{A}').$$

The latter space contains the natural embedding of $\mathrm{Map}(\Lambda, \mathcal{A}') \otimes \mathrm{Map}(\Lambda, \mathcal{A}')$ as a dense subspace in the topology of pointwise convergence, and thus may be viewed as the completed tensor product.

Recall any Abelian group with a homomorphism to its dual has an R-matrix associated to the homomorphism in the Hopf algebra of functions. In the case of the homomorphism $\lambda \mapsto s^{L\langle \lambda, \cdot \rangle}$, the R-matrix is $\sum_{\lambda,\gamma} s^{L\langle \lambda,\gamma \rangle} \delta_\lambda \otimes \delta_\gamma$, which once again is an element not of the tensor product of the Hopf algebra with itself but of the completion.

Definition 3.6. *Let $U_{\mathcal{A}'}^\dagger(\mathfrak{h})$ be* $\mathrm{Map}(\Lambda, \mathcal{A}')$ *viewed as a topological ribbon Hopf algebra.*

Notation 3.7. Defining $\lambda_i \in \Lambda$ by $\langle \lambda_i, \check{\alpha_j} \rangle = \delta_{i,j}$ we can write the canonical dual element to the pairing as $\sum_i \lambda_i \otimes \check{\alpha_i}$. We denote by q^λ the homomorphism

$$\sum_{\gamma \in \Lambda} s^{L\langle \lambda, \gamma \rangle} \delta_\gamma \in \mathrm{Map}(\Lambda, \mathcal{A}').$$

Likewise, we write

$$q^{\lambda \otimes \gamma} = \sum_{\lambda', \gamma'} s^{L\langle \lambda, \lambda' \rangle \langle \gamma, \gamma' \rangle} \delta_{\lambda'} \otimes \delta_{\gamma'} \in \mathrm{Map}(\Lambda, \mathcal{A}') \otimes \mathrm{Map}(\Lambda, \mathcal{A}'),$$

and then formally we can refer to the R-matrix above as $q^{\sum_i \lambda_i \otimes \check{\alpha_i}}$. We develop this equality with detail so as to meka the reader feel comfortable with the notation. Precisely,

$$q^{\lambda_i \otimes \check{\alpha_i}} = \sum_{\mu^i, \eta^i} q^{\langle \check{\alpha_i}, \mu^i \rangle \langle \lambda_i, \eta_i \rangle} \delta_{\eta^i} \otimes \delta_{\mu^i},$$

and so

$$\begin{aligned}
q^{\sum_i \lambda_i \otimes \check{\alpha}_i} &= \prod_i \sum_{\mu^i, \eta^i \in \Lambda} q^{\langle \check{\alpha}_i, \mu^i \rangle \langle \lambda_i, \eta^i \rangle} \delta_{\eta^i} \otimes \delta_{\mu^i} \\
&= \sum_{\substack{(\mu^1, \ldots, \mu^N), \\ (\eta^1, \ldots, \eta^N) \in \Lambda^N}} \prod_{j=1}^{N} \left(q^{\langle \check{\alpha}_j, \mu^i \rangle \langle \lambda_j, \eta^i \rangle} \delta_{\eta^i} \otimes \delta_{\mu^i} \right) \\
&= \sum_{\mu, \eta \in \Lambda} \prod_{j=1}^{N} q^{\langle \check{\alpha}_j, \mu \rangle \langle \lambda_j, \eta \rangle} \delta_\eta \otimes \delta_\mu \\
&= \sum_{\mu, \eta \in \Lambda} q^{\sum_{j=1}^{N} \langle \langle \check{\alpha}_i, \mu \rangle \lambda_j, \eta \rangle} \delta_\eta \otimes \delta_\mu \\
&= \sum_{\mu, \eta \in \Lambda} q^{\langle \mu, \eta \rangle} \delta_\mu \otimes \delta_\eta .
\end{aligned}$$

Define the weight w of a monomial in $\{E_i, F_i, K_i\}$ to be the sum of α_i for each factor of E_i and $-\alpha_i$ for each factor of F_i. $U^\dagger_{\mathcal{A}'}(\mathfrak{h})$ acts on $U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$ by Hopf automorphisms via the Λ-grading of $U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$.

Specifically, $f \in U^\dagger_{\mathcal{A}'}(\mathfrak{h})$ acts on a monomial X by $f[X] = f(w(X))X$ and extends linearly. This action is automorphic in the sense that $\Delta(h[X]) = h^{(1)}[X^{(1)}] \otimes h^{(2)}[X^{(2)}]$ and $h[XY] = h_{(1)}[X]h_{(2)}[Y]$. In fact,

$$\begin{aligned}
h[XY] &= h(w(XY))XY = h(w(X) + w(Y))XY = \Delta(h)(w(X) \otimes w(Y))XY \\
&= h_{(1)}(w(X))X h_{(2)}(w(Y))Y = h_{(1)}[X]h_{(2)}[Y],
\end{aligned}$$

$$\begin{aligned}
\Delta(h[X]) &= h(w(X))X_{(1)} \otimes X_{(2)} = h(w(X_{(1)}) + w(X_{(2)}))X_{(1)} \otimes X_{(2)} \\
&= h_{(1)}(w(X_{(1)}) \otimes h_{(2)}(w(X_{(2)}))X_{(1)} \otimes X_{(2)} = h^{(1)}[X^{(1)}] \otimes h^{(2)}[X^{(2)}],
\end{aligned}$$

since it is straightforward to check that $U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$ is a graded Hopf algebra, in the sense that

$$\Delta((U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g}))_\lambda) \subseteq \bigoplus_{\gamma + \eta = \lambda} (U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g}))_\gamma \otimes (U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g}))_\eta .$$

As such we can form the smash product Hopf algebra (see [Mon93] for a general definition, or think of it as a semidirect product) $U^\dagger_{\mathcal{A}'}(\mathfrak{h}) \ltimes U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$. It is clear that this Hopf algebra is (densely) generated by $\{E_i, F_i, K_i\} \cup \{\delta_\lambda\}_{\lambda \in \Lambda}$, with the standard quantum group relations together with

$$\delta_\lambda \delta_\gamma = \delta_{\lambda,\gamma} \delta_\lambda, \qquad \sum_{\lambda \in \Lambda} \delta_\lambda = 1$$

$$\delta_\lambda K_i = K_i \delta_\lambda \qquad \delta_\lambda E_i = E_i \delta_{\lambda - \alpha_i} \qquad \delta_\lambda F_i = F_i \delta_{\lambda + \alpha_i} .$$

If $U^\dagger_{\mathcal{A}'}(\mathfrak{h}) \ltimes U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$ acts on an $\mathcal{A}'$-module V, and $v \in V$, we say v is of weight $\lambda \in \Lambda$ if $K_i v = q^{\langle \lambda, \alpha_i \rangle} v$ and $fv = f(\lambda)v$ for $f \in U^\dagger_{\mathcal{A}'}(\mathfrak{h})$, and we say V is a λ weight space if it consists entirely of weight λ vectors.

Let $\mathfrak{W}$ be the direct product of all $U^\dagger_{\mathcal{A}'}(\mathfrak{h}) \ltimes U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$-modules which are a finite direct sum of $\mathcal{A}'$-free λ weight spaces for $\lambda \in \Lambda$. Of course $U^\dagger_{\mathcal{A}'}(\mathfrak{h}) \ltimes U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$ acts on $\mathfrak{W}$. The kernel of this action is a two-sided ideal (clearly it includes at least $K_i - q^{\alpha_i},$), and since the tensor product of two finite direct sums of $\mathcal{A}'$-free λ-spaces for $\lambda \in \Lambda$ is another such, it is a Hopf ideal. Thus the quotient is a Hopf algebra which embeds into the module of endomorphisms of $\mathfrak{W}$. The product topology on $\mathfrak{W}$ gives the space of endomorphisms a topology, one in which a sequence converges if and only if it converges on each finite-dimensional submodule (these being discrete, this happens when the sequence is eventually constant on each submodule).

Definition 3.8. *We define $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$ to be the closure of the image of $U^{\dagger}_{\mathcal{A}'}(\mathfrak{h}) \ltimes U^{res}_{\mathcal{A}'}(\mathfrak{g})$ in* $\mathrm{End}(\mathfrak{W})$ *in this topology. The product, coproduct, antipode and counit clearly extend to the completion (Δ has range the closure of $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g}) \otimes U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$ in* $\mathrm{End}_{\mathcal{A}'}(\mathfrak{W} \otimes \mathfrak{W})$, *which we will refer to as $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g}) \overline{\otimes} U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$, the completed tensor product).*

Remark 3.9. We remark again that, in this new algebra, we have relations $K_i = q^{\alpha_i}$.

Also, note that $W^{\lambda}_{\mathcal{A}'}$ ($\lambda \in \Lambda^+$) can be made into a $U^{\dagger}_{\mathcal{A}'}(\mathfrak{h}) \ltimes U^{\mathrm{res}}_{\mathcal{A}'}(\mathfrak{g})$-module by letting $f \in U^{\dagger}_{\mathcal{A}'}(\mathfrak{h})$ act on a weight λ vector by multiplication by $f(\lambda)$. $W^{\lambda}_{\mathcal{A}'}$ is a finite direct sum of free weight spaces, as above, so any pair of elements of the semidirect product that act as different endomorphisms on some $W^{\lambda}_{\mathcal{A}'}$ represents different elements of $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$.

This extended Hopf algebra $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$ is a ribbon Hopf algebra. Specifically notice that our earlier R-matrix $q^{\sum_i \check{\alpha}_i \otimes \lambda_i}$ is an element of $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g}) \overline{\otimes} U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$. Therefore so is

$$R = q^{\sum_i \check{\alpha}_i \otimes \lambda_i} \sum_{t_1, \dots t_N = 1}^{\infty} \prod_{r=1}^{N} q_{\beta_r}^{t_r(t_r+1)/2} (1 - q_{\beta_r}^{-2})^{t_r} [t_r]_{q_{\beta_r}}! E_{\beta_r}^{(t_r)} \otimes F_{\beta_r}^{(t_r)} \in U^{\dagger}_{\mathcal{A}'}(\mathfrak{g}) \overline{\otimes} U^{\dagger}_{\mathcal{A}'}(\mathfrak{g}) \tag{1}$$

where $q_{\beta_r} = q^{d_i}$ when β_r is the same length as α_i.

