From Hopf algebras to tensor categories

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Commutative Algebra and Its Interactions with Algebraic Geometry, Representation Theory, and Physics

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I. Introduction. $\Bbbk$ algebraically closed field.

A algebra: product $\mu : A \otimes A \to A$, unit $u : \Bbbk \to A$

Associative:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

Unitary:

$$\begin{tikzcd} \Bbbk \otimes A \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\sim"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \Bbbk \arrow[l, "\mathrm{id} \otimes u"'] \arrow[dl, "\sim"] \\ & A & \end{tikzcd}$$

C coalgebra: coproduct $\Delta : C \to C \otimes C$, counit $\varepsilon : C \to \Bbbk$

Co-associative:

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "\mathrm{id} \otimes \Delta"'] \\ C \otimes C \arrow[u, "\Delta \otimes \mathrm{id}"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

Co-unitary:

$$\begin{tikzcd} \Bbbk \otimes C \arrow[dr, "\sim"'] & C \otimes C \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \varepsilon"] & C \otimes \Bbbk \arrow[dl, "\sim"] \\ & C \arrow[u, "\Delta"] & \end{tikzcd}$$

Hopf algebra: $(H, \mu, u, \Delta, \varepsilon)$

- (H, μ, u) algebra
- (H, Δ, ε) coalgebra
- Δ, ε algebra maps
- There exists $\mathcal{S} : H \to H$ (the antipode) such that

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[dr, "\varepsilon"'] & H \otimes H \arrow[r, "\mathcal{S}\otimes\mathrm{id}", "\mathrm{id}\otimes\mathcal{S}"'] & H \otimes H \arrow[r, "\mu"] & H \\
& \Bbbk \arrow[urr, "u"'] & &
\end{tikzcd}$$

Example:

- Γ finite group
- $H = \mathcal{O}(\Gamma) =$ algebra of functions $\Gamma \to \mathbb{k}$
- $\Delta : H \to H \otimes H \simeq \mathcal{O}(\Gamma \times \Gamma)$, $\Delta(f)(x, y) = f(x.y)$.
- $\varepsilon : H \to \mathbb{k}$, $\varepsilon(f) = f(e)$.
- $\mathcal{S} : H \to H$, $\mathcal{S}(f)(x) = f(x^{-1})$.

Remark: $(H, \mu, u, \Delta, \varepsilon)$ finite-dimensional Hopf algebra
$\Longrightarrow$ $(H^*, \Delta^t, \varepsilon^t, \mu^t, u^t)$ Hopf algebra

Example: $H = \mathcal{O}(\mathit{\Gamma})$; for $x \in \mathit{\Gamma}$, $E_x \in H^*$, $E_x(f) = f(x)$. Then

$$E_x E_y = E_{xy}, \qquad \mathcal{S}(E_x) = E_{x^{-1}}.$$

Hence $H^* = \Bbbk\mathit{\Gamma}$, group algebra of $\mathit{\Gamma}$.

Remark: $(H, \mu, u, \Delta, \varepsilon)$ Hopf algebra with $\dim H = \infty$,
H^* NOT a Hopf algebra,
but contains a largest Hopf algebra with operations transpose to those of H.

Example:

- Γ affine algebraic group

- $H = \mathcal{O}(\Gamma) =$ algebra of regular (polynomial) functions $\Gamma \to \Bbbk$ is a Hopf algebra with analogous operations.

- $H^* \supset \Bbbk\Gamma$
- $H^* \supset \mathcal{U} :=$ algebra of distributions with support at e; this is a Hopf algebra

- If $\operatorname{char} \Bbbk = 0$, then $\mathcal{U} \simeq U(\mathfrak{g})$, $\mathfrak{g} =$ Lie algebra of Γ

- If $\mathfrak{g}$ is any Lie algebra, then the enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$.

Short history: See also [AF].

• Since the dictionary *Lie groups* $\leftrightsquigarrow$ *Lie algebras* fails when char > 0, Dieudonné studied in the early 50's the hyperalgebra $\mathcal{U}$. Pierre Cartier introduced the abstract notion of hyperalgebra (cocommutative Hopf algebra) in 1955.

• A. Borel considered algebras with a coproduct (1952) extending previous work of Hopf. He coined the expression *Hopf algebra*.

• Very influential paper by Milnor and Moore.

• George I. Kac introduced an analogous notion in the context of von Neumann algebras.

• The first appearance of the definition (that I am aware of) as it is known today is in a paper by Kostant (1965).

First invariants of a Hopf algebra H:
$G(H) = \{x \in H - 0 : \Delta(x) = x \otimes x\}$, group of grouplikes.

$\mathrm{Prim}(H) = \{x \in H : \Delta(x) = x \otimes 1 + 1 \otimes x\}$, Lie algebra of primitive elements.

$\tau : V \otimes W \to W \otimes W$, $\tau(v \otimes w) = w \otimes v$ the *flip*.
H is commutative if $\mu\tau = \mu$. H is cocommutative if $\tau\Delta = \Delta$.

Group algebras, enveloping algebras, hyperalgebras are cocommutative.

Theorem. (Cartier-Kostant, early 60's). $\mathrm{char}\, \Bbbk = 0$.
Any cocommutative Hopf algebra is of the form $U(\mathfrak{g})\#\Bbbk\Gamma$.

$H = \Bbbk[X]$, $\Delta(X) = X \otimes 1 + 1 \otimes X$. Then

$$\Delta(X^n) = \sum_{0 \le j \le n} \binom{n}{j} X^j \otimes X^{n-j}.$$

If $\operatorname{char} \Bbbk = p > 0$, then $\Delta(X^p) = X^p \otimes 1 + 1 \otimes X^p$.

Thus $\Bbbk[X]/\langle X^p \rangle$, $\Delta(X) = X \otimes 1 + 1 \otimes X$ is a Hopf algebra, commutative and cocommutative, dim p.

(Kulish, Reshetikhin and Sklyanin, 1981). Quantum $SL(2)$: if $q \in \Bbbk$, $q \neq 0, \pm 1$, set

$$
\begin{aligned}
U_q(\mathfrak{sl}(2)) = \Bbbk\langle E, F, K, K^{-1} | KK^{-1} = 1 = K^{-1}K \\
KE = q^2 EK, \\
KF = q^{-2} FK, \\
EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\rangle
\end{aligned}
$$

$$
\begin{aligned}
\Delta(K) &= K \otimes K, \\
\Delta(E) &= E \otimes 1 + K \otimes E, \\
\Delta(F) &= F \otimes K^{-1} + 1 \otimes F.
\end{aligned}
$$

These Hopf algebras, neither commutative nor cocommutative, are analogues of the enveloping algebra of $\mathfrak{sl}(2)$.

(Lusztig, 1989). If q is a root of 1 of order N odd, then

$$\begin{aligned} \mathfrak{u}_q(\mathfrak{sl}(2)) = \Bbbk\langle E, F, K, K^{-1} | &\text{same relations plus} \\