Proposition 3.10 ([CP94])**.** *$U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$ is a ribbon Hopf algebra with R-matrix R as above. Further, the grouplike element q^{ρ} is a charmed element of the Hopf algebra for this R.*

Remark 3.11. In particular, conjugation by $q^{2\rho}$ is the square of the antipode, so that for any finite-dimensional $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$ module V, free over $\mathcal{A}'$, the functional

$$\mathrm{qtr}_V : U^{\dagger}_{\mathcal{A}'}(\mathfrak{g}) \to \mathcal{A}'$$
$$\mathrm{qtr}_V(x) = \mathrm{tr}_V(q^{2\rho} x)$$

is an invariant functional on $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$ in the sense that $\mathrm{qtr}(a_{(1)} b S(a_{(2)})) = \epsilon(a)\,\mathrm{qtr}(b)$. In fact,

$$\begin{aligned} \mathrm{tr}_V(q^{2\rho} a_{(1)} b S(a_{(2)})) &= \mathrm{tr}_V(q^{2\rho} a_{(1)} q^{-2\rho} q^{2\rho} b S(a_{(2)})) = \mathrm{tr}_V(S^2(a_{(1)}) q^{2\rho} b S(a_{(2)})) \\ &= \mathrm{tr}_V(q^{2\rho} b S(a_{(2)}) S^2(a_{(1)})) = \mathrm{tr}_V(q^{2\rho} b S(S(a_{(1)}) a_{(2)})) = \epsilon(a)\,\mathrm{tr}_V(q^{2\rho} b). \end{aligned}$$

Define the *quantum dimension*

$$\mathrm{qdim}(V) = \mathrm{qtr}_V(1) = \mathrm{tr}(q^{2\rho})$$

and in particular define $\mathrm{qtr}_{\lambda} = \mathrm{qtr}_{W^{\lambda}_{\mathcal{A}'}}$ and $\mathrm{qdim}(\lambda) = \mathrm{qdim}(W^{\lambda}_{\mathcal{A}'})$. Notice that since $\mathrm{qtr}_{V \otimes W} = \mathrm{qtr}_V\, \mathrm{qtr}_W$,

$$\mathrm{qdim}(V \otimes W) = \mathrm{qdim}(V)\,\mathrm{qdim}(W).$$

Finally, a simple calculation modeled on the classical Weyl character formula gives ([CP94])

$$\mathrm{qdim}(\lambda) = \prod_{\beta > 0} \left(q^{\langle \lambda + \rho, \beta \rangle} - q^{-\langle \lambda + \rho, \beta \rangle} \right) / \left(q^{\langle \rho, \beta \rangle} - q^{-\langle \rho, \beta \rangle} \right) \in \mathcal{A}. \tag{2}$$

4. Quantum groups at roots of 1. The affine Weyl group

We now restrict the generic q to a root of unity. Specifically, let l be a positive integer, and consider the homomorphism $\mathcal{A}' \to \mathbb{Q}[\mathbf{s}]$, where $\mathbf{s}$ is an abstract primitive lLth root of unity (i.e. satisfies the lLth cyclotomic polynomial) given by $s \mapsto \mathbf{s}$. Write $\mathbf{q} = \mathbf{s}^L$. Likewise $q \mapsto \mathbf{q}$ gives a homomorphism $\mathcal{A} \to \mathbb{Q}[\mathbf{q}]$,

Definition 4.1. *As before, define* $U_{\mathbf{s}}^{\dagger}(\mathfrak{g}) = U_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$, *and* $U_{\mathbf{q}}^{res}(\mathfrak{g}) = U_{\mathcal{A}}^{res}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{Q}[\mathbf{q}]$.

We can define also $W_{\mathbf{s}}^{\lambda} = W_{\mathcal{A}'}^{\lambda} \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$.

Notation 4.2. Write $\mathbf{q}^{\lambda}$ for $q^{\lambda} \otimes 1 \in U_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) \otimes \mathbb{Q}[\mathbf{q}] = U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$.

Notice $U_{\mathbf{s}}^{\dagger} \otimes U_{\mathbf{s}}^{\dagger} \cong (U_{\mathcal{A}'}^{\dagger} \otimes U_{\mathcal{A}'}^{\dagger}) \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$ embeds naturally (and densely in the inherited topology) into $(U_{\mathcal{A}'}^{\dagger} \overline{\otimes} U_{\mathcal{A}'}^{\dagger}) \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$, and thus we may define the latter space as the completed tensor product $U_{\mathbf{s}}^{\dagger} \overline{\otimes} U_{\mathbf{s}}^{\dagger}$. $U_{\mathbf{s}}^{\dagger}$ then becomes a Hopf algebra, and in fact a ribbon Hopf algebra since the image of R is in $U_{\mathbf{s}}^{\dagger} \overline{\otimes} U_{\mathbf{s}}^{\dagger}$.

For each $i \leq n$ let l_i be $l/\gcd(l, d_i)$ (that is, the index of q_i) and let l_i' be l_i or $l_i/2$ according to whether l_i is odd or even (so that l_i' is the least natural number such that $q_i^{l_i'} \in \{\pm 1\}$. Likewise let l' be l or $l/2$ according to whether l is odd or even.

Definition 4.3. *The* affine Weyl group, $\mathcal{W}_l$, *is the group of isometries of* $\mathfrak{h}^*$ *generated by reflection about the hyperplanes*

$$\langle x, \alpha_i \rangle = \langle k l_i' \alpha_i/2, \alpha_i \rangle = k l_i' d_i$$

for each $k \in \mathbb{Z}$ *and each* $\alpha_i \in \Delta$.

Remark 4.4. $\mathcal{W}_l$ includes the Weyl group $\mathcal{W}$ as a subgroup (when $k = 0$).

Again we will usually be interested in the translated action of the affine Weyl group, given by $\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho$.

Lemma 4.5.

- *If* l' *is divisible by* D, *the affine Weyl group is the semidirect product of the ordinary Weyl group is the semidirect product of the ordinary Weyl group the group of translations* $l'\check{\Lambda}_r$. *In particular a set of generators consists of reflections* σ_{α_i}, $\alpha_i \in \Delta$ *together with translation by* $l'\theta/D$. *A fundamental domain for the translated action of the affine Weyl group is the* principal Weyl alcove, C_l, *which is the region* $\langle x + \rho, \alpha_i \rangle \geq 0$, $\langle x + \rho, \theta \rangle \leq l'$.
- *if* l' *is not divisible by* D, *the affine Weyl group is the semidirect product of the ordinary Weyl group is the semidirect product of the ordinary Weyl group the group of translations* $l'\Lambda_r$. *In particular a set of generators consists of reflections* σ_{α_i}, $\alpha_i \in \Delta$ *together with translation by* $l'\phi$. *A fundamental domain for the translated action of the affine Weyl group is the* principal Weyl alcove, C_l, *which is the region* $\langle x + \rho, \alpha_i \rangle \geq 0$, $\langle x + \rho, \phi \rangle \leq l'$.

Proof. Reflection about the hyperplanes $\langle x, \alpha_i \rangle = 0$ followed by reflection about $\langle x, \alpha_i \rangle = \langle l_i' \alpha_i/2, \alpha_i \rangle$ gives translation by $l_i' \alpha_i$. In fact, this last reflection is given by

$$z \mapsto \sigma_i\left(z - \frac{l_i' d_i}{\langle \alpha_i, \alpha_i \rangle}\right) + \frac{l_i' d_i}{\langle \alpha_i, \alpha_i \rangle}$$

and then the composition of the two reflections gives, for $z = \sigma_i(y)$

$$\begin{aligned} y \mapsto & \sigma_i\left(\sigma_i(y) - \frac{l_i' d_i}{\langle \alpha_i, \alpha_i \rangle}\right) + \frac{l_i' d_i}{\langle \alpha_i, \alpha_i \rangle} \\ &= y + \frac{2 l_i' d_i}{\langle \alpha_i, \alpha_i \rangle} \alpha_i = y + l_i' \alpha_i. \end{aligned}$$

This translation, since $d_i = 1$ or $d_i = D$ and D is prime, is translation by $l'\alpha_i$ or $l'\check{\alpha}_i$ according to the divisibility of l'. Conjugation by σ_β for various β gives translation by $l'\beta$ or $l'\check{\beta}$. Thus $\mathcal{W}_l$ contains the groups mentioned, and clearly is generated by them. Since the Weyl group acts by conjugation on the group of translations, the full group is a semidirect product.

The subgroup of translations is generated by $l'\check{\beta}$ (resp. $l'\beta$) for β a long root of Φ (resp. β a short root of Φ). Thus a fundamental domain for the group of translations would be the polygon bounded by the hyperplanes $\langle x+\rho, \check{\beta}\rangle \le l'$ (resp. $\langle x+\rho, \beta\rangle \le l'$) for all such β. Now, this region is invariant under the translated action of $\mathcal{W}$: if x is on the region, then so it is $\sigma \cdot x$, for $\sigma \in \mathcal{W}$:

$$\langle \sigma\cdot x-\rho, \beta\rangle = \langle \sigma(x-\rho), \beta\rangle = \langle x-\rho, \sigma(\beta)\rangle = \langle x-\rho, \beta'\rangle \le l'.$$

Therefore, a fundamental region for $\mathcal{W}_l$ is given by the intersection of this region with a fundamental region of this action of $\mathcal{W}$ which is exactly the region given. □

Remark 4.6. When $\mathfrak{g}$ is not simply-laced the affine Weyl group's action is distinctly different when l' is divisible by D and when it is not. When l' is divisible by D, this is the affine Weyl group relevant to affine Lie algebras, and the affine Weyl group of the root system Φ, but where the translations are multiplied by l'/D. When l' is not divisible by D it is (with multiplication by l') the affine Weyl group of the dual root system $\check{\Phi}$. Therefore, in the context of quantum groups, both Weyl groups appear with identical meaning.

Lemma 4.7. *The affine Weyl group $\mathcal{W}_l$ is the largest subgroup of $\mathcal{W}^\dagger \overset{def}{=} \mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$ which fixes the root lattice Λ_r under the translated action. In particular these two are equal when $2D|l$.*

Proof. About the last claim, note simply that if $2D|l$, then $\mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r = \mathcal{W} \ltimes l'\check{\Lambda}_r = \mathcal{W}_l$, since $D|l$. Now we focus on the main claim: The subgroup of the group of translations $\frac{l}{2}\check{\Lambda}_r$ which preserves Λ_r is $\frac{l}{2}\check{\Lambda}_r \cap \Lambda_r$, and, since a half dual root is never a dual root, the latter equals $l'\check{\Lambda}_r \cap \Lambda_r$. In fact, if l is even, then $\frac{l}{2} = l'$ and there's nothing to say. If l is odd, and $x = \frac{l}{2}y$, $y \in \check{\Lambda}_r$, then $\frac{l}{2}y \in \Lambda_r$ and so $y \in 2\check{\Lambda}_r$. Therefore, $x \in l\check{\Lambda}_r \cap \Lambda$. Now, if $D|l'$, then $l'\check{\Lambda}_r \subset \Lambda_r$, and therefore $l'\check{\Lambda}_r \cap \Lambda_r = l'\check{\Lambda}_r$. If D does not divide l', then they are relatively prime, so for short roots $l'\check{\beta} = l'\beta \in \lambda_r$, but for long roots the smallest multiple in Λ_r is $Dl'\check{\beta} = l'\beta$, so $\frac{l}{2}\check{\Lambda}_r \cap \Lambda_r \subset l'\lambda_r$. Since the translated action of the ordinary weyl group preserves the root lattice, we have that the largest subgroup which preserves the root lattice is precisely $\mathcal{W} \ltimes l'\Lambda_r$ in the case in which $D \nmid l'$ and $\mathcal{W} \ltimes l'\Lambda_r$ when $D|l'$, that is, the result follows. □

The aim now is to give a proof of the quantum version of Harish-Chandra's Theorem for $U_{\mathbf{s}}^\dagger(\mathfrak{g})$.

Consider the action of the center Z of $U_{\mathbf{s}}^\dagger$ on a highest weight module of highest weight λ. If v is a vector of weight λ then so is zv for any $z \in Z$, so z must act as multiplication by an element of $\mathbb{Q}[\mathbf{s}]$ on v, say $zv = \chi_\lambda(z)v$. Since every element of the highest weight module is of the form Fv for some $F \in U_{\mathbf{s}}^{\dagger-}$ (the subalgebra of $U_{\mathbf{s}}^\dagger$ generated by $\{1, F_i^{(k)}\}$) we have $zFv = Fzv = \chi_\lambda(z)Fv$, so that in fact z acts as multiplication by $\chi_\lambda(z)$ on the entire highest weight module. Thus each λ gives us an algebra homomorphism χ_λ from the center Z to $\mathbb{Q}[\mathbf{s}]$.

Definition 4.8. *If $\lambda, \gamma \in \Lambda$ say that $\lambda \sim \gamma$ if $\chi_\lambda = \chi_\gamma$. This is an equivalence relation.*

Of course $\chi_\lambda = \chi_\gamma$ if λ occurs as a highest weight in a highest weight γ module, so $\sim$ includes at least the extension of this relation to an equivalence relation.

It will be useful to understand the relation $\sim$ in the case $\mathfrak{g} = \mathfrak{sl}_2$. in this case, it is known that the Verma module of weight $j\theta$, $j \in \mathbb{Z}^{\ge 0}/2 \cong \Lambda^+$, is a $U_{\mathbf{s}}^\dagger$-module spanned by $\{F^{(k)}v \,|\, k \ge 0\}$, where v is of weight $j\theta$ and $Ev = 0$. For this module:

$$EF^{(s)}v = [2j-s+1]_{\mathbf{q}}\, F^{(s-1)}v.$$

To show this, we will first prove by induction the following commutation formula:

$$EF^s = F^sE + F^{s-1}\sum_{i=0}^{s-1}\frac{\mathbf{q}^{-2i}K - \mathbf{q}^{2i}K^{-1}}{\mathbf{q}-\mathbf{q}^{-1}}.$$

The case $s=1$ is just the defining relation $EF - FE = \frac{K^2-K^{-2}}{\mathbf{q}-\mathbf{q}^{-1}}$. Now, assuming the formula holds for a given s, we get

$$\begin{aligned}
EF^{s+1}v &= F^sEF + F^s\sum_{i=0}^{s-1}\frac{\mathbf{q}^{-2i}K - \mathbf{q}^{2i}K^{-1}}{\mathbf{q}-\mathbf{q}^{-1}}F \\
&= F^s(FE + \frac{K-K^{-1}}{\mathbf{q}-\mathbf{q}^{-1}}) + F^s\sum_{i=0}^{s-1}\frac{\mathbf{q}^{-2i-2} - \mathbf{q}^{2i+2}}{\mathbf{q}-\mathbf{q}^{-1}} \\
&= F^{s+1}E + F^s\frac{K-K^{-1}}{\mathbf{q}-\mathbf{q}^{-1}} + F^s\sum_{i=1}^{s}\frac{\mathbf{q}^{-2i} - \mathbf{q}^{2i}}{\mathbf{q}-\mathbf{q}^{-1}} \\
&= F^{s+1}E + F^s\sum_{i=0}^{s}\frac{\mathbf{q}^{-2i}K - \mathbf{q}^{2i}K^{-1}}{\mathbf{q}-\mathbf{q}^{-1}}.
\end{aligned}$$

If we apply this relation to v, recalling that $Ev=0$ and $Kv = -q^{\langle\theta,j\theta\rangle} = \mathbf{q}^{2j}$, we get:

$$\begin{aligned}
EF^sv &= F^sEv + F^{s-1}\sum_{i=0}^{s-1}\frac{\mathbf{q}^{-2i}K - \mathbf{q}^{2i}K^{-1}}{\mathbf{q}-\mathbf{q}^{-1}}v = F^{s-1}\sum_{i=0}^{s-1}\frac{\mathbf{q}^{2j-2i} - \mathbf{q}^{2i-2j}}{\mathbf{q}-\mathbf{q}^{-1}}v \\
&= \frac{1}{\mathbf{q}-\mathbf{q}^{-1}}\left[\mathbf{q}^{2j}\sum_{i=0}^{s-1}(\mathbf{q}^{-2})^i - \mathbf{q}^{-2j}\sum_{i=0}^{s-1}(\mathbf{q}^2)^i\right]F^{s-1}v = \frac{1}{\mathbf{q}-\mathbf{q}^{-1}}\left[\mathbf{q}^{2j}\frac{1-\mathbf{q}^{-2s}}{1-\mathbf{q}^{-2}} - \mathbf{q}^{-2j}\frac{1-\mathbf{q}^{2s}}{1-\mathbf{q}^2}\right]F^{s-1}v.
\end{aligned}$$

Now, since $[s]_{\mathbf{q}}! = [s]_{\mathbf{q}}[s-1]_{\mathbf{q}}! = \frac{q^s - \mathbf{q}^{-s}}{q-\mathbf{q}^{-1}}$, we have that:

$$\begin{aligned}
EF^{(s)}v &= \frac{1}{\mathbf{q}^s-\mathbf{q}^{-s}}\left[\mathbf{q}^{2j}\mathbf{q}^{-s}\frac{\mathbf{q}^s-\mathbf{q}^{-s}}{1-\mathbf{q}^{-2}} - \mathbf{q}^{-2j}\mathbf{q}^{s}\frac{\mathbf{q}^{-s}-\mathbf{q}^{s}}{1-\mathbf{q}^{2}}\right]F^{(s-1)}v \\
&= (\frac{\mathbf{q}^{2j-s}}{1-\mathbf{q}^{-2}} + \frac{\mathbf{q}^{s-2j}}{1-\mathbf{q}^{2}})F^{(s-1)}v = (\frac{\mathbf{q}^{2j-s+1}}{\mathbf{q}-\mathbf{q}^{-1}} - \frac{\mathbf{q}^{s-2j-1}}{\mathbf{q}-\mathbf{q}^{-1}})F^{(s-1)}v \\
&= \frac{\mathbf{q}^{2j-s+1} - \mathbf{q}^{s-2j-1}}{\mathbf{q}-\mathbf{q}^{-1}}F^{(s-1)}v = [2j-s+1]_{\mathbf{q}}F^{(s-1)}v.
\end{aligned}$$

Notice $[r]_{\mathbf{q}} = 0$ if and only if r is a multiple of l'. Thus $F^{(s)}v$ is a highest weight vector when either $s = 2j+1$ or $s = 2j+1-kl'$ and $s < l'$.

Now,

$$\begin{aligned}
KF^{(2j+1)}v &= \mathbf{q}^{-2(2j+1)}F^{(2j+1)}Kv = \mathbf{q}^{-2(2j+1)}\mathbf{q}^{2j}F^{(2j+1)}v \\
&= \mathbf{q}^{(-2j-2)}F^{(2j+1)}v = \mathbf{q}^{\langle\theta,-(j+1)\theta}F^{(2j+1)}v,
\end{aligned}$$

and

$$\begin{aligned}
KF^{(2j+1-kl')}v &= \mathbf{q}^{-2(2j+1-kl')}F^{(2j+1-kl')}Kv = \mathbf{q}^{-2(2j+1-kl')}\mathbf{q}^{2j}F^{(2j+1-kl')}v \\
&= \mathbf{q}^{-2j-2+2kl'}F^{(2j+1-kl')}v = \mathbf{q}^{\langle\theta,(kl'-j-1)\theta}F^{(2j+1)}v.
\end{aligned}$$

So $j\theta \sim -(j+1)\theta$ when $j \geq 0$ and $j\theta \sim (kl'-j-1)\theta$ when $2j < (k+1)l'$. Note that this last relation gives $j\theta \sim (j+kl')\theta$ By transitivity $j \sim j'$ whenever $j\theta$ is connected to $j'\theta$ by the quantum Weyl group, since $\sigma_\theta \cdot j\theta = \sigma_\theta(j\theta + \frac{1}{2}\theta) - \frac{1}{2}\theta = -(j+1)\theta$ and the group of translations is generated by $\theta \mapsto \theta + kl'\theta$.

Now consider a general $\mathfrak{g}$. In the proof of the next proposition it will very useful the detail in which we've developed the $\mathfrak{sl}_2$ case.

Proposition 4.9. *If $\lambda, \gamma \in \Lambda$ and there is a $\sigma \in \mathcal{W}_l$ such that $\gamma = \sigma \cdot \lambda$, then $\lambda \sim \gamma$.*

Proof. Let $\lambda \in \Lambda$. Recall that the Verma module of highest weight λ can be constructed as follows. Consider U as a U-module under the adjoint action, and quotient it by the left ideal generated by $E_i^{(k)}$ and $K_i - q^{(\lambda,\alpha_i)})$ for all i. It is easy to see that the vector 1 (which above was called v) is a highest weight vector of weight λ. Now for each i $\{E_i^{(k)}, F_i^{(k)}, K_i\}$ generate a subalgebra isomorphic to $U^\dagger_{s_i}(\mathfrak{sl}_2)$ (here $\mathbf{s}_i = \mathbf{s}^{d_i}$) and the vectors $F_i^{(k)}v$ span a $U^\dagger_{\mathbf{s}_i}(\mathfrak{sl}_2)$ module isomorphic to the Verma module of weight $\langle \lambda, \check{\alpha}_i \rangle/2$. Therefore the vectors $F_i^{\langle \lambda, \check{\alpha}_i \rangle + 1}v$ and $F_i^{\langle \lambda, \check{\alpha}_i \rangle + 1 - kl'_i}v$ where $\langle \lambda, \check{\alpha}_i \rangle < (k+1)l'_i$ give highest weight vectors. Thus $\lambda \sim \sigma \cdot \lambda$, for σ a generator of the affine Weyl group. The result follows by the transitivity of the $\sim$ relation. $\square$

Let R be the R-matrix in $U^\dagger_{\mathbf{s}}(\mathfrak{g})$ and let $R_{21} = \tau(R)$, where $\tau : U^\dagger_{\mathbf{s}}(\mathfrak{g}) \overline{\otimes} U^\dagger_{\mathbf{s}}(\mathfrak{g}) \to U^\dagger_{\mathbf{s}}(\mathfrak{g}) \overline{\otimes} U^\dagger_{\mathbf{s}}(\mathfrak{g})$ is the standard flip. Let $D = R_{21}R$. We write $D = \sum_i x_i \otimes y_i$ (infinite sum).

Let

$$\Psi \colon (U^\dagger_{\mathbf{s}}(\mathfrak{g}))^* \to U^\dagger_{\mathbf{s}}(\mathfrak{g})$$
$$\Psi(z^*) = \sum_i z^*(y_i) x_i,$$

where $(U^\dagger_{\mathbf{s}}(\mathfrak{g}))^*$ is the direct sum over all $\lambda \in \Lambda^+$ of the set $(U^\dagger_{\mathbf{s}}(\mathfrak{g}))_\lambda$ of functionals on $U^\dagger_{\mathbf{s}}(\mathfrak{g})$ which factor through the representation on $W^\lambda_{\mathbf{s}}$:

$$(U^\dagger_{\mathbf{s}}(\mathfrak{g}))_\lambda = \{x \mapsto \phi(x \cdot w)/\phi \in (W^\lambda_{\mathbf{s}})^*, w \in W^\lambda_{\mathbf{s}}\}.$$

This map is called the Drinfel'd map. We will also be interested in

$$D_{\mathfrak{h}} = \mathbf{q}^{\sum_i \lambda_i \otimes \check{\alpha}_i} \mathbf{q}^{\sum_i \check{\alpha}_i \otimes \lambda_i} = \mathbf{q}^{2\sum_i \lambda_i \otimes \check{\alpha}_i}$$

and the associated

$$\Psi_{\mathfrak{h}} \colon (U^\dagger_{\mathbf{s}}(\mathfrak{g}))^* \to U^\dagger_{\mathbf{s}}(\mathfrak{h})$$
$$\Psi_{\mathfrak{h}}(z^*) = \sum_i z^*(y_i) x_i$$

writing $D_{\mathfrak{h}} = \sum_i x_i \otimes y_i$.

By the PBW theorem there is a well-defined map

$$\Theta \colon U^\dagger_{\mathbf{s}}(\mathfrak{g}) \to U^\dagger_{\mathbf{s}}(\mathfrak{h})$$

given by sending all products in the PBW basis which contain factors of E_i or F_i to zero and all other products to themselves. Thus $\chi_\lambda = \lambda \circ \Theta$ on the center Z. What's more, since the only terms in D which do not contain factors of the form E and F are those in $D_{\mathfrak{h}}$,

$$\Theta\Psi = \Psi_{\mathfrak{h}}. \tag{3}$$

Consider the adjoint action of $U^\dagger_{\mathbf{s}}(\mathfrak{g})$ on itself, given by, for $a, x \in U^\dagger_{\mathbf{s}}(\mathfrak{g})$,

$$\mathrm{ad}_a(x) = a_{(1)} x S(a_{(2)}).$$

An invariant element of $U^\dagger_{\mathbf{s}}(\mathfrak{g})$ is then an x such that $\mathrm{ad}_a(x) = \epsilon(a)x$, $\forall a \in U^\dagger_{\mathbf{s}}(\mathfrak{g})$. We will need the following general lemma:

Lemma 4.10. *Let H be a (topological) Hopf algebra. Then the map $H \otimes H \to H \otimes H$ given by $a \otimes b \mapsto a_{(1)} \otimes a_{(2)} b$ is a bijection.*

Proof. It is straightforward to check that the map $H \otimes H \to H \otimes H$ given by $a \otimes b \mapsto a_{(1)} \otimes S(ba_{(2)})$ gives an inverse to the map defined above. □

We consider now the inverse of the map defined in the Lemma, for $H = U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$. We can, thus, for a given $u \in U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$, choose an element $a \otimes b \in U_{\mathbf{s}}^{\dagger}(\mathfrak{g}) \otimes U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$ such that $a_{(1)} \otimes S(ba_{(2)}) = u \otimes 1$. Now, if z is an ad-invariant element, we have

$$uz = uz1 = a_{(1)} z S(ba_{(2)}) = a_{(1)} z S(a_{(2)}) S(b) = \epsilon(a) z S(b) = z a_{(1)} S(a_{(2)}) S(b) = zu.$$

So every ad-invariant element lies in the center of $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$. Since the converse is clearly true, we conclude that the invariants elements of $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$ (or of any Hopf algebra) are exactly the elements of the center.

Likewise, the coadjoint action on $(U_{\mathbf{s}}^{\dagger}(\mathfrak{g}))^*$ send an element z^* to

$$\mathrm{coad}_a(z^*) = z^*(a_{(1)} \cdot S(a_{(2)})).$$

Lemma 4.11. *The Drinfel'd map Ψ takes invariant functionals to the center of $(U_{\mathbf{s}}^{\dagger}(\mathfrak{g}))^*$.*

Proof. Remember that if a Hopf algebra H is quasitriangular, then so is the co-opposite Hopf algebra H^{cop}, with quasitriangular element $R^{\mathrm{cop}} = R_{21}$. Therefore, we have

$$D\Delta(a)D = R_{21}R\Delta(a) = R_{21}\Delta^{\mathrm{cop}}(a)R = (\Delta^{\mathrm{cop}})^{\mathrm{cop}}(a)R_{21}R = \Delta(a)R_{21}R = \Delta(a)D, \tag{4}$$

for all $a \in (U_{\mathbf{s}}^{\dagger}(\mathfrak{g}))^*$. Then if z^* is an invariant functional,

$$\begin{aligned}
\epsilon(a)\Psi(z^*) = \epsilon(a)\sum_i z^*(y_i)x_i &= \sum_i z^*(y_i)x_i a_{(1)} S(a_{(2)}) = \\
&= \sum_i z^*(y_i)x_i a_{(1)} \epsilon(a_{(2)}) S(a_{(3)}) \\
&= \sum_i z^*(y_i a_{(2)} S(a_{(3)})) x_i a_{(1)} S(a_{(4)}) \\
&\overset{(4)}{=} \sum_i z^*(a_{(2)} y_i S(a_{(3)})) a_{(1)} x_i S(a_{(4)}) \\
&= \sum_i \mathrm{ad}_{a_{(2)}}(z^*)(y_i) a_{(1)} x_i S(a_{(3)}) \\
&= \sum_i \epsilon(a_{(2)}) z^*(y_i) a_{(1)} x_i S(a_{(3)}) \\
&= \sum_i z^*(y_i) a_{(1)} x_i S(a_{(2)}) = \mathrm{ad}_a(\Psi(z^*)).
\end{aligned}$$

Thus $\Psi(z^*)$ is an invariant element of $(U_{\mathbf{s}}^{\dagger}(\mathfrak{g}))^*$. □

Corollary 4.12. *If $\lambda \sim \gamma$, then $\lambda \circ \Psi_{\mathfrak{h}} = \gamma \circ \Psi_{\mathfrak{h}}$ on invariant functionals.*

Proposition 4.13. *If $\chi_\lambda \Psi = \chi_\gamma \Psi$ on invariant functionals, then λ is mapped to γ by a transformation in the semidirect product of the translated action of the Weyl group with the group of translations $\frac{l}{2}\check{\Lambda}_r$.*

Further, $\{\chi_\lambda \Psi\}$ where λ runs through a choice of representative of each equivalence class of weights in Λ related by this group, is a set of linearly independent functionals on the quantum traces.

Proof. For each $\nu \in \Lambda^+$ the functional qtr_ν is an invariant functional. By induction on the ordering we can form a linear combination of these qtr_ν to produce an invariant functional which on $U_{\mathbf{s}}^\dagger(\mathfrak{h})$ acts as $\sum_{\sigma \in \mathcal{W}} \sigma(\nu)(\mathbf{q}^{2\rho} \cdot)$ for each $\nu \in \Lambda^+$. Notice that $(\mu \otimes \mu')(D_{\mathfrak{h}}) = \mathbf{q}^{2\langle \mu, \mu' \rangle}$. Now,

$$\begin{aligned}
\chi_\lambda(\Psi_{\mathfrak{h}}(\sum_{\sigma \in \mathcal{W}} \sigma(\nu)(\mathbf{q}^{2\rho} \cdot))) &= \lambda(\sum_{\sigma \in \mathcal{W}} \sigma(\nu) \left(\mathbf{q}^{2\rho}(D_{\mathfrak{h}}^2)\right) D_{\mathfrak{h}}^1) \\
&= \lambda(\sum_{\sigma \in \mathcal{W}} \sum_{\eta, \gamma} \sigma(\nu)(\mathbf{q}^{2\rho}\delta_\eta) q^{2\langle \gamma, \eta \rangle} \delta_\gamma) \\
&= \lambda(\sum_{\sigma \in \mathcal{W}} \sum_{\eta, \gamma, \mu} \sigma(\nu)(\delta_\mu \delta_\eta) q^{\langle 2\rho, \mu \rangle} q^{2\langle \gamma, \eta \rangle} \delta_\gamma) \\
&= \lambda(\sum_{\sigma \in \mathcal{W}} \sum_{\eta, \gamma} \sigma(\nu)(\delta_\eta) q^{\langle 2\rho, \eta \rangle} q^{2\langle \gamma, \eta \rangle} \delta_\gamma) \\
&= \sum_{\sigma \in \mathcal{W}} \sum_{\eta} \sum_{\eta} \sigma(\nu)(\delta_\eta) q^{\langle 2\rho, \eta \rangle} q^{2\langle \lambda, \eta \rangle} \delta_\lambda \\
&= \sum_{\sigma \in \mathcal{W}} \sum_{\eta} \sigma(\nu)(\delta_\eta) q^{2\langle \rho + \lambda, \eta \rangle} \delta_\lambda \\
&= \sum_{\sigma \in \mathcal{W}} \sum_{\eta} \delta_\eta(\sigma(\nu)) q^{2\langle \rho + \lambda, \eta \rangle} \delta_\lambda \\
&= \sum_{\sigma \in \mathcal{W}} \mathbf{q}^{2\langle \lambda + \rho, \sigma(\nu) \rangle}.
\end{aligned}$$

Therefore, we have

$$\chi_\lambda(\Psi(\sum_{\sigma \in \mathcal{W}} \sigma(\nu)(\mathbf{q}^{2\rho} \cdot))) = \sum_{\sigma \in \mathcal{W}} \mathbf{q}^{2\langle \lambda + \rho, \sigma(\nu) \rangle}.$$

The set $\{\mathbf{q}^{2\langle \lambda + \rho, \nu \rangle} : \lambda \in \frac{1}{2L}\check{\Lambda}_r / \frac{l}{2}\check{\Lambda}_r\}$ is a basis for maps from $\Lambda/(lL\Lambda)$ to $\mathbb{Q}[\mathbf{s}]$. $\mathcal{W}$ permutes this basis (with the translated action), so $\sum_{\sigma \in \mathcal{W}} \mathbf{q}^{2\langle \lambda + \rho, \sigma(\nu) \rangle}$ forms a basis for maps from $(\Lambda/(lL\Lambda))^{\mathcal{W}}$ to $\mathbb{Q}[\mathbf{s}]$, when λ ranges over representatives of each Weyl orbit in $\frac{1}{2L}\check{\Lambda}_r / \frac{l}{2}\check{\Lambda}_r$. Thus $\chi_\lambda \Psi$ is unchanged by $\mathcal{W}^\dagger \mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$, and any set of orbit representatives is linearly independent.

□

Lemma 4.14. *If $\lambda \sim \gamma$ then $\lambda - \gamma \in \Lambda_r$.*

Proof. Notice an element $f \in U_{\mathbf{s}}^\dagger(\mathfrak{h})$ is in the center of $U_{\mathbf{s}}^\dagger(\mathfrak{g})$ if and only if $f(\lambda) = f(\lambda + \alpha_i)$ for all $\lambda \in \Lambda$ and all i. The sub-Hopf algebra of such functions is isomorphic to the Hopf algebra of functions on the fundamental group Λ/Λ_r. Such an f acts on λ by multiplication by $f(\lambda)$, so every such f will agree on λ and γ if an only if $\lambda - \gamma \in \Lambda_r$.

□

Theorem 4.15. *$\lambda \sim \gamma$ if and only if $\lambda = \sigma \cdot \gamma$ for some $\sigma \in \mathcal{W}_l$.*

Proof. That the latter implies the former is exactly Proposition 4.9.

If $\lambda \sim \gamma$, then by Proposition 4.9, they are connected by an element of $\mathcal{W}^\dagger$. On the other hand, by Lemma 4.14 they differ by an element of the root lattice, since if an element of $\mathcal{W}^\dagger$ takes one vector to another vector that differs from it by a root vector, the difference of any vector and its image is a root vector, since the obstruction for this does not depend on the vector of the argument, but on the element itself. Thus, by Lemma 4.7 they are connected by an element of the affine Weyl group $\mathcal{W}_l$. □

Corollary 4.16. *The set $\{\chi_\lambda\}$, choosing one λ from each translated $\mathcal{W}_l$ equivalence class, is a linearly independent set of functionals on the center.*

Proof. By Proposition 4.13 a linear relation between these would reduce to a linear relation between those in the translated $\mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$ orbit of some λ. Since elements of this orbit which are not $\mathcal{W}_l$ equivalent must be in distinct classes of Λ/Λ_r, computing them on the center intersected with $U_{\mathbf{s}}^\dagger(\mathfrak{h})$ shows that no such nontrivial relation exists. $\square$

5. Quantum groups at roots of 1. The tilting modules

In this section we will use U to refer to any of the forms of the quantum group defined in the last two sections: $U_q(\mathfrak{g})$, $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$, $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$, $U_{\mathbf{q}}^{\mathrm{res}}(\mathfrak{g})$ or $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$, and use "the ground ring" to refer to $\mathbb{Q}(q)$, $\mathcal{A}$, $\mathcal{A}'$, $\mathbb{Q}[\mathbf{q}]$, or $\mathbb{Q}[\mathbf{s}]$ as appropriate. We also drop the subscripts from such notation as $W_{\mathcal{A}'}^{\lambda}$ when no confusion would ensue.

5.1. Weyl filtrations and tilting modules.

Definition 5.1. *A U-module V is said to have a* Weyl filtration *if there exists a sequence of submodules*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

such that for each $1 \le i \le n$, V_i/V_{i-1} is isomorphic to the Weyl module W^λ for some $\lambda \in \Lambda^+$.

Proposition 5.2. *Suppose W is a $U_{\mathcal{A}}^{res}(\mathfrak{g})$-module such that $W \otimes_{\mathcal{A}} \mathbb{Q}(q) = \bigoplus_i W_q^{\lambda_i}$. Then $W \otimes_{\mathcal{A}} \mathbb{Q}[\mathbf{q}]$ and $(W \otimes_{\mathcal{A}} \mathcal{A}') \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$ admit Weyl filtrations with the ith factor of highest weight λ_i, where the λ_i are assumed to be ordered so that λ_i is never greater than λ_j for $j < i$.*

Proof. We shall prove the Proposition over $\mathbb{Q}[\mathbf{q}]$, the argument is exactly the same for $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$, over $\mathbb{Q}[\mathbf{s}]$.

Decomposing $W = W_{\mathrm{tor}} \oplus W_{\mathrm{free}}$ into its torsion and free parts over $\mathcal{A}$, notice that $W \otimes_{\mathcal{A}} \mathbb{Q}(q) = W_{\mathrm{free}} \otimes_{\mathcal{A}} \mathbb{Q}(q)$ and likewise for $\mathbb{Q}[\mathbf{q}]$. Since W_{tor} is a $U_{\mathcal{A}}^{\mathrm{res}}$-module, the quotient by it is a free $\mathcal{A}$-module and a $U_{\mathcal{A}}^{\mathrm{res}}$ module whose tensor with $\mathbb{Q}(q)$ and $\mathbb{Q}[\mathbf{q}]$ are isomorphic to that of W. Thus we can assume W is a free $\mathcal{A}$-module. Notice in this case the maps $v \mapsto v \otimes 1$ are injective maps from W to $W \otimes_{\mathcal{A}} \mathbb{Q}(q)$ and $W \otimes_{\mathcal{A}} \mathbb{Q}[\mathbf{q}]$ whose ranges span.

Let $w \in W$ be such that $w \otimes 1 \in W \otimes_{\mathcal{A}} \mathbb{Q}(q)$ is a vector of weight λ_1. By the maximality of λ_1 w must be a highest weight vector. Then $U_{\mathcal{A}}^{\mathrm{res}} w$ is a $U_{\mathcal{A}}^{\mathrm{res}}$-module, free over $\mathcal{A}$, whose tensor product with $\mathbb{Q}(q)$ yields a U_q-module isomorphic to $W_q^{\lambda_1}$. Thus $U_{\mathcal{A}}^{\mathrm{res}} w$ must be isomorphic to $W_{\mathcal{A}}^{\lambda_1}$. Its tensor product with $\mathbb{Q}[\mathbf{q}]$ thus gives a submodule isomorphic to $W_{\mathbf{q}}^{\lambda_1}$. Therefore the quotient $W/W_{\mathcal{A}}^{\lambda_1}$ is a module whose tensor product with $\mathbb{Q}(q)$ is isomorphic to $\bigoplus_{i>1} W_q^{\lambda_i}$ and whose tensor product with $\mathbb{Q}[\mathbf{q}]$ is $(W \otimes_{\mathcal{A}} \mathbb{Q}[\mathbf{q}])/W_{\mathbf{q}}^{\lambda_1}$. By induction the proposition follows. $\square$

Corollary 5.3. *The tensor product of two $U_{\mathbf{q}}^{res}$ or $U_{\mathbf{s}}^{\dagger}$ modules with a Weyl filtration admits a Weyl filtration.*

Proof. By induction it suffices to prove that $W^\lambda \otimes W^\gamma$ admits a Weyl filtration, since if A and B admit Weyl filtrations and $V \subset B$ admits a Weyl filtration such that $B/V \cong W^\lambda$, for some λ, then, if $A \otimes V$ admits a Weyl filtration, this filtration and $A \otimes V \subset A \otimes B$ give a Weyl filtration for the tensor product. The $W^\lambda \otimes W^\gamma$ case follows from the previous Proposition, since, as we have recalled in Section 3, the tensor product of $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ Weyl modules decomposes into a direct sum of $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ Weyl modules. $\square$

Remark 5.4. Notice the entries in that Weyl filtration are the same as the entries in the classical decomposition of the tensor product of classical modules which were direct sums with the same entries as the original Weyl filtrations. Thus if we restrict our attention to modules with a Weyl filtration the category of such modules forms a monoidal category which reminds us of the tensor category of classical finite-dimensional modules if we replace the notion of direct sum decomposition of modules with that of Weyl filtration.

Definition 5.5. *A U-module V is said to have a* dual Weyl filtration *if there exists a sequence of submodules*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

such that for each $1 \leq i \leq n$ V_i/V_{i-1} is isomorphic to the dual of a Weyl module $(W^\lambda)^*$ for some $\lambda \in \Lambda^+$.

Lemma 5.6. *V admits a dual Weyl filtration if and only if V^* admits a Weyl filtration.*

Proof. This follows from the following linear algebra fact: let $W \subset V$ two (finite dimensional) vector spaces, then the map

$$V^*/W^* \to (V/W)^*, \; f + W^* \mapsto (x + W \mapsto f(x)),$$

for $f \in V^*$, $x \in V$, is a linear isomorphism. □

Definition 5.7. *A U-module V is a* tilting module *if it admits both a Weyl filtration and a dual Weyl filtration.*

Corollary 5.8. *The properties of admitting a Weyl filtration, admitting a good filtration or being a tilting module are preserved by tensor product.*

Remark 5.9. The category of tilting modules forms a rigid monoidal category, though not a semisimple one. It is nevertheless true that, because of the existence of Weyl and dual Weyl filtrations, behaves in many respects like the semisimple tensor category of classical $\mathfrak{g}$ modules.

5.2. **Indecomposable tilting modules.** The following Proposition will lead us to the study of indecomposable tilting modules. It addresses to one of the most relevant facts on tilting modules.

Proposition 5.10. *If $V \cong W \bigoplus W'$, then V is tilting if and only if W and W' are tilting*

To understand indecomposable tilting modules, and in order to prove the Proposition, we need to recall some notions from Homological algebra. For precise definitions and statements, we refer the reader to the book [Mac63].

Let A and C be two modules over a ring R, Consider long exact sequences of R-modules

$$0 \to B_{n-1} \to \cdots \to B_1 \to B_0 \to C \to 0$$

running from A to C with n intermediate modules. These extensions, suitably classified by a congruence relation, are the elements of a group $\mathrm{Ext}^n(C, A)$. To calculate this group, we present C as a quotient $C = F_0/S_0$ of a free module F_0. This process can be iterated as $S_0 = F_1/S_1$, $S_1 = F_2/S_2$, ... to give an exact sequence:

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to C \to 0$$

called a "free resolution" of C. The complex $\mathrm{Hom}(F_n, A)$ has cohomology $\mathrm{Ext}^n(C, A)$. Alternatively, one may imbed A in an injective module J_0 and then J_0/A in an injective module J_1 and in this way iterate this process to get an exact sequence:

$$0 \to A \to J_0 \to J_1 \to \cdots \to J_n \to \cdots$$

called an "injective resolution" of A. The complex $\mathrm{Hom}(C, J_n)$ has cohomology $\mathrm{Ext}^n(C, A)$. We do not give details here about the equivalence of both definitions, nor about the congruence relation mentioned above. We restrict ourselves to providing the reader a list of classical properties about the Ext groups that will be needed in the sequel. We define $\mathrm{Ext}^0(A, C) \cong \mathrm{Hom}(A, C)$.

Proposition 5.11 ([Mac63]). (1) *$\mathrm{Ext}^n(A, C)$ is functorial on each variable.*

(2) *If $0 \to X \to Y \to Z \to 0$ is a short exact sequence then we get the long exact sequences of homology:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^0(Z, C) & \longrightarrow & \mathrm{Ext}^0(Y, C) & \longrightarrow & \mathrm{Ext}^0(X, C) & \longrightarrow \\ & & \mathrm{Ext}^1(Z, C) & \longrightarrow & \mathrm{Ext}^1(Y, C) & \longrightarrow & \cdots & \end{array} \tag{5}$$

and

$$
(6)\qquad \begin{array}{l} 0 \longrightarrow \operatorname{Ext}^0(A,X) \longrightarrow \operatorname{Ext}^0(A,Y) \longrightarrow \operatorname{Ext}^0(A,Z) \longrightarrow \\ \qquad \operatorname{Ext}^1(A,X) \longrightarrow \operatorname{Ext}^1(A,Y) \longrightarrow \quad \cdots . \end{array}
$$

There is another approach to the $\operatorname{Ext}^1(C,A)$ group, in terms of short exact sequences $0 \to A \to B \to C \to 0$ which will be very useful in what follows. We let $\operatorname{Ext}(C,A)$ be the set of all congruence classes of extensions of A by C, via the following equivalence relation: two sequences

$$E : 0 \to A \to B \to C \to 0, \text{ and } E' : 0 \to A \to B' \to C \to 0$$

are said to be equivalent if there exists a monomorphism $\beta : B \to B'$ such that the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
$$

We now that in this situation β must be an isomorphism (Five Lemma) and thus the relation is seen to be an equivalence relation. It is possible to give to this congruence set a group structure, with the addition operation being the so-called "Baer sum" (cf. [Mac63]), which renders the trivial extension

$$0 \to A \to A \oplus C \to C \to 0$$

as the zero element of the group. With this in mind, we arrive to:

Theorem 5.12 ([Mac63])**.** *Let A and C be two modules over a given ring R. If $K \xrightarrow{\iota} P \to C$ is an exact sequence, with P projective, then*

$$\operatorname{Ext}(C,A) \cong \operatorname{Hom}_R(K,A)/\iota^* \operatorname{Hom}(P,A).$$

In particular, $\operatorname{Ext}(C,A) \cong \operatorname{Ext}^1(C,A)$*.*

Corollary 5.13. *If A and C are two modules over a ring R and such that every short exact sequence $0 \to A \to B \to C \to 0$ splits, then* $\operatorname{Ext}^1(C,A) = 0$

We now focus our attention back on tilting modules.

Lemma 5.14. *If V is a U-module and $\lambda \in \Lambda^+$ is such that no weight appearing in V is greater than λ then any quotient*

$$V \xrightarrow{\ g\ } W^\lambda \longrightarrow 0$$

splits.

Proof. Let $v \in V$ be an homogeneous vector in the preimage of a highest weight vector in W^λ under g., then v is a vector of weight λ By the condition on λ, v is of maximal weight in V and hence is a highest weight vector, so there is a map $g' : W^\lambda \to V$ sending a highest weight vector to v. Clearly gg' is nonzero on the highest weight vector and hence is a multiple of 1, so the sequence splits. □

Lemma 5.15. *If A admits a Weyl filtration, and C admits a dual Weyl filtration then* $\operatorname{Ext}^1(A,C) = 0$.

Proof. We will prove first the base case, then do induction on the filtration of A, and then induction on the filtration of C.

- $\operatorname{Ext}^1(W^\lambda, (W^\gamma)^*) = 0$ *for all* $\lambda, \gamma \in \Lambda^+$. Suppose first that $\lambda \not< \gamma^*$, where γ^* (minus the image of γ under the action of the longest element of the Weyl group) is the maximal highest weight of $(W^\gamma)^*$. Then if
$$0 \to C \to B \to W^\lambda \to 0$$

with C a submodule of $(W^\gamma)^*$ then the result follows because the sequence splits by Lemma 5.14, since if a weight η in B were greater than λ, it would be in $\ker(B \to W^\lambda) = \mathrm{img}((W^\gamma)^* \to B)$ and this would yield an element in $(W^\gamma)^*$ of weight η and consequently we would have $\lambda < \eta < \gamma^*$. On the other hand if $\lambda < \gamma^*$ and

$$0 \to (W^\gamma)^* \to B \to W^\lambda \to 0$$

then dualizing

$$0 \to (W^\lambda)^* \to B^* \to W^\gamma \to 0$$

and again by Lemma 5.14 the result follows.

- $\mathrm{Ext}(A, (W^\gamma)^*) = 0$ *if A admits a Weyl filtration.* Inductively there is a short exact sequence

$$0 \to W^\lambda \to A \to A' \to 0$$

with A' admitting a Weyl filtration and hence $\mathrm{Ext}^1(A', (W^\gamma)^*) = 0$. From the long exact sequence (5)

$$\to [\mathrm{Ext}^1(W^\lambda, (W^\gamma)^*) = 0] \to \mathrm{Ext}^1(A, (W^\gamma)^*) \to [0 = \mathrm{Ext}^1(A', (W^\gamma)^*)] \to$$

from which it follows $\mathrm{Ext}(A, (W^\gamma)^*) = 0$.

- *$Ext(A, C) = 0$ if A admits a Weyl filtration and C admits a dual Weyl filtration.* Again inductively we have a sequence

$$0 \to (W^\gamma)^* \to C \to C' \to 0$$

with C' admitting a dual Weyl filtration and hence $\mathrm{Ext}^1(A, C) = 0$. Again the long exact sequence 6 and the previous two items give $\mathrm{Ext}(A, C) = 0$.

□

The converse of this later Lemma also holds, in the sense stated in the following Lemma, for whose proof in the totally analogue case of algebraic groups in finite characteristic, we refer to [Don93], [Don81]:

Lemma 5.16. *A highest weight module M has a Weyl filtration if and only if $\mathrm{Ext}^1(M, B) = 0$ for every highest weight module B with a dual Weyl filtration. Similarly, M has a dual Weyl filtration if and only if $Ext^1(A, M) = 0$ for all A which have a Weyl filtration.*

Corollary 5.17. *A $U-$module T is tilting if and only if $\mathrm{Ext}^1(T, M) = 0 = \mathrm{Ext}^1(N, T)$ for every highest weight module M admitting a dual Weyl filtration and every highest weight module N with a Weyl filtration.*

It is readily seen now that Proposition 5.10 follows by the additivity of the Ext functor.

Proposition 5.18. *If Q, Q' are indecomposable tilting modules over U each with a maximal vector of weight λ, then $Q \cong Q'$.*

Proof. Suppose v is a weight λ vector in Q and v' is a weight λ vector in Q'. Let $f : W^\lambda \to Q$ and $f' : W^\lambda \to Q'$ send a particular highest weight vector to v and v' respectively. Let j be the smallest integer such that V_j contains v in a Weyl filtration of Q. Then $V_j/V_{j-1} \cong W^\lambda$. By the maximality of λ, $V_{j-1}/V_{j-2} \cong W^\gamma$ with $\lambda \not< \gamma$. Now, W^λ is a quotient of V_j and no weight appearing in V_j is greater than λ, by maximality. Then, by Lemma 5.14 the map $V_j \to W^\lambda$ has a section s and therefore $V_j = V_{j-1} \oplus W^\lambda$. We can thus find a new $V'_{j-1} = V_{j-2} \oplus W^\lambda$ (without changing V_{j-2}) such that the filtration is still Weyl but v is now an element of V_{j-1}. Inductively, there exists a Weyl filtration with the image of f being V_1, which is to say there is a short exact sequence

$$0 \longrightarrow W^\lambda \xrightarrow{f} Q \longrightarrow N \longrightarrow 0$$

with N admitting a Weyl filtration. The long exact sequence (5) gives

$$0 \to \mathrm{Hom}(N, Q') \to \mathrm{Hom}(Q, Q') \to \mathrm{Hom}(W^\lambda, Q') \to [0 = \mathrm{Ext}^1(N, Q')] \to$$

by Lemma 5.15, so that the sequence is in fact short exact, and $f' \in \mathrm{Hom}(W^\lambda, Q')$ must factor through a map $g' : Q \to Q'$ which takes v to a nonzero multiple of v'. By the same argument with Q and Q' reversed there is a map $g : Q' \to Q$ taking v' to a nonzero multiple of v. Thus gg' is a map from Q to itself taking v to a nonzero multiple of itself.

$\{(gg')^n[Q]\}_{n\in\mathbb{N}}$ is a nested sequence of submodules and thus by finite-dimensionality must stabilize on some submodule $(gg')^M[Q]$ such that gg' is onto when restricted to this submodule. So $Q = (gg')^M[Q] \bigoplus \ker((gg')^M)$. Since Q is indecomposable one of these summands must be zero, and since $v \in (gg')^M Q$ one has $Q = (gg')^M[Q]$ so gg' is invertible. $\square$

Corollary 5.19. *For every $\lambda \in \Lambda^+$, there exist a unique, up to isomorphism, indecomposable tilting module with a maximal vector of weight λ. We call this tilting module T_λ*

We will refer to the result of the following Corollary, as the *Linkage Principle*.

Corollary 5.20. *A simple module with highest weight λ can occur as a composition factor in the Weyl or indecomposable tilting module of highest weight γ only if $\lambda \leq \gamma$ and $\lambda = \sigma \cdot \gamma$ for some $\sigma \in \mathcal{W}_l$.*

Proof. If $\{V_i\}_{i=0}^n$ is a Weyl filtration for T_γ, then we have seen that we may assume $V_1 = W^\lambda$, and therefore $\lambda \leq \gamma$ and $\lambda \sim \gamma$. $\square$

6. Quantum groups at roots of 1. The negligible modules

As in Section 3 we work with $\mathbf{s}$ a primitive lLth root of unity, and consider the quantum group $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$.

Definition 6.1. *A U module V is called* negligible *if every intertwiner $\phi : V \to V$ has quantum trace 0.*

Proposition 6.2. *An indecomposable tilting module is negligible if and only if its quantum trace is 0.*

Proof. The algebra of intertwiners from an indecomposable module to itself consists of multiples of the identity and nilpotent intertwiners (Fitting Lemma). Since nilpotent interwiners commute with $\mathbf{q}^{2\rho}$, they have quantum trace zero. If the module has quantum dimension zero, then multiples of the identity have quantum trace zero. □

Our pourpose is to describe the region in Λ^+ where indecomposable negligible tilting modules can be found.

Let M be the lattice of translations $l'\check{\Lambda}_r$ or $l'\Lambda_r$ according to whether D divides l' or not, so that $\mathcal{W}_l = \mathcal{W} \ltimes M$.

Definition 6.3. *We define the* walls *of the (translated) affine Weyl group to be the hyperplanes*

$$w_{k,\alpha} = \begin{cases} \{x \in \mathfrak{h}^* \,,\, \langle x+\rho, \alpha\rangle = kl'\} & \text{if } D|l \\ \{x \in \mathfrak{h}^* \,,\, \langle x+\rho, \check{\alpha}\rangle = kl'\} & \text{else,} \end{cases}$$

for $\alpha \in \Phi^+$.

Likewise the walls *of $\mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$ are the hyperplanes $\langle x+\rho, \alpha\rangle = kl/2$.*

Note that the translated action of $\mathcal{W}_l$ is generated by reflections $\sigma_{k,\alpha}$ about $w_{k,\alpha}$. If x is on the wall $w_{k,\alpha}$ and no other wall then the stabilizer of x in $\mathcal{W}_l$ is $\{1, \sigma_{k,\alpha}\}$.

We call *alcoves* to the open regions into which these hyperplanes divide $\mathfrak{h}^*$, including the principal alcove C_l, such that $\sigma \mapsto \sigma \cdot C_l$ is a bijection between elements of $\mathcal{W}_l$ and alcoves. A wall $w_{k,\alpha}$ of an alcove $\sigma \cdot C_l$ is called a *lower wall* if every point y in the interior satisfies $\langle y+\rho, \alpha\rangle$ is greater than the corresponding quantity for the points on the wall, and an *upper wall* otherwise.

Finally, let θ_0 be θ if D divides l' and ϕ otherwise. Notice that w_{1,θ_0} is the unique upper wall of C_l.

Definition 6.4. *Let Λ^l be the intersection of the interior of C_l with Λ:*

$$\Lambda^l = \{\lambda \in \Lambda^+ \,|\, \langle \lambda+\rho, \theta_0\rangle < l'\}.$$

The following Theorem, whose proof we will give after several lemmas, characterizes the negligible region.

Theorem 6.5. *In $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$, every T_λ with λ not in Λ^l is negligible, provided $l' \geq D\check{h}$ if $D|l'$ or $l' > h$ otherwise.*

Lemma 6.6. *If $\sigma \in \mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$, then $\mathrm{qdim}(\lambda) = (-1)^\sigma\, \mathrm{qdim}(\sigma\cdot\lambda)$ whenever $\lambda, \sigma\cdot\lambda \in \Lambda^+$, where $(-1)^\sigma$ represents the orientation of σ. In particular this is true of $\sigma \in \mathcal{W}_l$.*

Proof. By the Weyl formula (2), $\mathrm{qdim}(\lambda)$ is

$$\mathrm{qdim}(\lambda) = \prod_{\beta>0} \left(\mathbf{q}^{\langle\lambda+\rho,\beta\rangle} - \mathbf{q}^{-\langle\lambda+\rho,\beta\rangle}\right) / \left(\mathbf{q}^{\langle\rho,\beta\rangle} - \mathbf{q}^{-\langle\rho,\beta\rangle}\right).$$

In fact we can interpret $\mathrm{qdim}(\lambda)$ by this formula even when λ is not in Λ^+. It suffices to prove the first sentence when σ is a generator of the classical Weyl group σ_{0,α_i} and when σ is translation by $l\check{\theta}/2$.

Suppose first that σ is σ_{0,α_i}, then

$$\begin{aligned}
\operatorname{qdim}(\sigma\cdot\lambda) &= \prod_{\beta>0}\left(\mathbf{q}^{\langle\sigma\cdot\lambda+\rho,\beta\rangle}-\mathbf{q}^{-\langle\sigma\cdot\lambda+\rho,\beta\rangle}\right)/\left(\mathbf{q}^{\langle\rho,\beta\rangle}-\mathbf{q}^{-\langle\rho,\beta\rangle}\right)\\
&= \prod_{\beta>0}\left(\mathbf{q}^{\langle\sigma_{\alpha_i}(\lambda+\rho),\beta\rangle}-\mathbf{q}^{\langle-\sigma_{\alpha_i}(\lambda+\rho),\beta\rangle}\right)/\left(\mathbf{q}^{\langle\rho,\beta\rangle}-\mathbf{q}^{-\langle\rho,\beta\rangle}\right)\\
&= \prod_{\beta>0}\left(\mathbf{q}^{\langle\lambda+\rho,\sigma_{\alpha_i}(\beta)\rangle}-\mathbf{q}^{-\langle\lambda+\rho,\sigma_{\alpha_i}(\beta)\rangle}\right)/\left(\mathbf{q}^{\langle\rho,\beta\rangle}-\mathbf{q}^{-\langle\rho,\beta\rangle}\right)
\end{aligned}$$

since σ_{α_i} is a unipotent isometry. Notice that σ_{α_i} permutes the positive roots of Φ except for α_i which it reverses ([Hum72][10.2]) so all factors above stay the same except for one which changes sign. Thus the formula above gives $-\operatorname{qdim}(\sigma\cdot\lambda)=(-1)^\sigma\operatorname{qdim}(\lambda)$.

Now suppose σ is translation by $l\check{\theta}/2$. Then

$$\begin{aligned}
\operatorname{qdim}(\sigma\cdot\lambda) &= \prod_{\beta>0}\left(\mathbf{q}^{\langle\sigma\cdot\lambda+\rho,\beta\rangle}-\mathbf{q}^{-\langle\sigma\cdot\lambda+\rho,\beta\rangle}\right)/\left(\mathbf{q}^{\langle\rho,\beta\rangle}-\mathbf{q}^{-\langle\rho,\beta\rangle}\right)\\
&= \prod_{\beta>0}\left(\mathbf{q}^{\langle l\check{\theta}/2+\lambda+\rho,\beta\rangle}-\mathbf{q}^{-\langle l\check{\theta}/2+\lambda+\rho,\beta\rangle}\right)/\left(\mathbf{q}^{\langle\rho,\beta\rangle}-\mathbf{q}^{-\langle\rho,\beta\rangle}\right)\\
&= \prod_{\beta>0}\left(\mathbf{q}^{l\langle\beta,\check{\theta}\rangle/2}\mathbf{q}^{\langle\lambda+\rho,\beta\rangle}-\mathbf{q}^{-l\langle\beta,\check{\theta}\rangle/2}\mathbf{q}^{-\langle\lambda+\rho,\beta\rangle}\right)/\left(\mathbf{q}^{\langle\rho,\beta\rangle}-\mathbf{q}^{-\langle\rho,\beta\rangle}\right)\\
&= \left(\prod_{\beta>0}\mathbf{q}^{l\langle\beta,\check{\theta}\rangle/2}\right)\prod_{\beta>0}\left(\mathbf{q}^{\langle\lambda+\rho,\beta\rangle}-\mathbf{q}^{-\langle\lambda+\rho,\beta\rangle}\right)/\left(\mathbf{q}^{\langle\rho,\beta\rangle}-\mathbf{q}^{-\langle\rho,\beta\rangle}\right)\\
&= \mathbf{q}^{l\langle\rho,\check{\theta}\rangle}\operatorname{qdim}(\lambda)\\
&= \operatorname{qdim}(\lambda).
\end{aligned}$$

Since the affine Weyl group is a subgroup of $\mathcal{W}\ltimes\frac{l}{2}\check{\Lambda}_r$, the same result applies to the smaller group. ☐

Corollary 6.7. *As functionals on the center*

$$\operatorname{qtr}_{\sigma\cdot\lambda}=(-1)^\sigma\operatorname{qtr}_\lambda$$

when $\sigma\in\mathcal{W}_l$. As functionals on the image of quantum traces under Ψ the same is true when $\sigma\in\mathcal{W}\ltimes\frac{l}{2}\check{\Lambda}_r$ and $\sigma\cdot\lambda\in\Lambda$.

Lemma 6.8. *Let $\lambda\in\Lambda^+$. Then* $\operatorname{qdim}(\lambda)=0$ *if and only if λ has nontrivial stabilizer in $\mathcal{W}\ltimes\frac{l}{2}\check{\Lambda}_r$.*

Proof. Of course if λ has nontrivial stabilizer than it lies on some wall so there is a reflection σ which fixes λ and $\operatorname{qdim}(\lambda)=\operatorname{qdim}(\sigma\cdot\lambda)=-\operatorname{qdim}(\lambda)$, so $\operatorname{qdim}(\lambda)=0$. λ has no stabilizer $\operatorname{qdim}(\lambda)$ is a product of nonzero quantities, and thus nonzero. ☐

Proposition 6.9. *Every T_λ where λ is on a wall of $\mathcal{W}_l$ is negligible.*

Proof. By the Linkage Principle, Corollary 5.20, T_λ has a Weyl filtration all of whose entries are affine Weyl equivalent to λ. If λ is on a wall, so are all weights in its affine orbit, and hence the quantum dimension of T_λ, which is a sum of the quantum dimensions of the entries of the Weyl filtration, is zero. Therefore T_λ is negligible.

☐

Lemma 6.10.

(1) *If W_μ appears in a Weyl filtration of a tilting module T, and no μ' in the translated $\mathcal{W}_l$ orbit of μ with $\mu' > \mu$ appears in that filtration, then T_μ is a direct summand of T.*
(2) *If μ appears as a highest weight in the* classical *module $W^\lambda \otimes W^\gamma$ and no $\mu' > \mu$ in the translated $\mathcal{W}_l$ orbit of μ appears in the classical module $W^{\lambda'} \otimes W^{\gamma'}$ for $\lambda' \leq \lambda$ and $\gamma' \leq \gamma$ in the translated $\mathcal{W}_l$ orbits of λ and γ respectively, then T_μ is a direct summand of $T_\lambda \otimes T_\gamma$.*

Proof.
(1) By the Linkage Principle, Corollary 5.20, W_μ appears in the filtration of an indecomposable direct summand whose Weyl decomposition contains only modules with highest weights in the translated $\mathcal{W}_l$ orbit of μ. By the assumption on μ the weight μ is maximal in this summand, which must thus be isomorphic to T_μ by Proposition 5.18.
(2) Of course a factor of W_μ must appear in a filtration of $T_\lambda \otimes T_\gamma$ by Proposition 5.2 and Corollary 5.3. If a larger μ' in the orbit of μ also appeared in the filtration, it would appear in the classical decomposition of some $W^{\lambda'} \otimes W^{\gamma'}$ with λ' and γ' in a Weyl filtration of T_λ and T_γ respectively. This is ruled out by the assumption, so by part (a) we are done.

□

Lemma 6.11. *Suppose $\lambda, \gamma, \lambda+\sigma(\gamma) \in \Lambda^+$ for some σ in the classical Weyl group $\mathcal{W}$, $\gamma \in \Lambda^l$, suppose λ is on exactly one wall $w_{k,\alpha}$ and $\lambda + \sigma(\gamma)$ is in the interior of an alcove for which $w_{k,\alpha}$ is a lower wall. Then $T_{\lambda+\sigma(\gamma)}$ is a direct summand of $T_\lambda \otimes T_\gamma$.*

Proof. By Lemma 6.10(b), we must check that $\lambda + \sigma(\gamma)$ occurs as a highest weight in the classical decomposition of $W^\lambda \otimes W^\gamma$, and that nothing greater in its $\mathcal{W}_l$ orbit occurs as a highest weight in classical $W^{\lambda'} \otimes W^\gamma$ with λ' $\mathcal{W}_l$-equivalent to and less than λ (nothing is $\mathcal{W}_l$-equivalent to and less than γ because it is in the Weyl alcove).

For the first point, consider the classical Racah formula, Equation (11). Note that the result is true unless $\lambda + \sigma(\gamma)$ is the result of the translated action of a nontrivial element τ of the classical Weyl group on $\lambda + \mu$ for some μ that occurs as a weight of W^γ. It is easy to see that if λ, λ' are in the Weyl chamber then $\tau \cdot \lambda'$ is strictly further from λ than λ' for any $\tau \in \mathcal{W}$, so the length of μ must be strictly greater than the length of $\sigma(\gamma)$, which is not possible if μ is a weight of W^γ.

Essentially the same argument applies for the second point. Since $\lambda, \lambda+\sigma(\gamma)$ are in the same alcove, any λ', μ' in the translated $\mathcal{W}_l$ orbits respectively of λ and $\lambda + \sigma(\gamma)$ must be at least as far away from each other as $\lambda + \sigma(\gamma)$ and λ are, with equality only when μ', λ' are in the same alcove. But if μ' is a weight in classical $W^{\lambda'} \otimes W^\gamma$, it must be at most $||\gamma||$ away from λ', with that distance only achieved if $\mu' - \lambda' = \sigma'(\gamma)$ for some $\sigma' \in \mathcal{W}$. Thus if $\lambda' < \lambda$, λ', μ' are in the orbits of λ and $\lambda + \sigma(\lambda)$, and μ' is in $W^\lambda \otimes W^\gamma$, then μ' is in the same alcove as λ', so there is a single $\tau \in \mathcal{W}_l$ such that $\tau \cdot \lambda = \lambda'$ and $\tau \cdot (\lambda + \sigma(\lambda)) = \mu'$. If $\mu' > \lambda + \sigma(\gamma)$, then $\lambda' \geq \lambda$, so we must have $\lambda = \lambda'$. In this case $\tau = \sigma_{n,\alpha}$, the reflection about the wall on which λ lies. since λ is on a lower wall this would make $\mu' \leq \lambda + \sigma(\gamma)$. Thus by contradiction the result is proven. □

Now we are ready to give the proof of Theorem 6.5:

Proof. In light of Lemma 6.11 and Lemma 6.6, it suffices to find for each alcove other than C_l with nonempty intersection with Λ^+ a dominant weight λ on the interior of a lower wall of that alcove. Then each μ in the interior of this alcove, since $\mu - \lambda$ is Weyl conjugate to something in C_l, would have a T_μ as a summand in some $T_\lambda \otimes T_\gamma$, and thus would be negligible. This requires that every such alcove have a lower wall whose intersection with the weight lattice consists of dominant weights, and that on the interior of each wall of the fundamental alcove there is a weight.

For the first, notice that every wall of an alcove is either a wall of the principal chamber for the translated action of the classical Weyl group or is transverse to it, so every wall of every alcove either contains no dominant weights or all the weights in its interior are dominant. If the alcove intersects Λ^+ all of its walls that do not intersect Λ^+ must be part of the walls of the chamber. If all the lower walls of an alcove are walls of the chamber, the alcove clearly must be C_l.

For the second, if the the wall is w_{0,α_i}, one can readily check that $-\lambda_i$ lies in the interior of the wall (under the restriction on l). If the wall is $w_{1,\theta}$ ($D|l'$ case), notice there is always a fundamental weight λ_i such that $\langle \lambda_i, \check{\theta} \rangle = 1$ (Check [Hum72][p. 66]), so $(l' - \check{h}/D)\lambda_i$ lies on $w_{1,\theta}$. Since it is a dominant weight it lies on no other walls. If the wall is $w_{1,\phi}$ ($D\not| l'$ case), we can find λ_i such that $\langle \lambda_i, \phi \rangle = 1$ for B_n and C_n, and therefore $(l' - h)\lambda_i$ will do the trick. There remains only G_2 and F_4 to consider.

For G_2, we check that $\langle \lambda_1, \phi \rangle = 2$, $\langle \lambda_2, \phi \rangle = 3$. Now every integer greater than 1 can be written as a nonnegative integer combination of 2 and 3, and every number greater than 6 can be written so with neither coefficient equal to zero. Thus if $l > 6 = h$ there exists a positive integer combination of λ_1 and λ_2 whose inner product with ϕ is l. Thus this integer combination minus ρ lies on $w_{l,\phi}$ and no other wall.

For F_4, we check that $\langle \lambda_i, \phi \rangle$ gives $2, 4, 3, 2$ for $i = 1 \ldots 4$. Again if $l > 12 - h$, then l can be written as a positive linear combination of these four numbers, and thus the same combination of λ_1 through λ_4 gives a weight on the interior of $w_{l,\phi}$. $\square$

7. Quantum groups at roots of 1. The semisimple category of representations

Theorem 7.1. *The category of modules of $U^{\dagger}_{\mathcal{A}'}(\mathfrak{g})$ which are free and finite-dimensional over $\mathcal{A}'$ is an Abelian ribbon category.*

Proof. Kassel in [Kas95][XI-XIV] defines ribbon categories and proves that the category of finite-dimensional modules of a ribbon Hopf algebra over a field forms a ribbon category. One can easily check that the proof goes through unchanged for topological Hopf algebras, and for Hopf algebras over a p.i.d., provided we restrict to free modules (the freeness is required to define a map from the trivial module to the tensor product of a module with its dual). □

Theorem 7.2. *The category of all finite-dimensional $U^{\dagger}_{\mathbf{s}}(\mathfrak{g})$-modules is an Abelian ribbon category enriched over $\mathbb{Q}[\mathbf{s}]$, and the full subcategory of tilting modules is a ribbon subcategory.*

Proof. Again this is a corollary of Kassel's proof. A subcategory of a ribbon category is ribbon so long as it is closed under tensor product and left duals, and this is the content of Corollary 5.8 (together with the obvious fact that the set of tilting modules is closed under taking duals). □

We recall here the quotient construction of Mac Lane [Mac71][II.8]. Specifically, if $\mathcal{D}$ is a category and $f, g \in \mathrm{Hom}_{\mathcal{D}}(V, W)$, we say that $f \sim g$ if, for all $h \in \mathrm{Hom}_{\mathcal{D}}(W, V)$, $\mathrm{qtr}_V(hf) = \mathrm{qtr}_V(hg)$. Such an equivalence relation defines a functor to a quotient category $\mathcal{C}$ such that $f \sim g$ implies f and g are equal under the functor, and $\mathcal{C}$ is universal for this property.

Theorem 7.3. *The category of tilting modules has a full ribbon functor to a semisimple ribbon category $\mathcal{C}$ whose nonisomorphic simple objects are the image of the tilting modules with highest weight in Λ^l.*

Proof. We construct the functor via the quotient construction described above. It is clear that the image of any negligible tilting module is null, i.e. isomorphic to the null module $\{0\}$. Since $\mathrm{qdim}(\lambda) \neq 0$ for all $\lambda \in \Lambda^l$, each such module is mapped to a non-null object and thus $\mathcal{C}$ is semisimple with these as simple objects.

Because $f \sim g$ implies $f \otimes h \sim g \otimes h$ and $h \otimes f \sim h \otimes g$ for all h, $\mathcal{C}$ inherits a tensor product structure making the functor a tensor functor. The image of the braiding morphisms, duality morphisms and twist morphisms are braiding, duality and twist morphisms for $\mathcal{C}$. □

We aim to describe the quotient category in more detail. The following results, based on the previous two sections, will provide us an explicit description of the tilting modules in Λ^l.

Lemma 7.4. *Let $\lambda, \mu \in \Lambda^l$. Then, $W^\mu \subset T^\lambda$ if and only if $\lambda = \mu$.*

Proof. We have already seen that $W^\lambda \subset T^\lambda$, as the first term in a Weyl filtration of T^λ. But, the same argument shows that if $W^\mu \subset T^\lambda$, we can construct a Weyl filtration starting on W^μ. Therefore, by the Linkage principle Corollary 5.20 there is a $\sigma \in \mathcal{W}_l$ such that $\mu = \sigma \cdot \lambda$. Since $\lambda, \mu \in C_l$ and W_l permutes the alcoves, we are left with $\sigma = 1$, $\lambda = \mu$. □

Theorem 7.5. *$T^\lambda = W^\lambda$, for $\lambda \in \Lambda^l$.*

Proof. By the linkage principle Propostion 5.20 it is clear that the Weyl module W^λ with highest weight $\lambda \in \Lambda^l$ is irreducible, since Λ^l is a fundamental domain of W_l. As T^λ is irreducible, the result follows. □

Now we address to the problem of describing the tensor structure on the quotient category.

For classical Lie algebras or generic q write

$$W^\lambda \otimes W^\gamma \cong \bigoplus_{\mu \in \Lambda^+} N^{\mu}_{\lambda,\gamma} W^\mu \tag{7}$$

where $N^{\mu}_{\lambda,\gamma}$ are nonnegative integers representing multiplicities.

For $\mathbf{q}$ an lth root of unity, if $\lambda, \gamma \in \Lambda^l$, then

$$W^{\lambda} \otimes W^{\gamma} \cong \bigoplus_{\mu \in \Lambda^l} M^{\mu}_{\lambda,\gamma} W^{\mu} \oplus N$$

where N is a negligible tilting module and each $M^{\mu}_{\lambda,\gamma}$ is a nonnegative integer representing multiplicity, since W^{λ}, W^{γ} are tilting modules. We define the *truncated tensor product* $\hat{\otimes}$ on direct sums of modules in Λ^l by extending the following to direct sums:

$$W^{\lambda} \hat{\otimes} W^{\gamma} \cong \bigoplus_{\mu \in \Lambda^l} M^{\mu}_{\lambda,\gamma} W^{\mu}. \tag{8}$$

Remark 7.6. Since each tilting module can be written uniquely as a direct sum $M \oplus N$, where M is isomorphic to a direct sum of modules in Λ^l and N is negligible, we can define an isomorphic functor from $\mathcal{C}$ to the full subcategory of the tilting module category consisting of modules isomorphic to a direct sum of modules in Λ^l. Unfortunately this functor does not preserve the tensor product. However this is a tensor isomorphism if the range category uses the truncated tensor product $\hat{\otimes}$ for its monoidal structure. This proves the truncated tensor product is a monoidal structure.

Our aim now is to derive a formula to compute the numbers $M^{\mu}_{\lambda,\gamma}$.

Proposition 7.7.

$$M^{\mu}_{\lambda,\gamma} = \sum_{\substack{\sigma \in \mathcal{W}_l \\ \sigma \cdot \mu \in \Lambda^+}} (-1)^{\sigma} N^{\sigma \cdot \mu}_{\lambda,\gamma}. \tag{9}$$

Proof. Over $\mathcal{A}$,

$$\operatorname{qtr}_{W^{\lambda} \otimes W^{\gamma}} = \sum_{\mu} N^{\mu}_{\lambda,\gamma} \operatorname{qtr}_{W^{\mu}}$$

so in particular the same holds over $\mathbb{Q}[\mathbf{s}]$. As a functional on the center this is equal to

$$\sum_{\mu \in \Lambda^l} \sum_{\substack{\sigma \in \mathcal{W}_l \\ \sigma \cdot \mu \in \Lambda^+}} (-1)^{\sigma} N^{\sigma \cdot \mu}_{\lambda,\gamma} \operatorname{qtr}_{\mu}.$$

On the other hand as a functional on the center

$$\operatorname{qtr}_{W^{\lambda} \otimes W^{\gamma}} = \operatorname{qtr}_{W^{\lambda} \hat{\otimes} W^{\gamma}} = \sum_{\mu \in \Lambda^l} M^{\mu}_{\lambda,\gamma} \operatorname{qtr}_{\mu}.$$

Since $\{\operatorname{qtr}_{\mu}\}_{\mu \in \Lambda^l}$ are linearly independent as functionals on the center (Corollary 4.16), the result follows. $\square$

Corollary 7.8 (Quantum Racah Formula)**.**

$$M^{\mu}_{\lambda,\gamma} = \sum_{\sigma \in \mathcal{W}_l} (-1)^{\sigma} \dim(W^{\lambda}(\sigma \cdot \mu - \gamma)) \tag{10}$$

where $W^{\lambda}(\gamma)$ is the subspace of W^{λ} of weight γ.

Proof. This result relies on the *classical Racah formula,* which says that

$$N^{\mu}_{\lambda,\gamma} = \sum_{\tau\in\mathcal{W}} (-1)^{\tau} \dim(W^{\lambda}(\tau\cdot\mu - \gamma)). \tag{11}$$

Recall that Λ^+ is a fundamental domain for the (standard) action of $\mathcal{W}$ and that only the identity fixes it. Suppose $\sigma \in \mathcal{W}_l$ takes the principal Weyl alcove C_l to some domain C. There is a unique element $\tau \in \mathcal{W}$ such that τ^{-1} of C intersects Λ^+. Thus $\tau^{-1}\sigma$ takes Λ^l to some fundamental domain in Λ^+. We conclude that every element of the affine Weyl group can be written uniquely as $\tau\eta$, where $\tau \in \mathcal{W}$ and $\eta[\Lambda^l] \subset \Lambda^+$.

Note now that in our formula (9), the range of summation was the set $\{\sigma \in \mathcal{W}_l,\, \sigma\cdot\mu \in \Lambda^+\}$, for a fixed $\mu \in \Lambda^l$. If σ belongs to this set, then $\sigma[\Lambda^l] \subset \Lambda^+$, since it preserves connected components on the complement of the hyperplanes. In this case, the (τ, η) factorization described in the previous paragraph is given by $\tau = 1$, $\eta = \sigma$. Therefore,

$$\begin{aligned} M^{\mu}_{\lambda,\gamma} &= \sum_{\substack{\sigma\,\in\,\mathcal{W}_l \\ \sigma\cdot\mu\,\in\,\Lambda^+}} (-1)^{\sigma} N^{\sigma\cdot\mu}_{\lambda,\gamma} = \sum_{\eta[\Lambda^l]\subset\Lambda^+} (-1)^{\eta} N^{\eta\cdot\mu}_{\lambda,\gamma} \\ &= \sum_{\eta[\Lambda^l]\subset\Lambda^+} (-1)^{\eta} \sum_{\tau\in\mathcal{W}} (-1)^{\tau} \dim(W^{\lambda}(\tau\eta\cdot\mu - \gamma) \\ &= \sum_{\sigma\in\mathcal{W}_l} (-1)^{\sigma} \dim(W^{\lambda}(\sigma\cdot\mu - \gamma). \end{aligned}$$

$\square$

8. Quantum groups at roots of 1. Conclusion and applications

We have constructed a modular tensor category from a quotient of a certain subcategory of the category of representations of the corresponding quantum group of a compact, simple, simply-connected Lie group and each integer level. This process of "semisimplification" has interest by itself, and has been considered by many authors, including a generalization on [O05]. It is of interest to consider which categoris can be made into semisimple ones, and relating the onformation from one to the other. The semisimple category constructed is not always modular, and thus not always give invariants of links and 3 manifolds. Nevertheless, this last issue is largely studied and Sawin, in [S05] describe exactly the cases in which the S matrix is nondegerante.

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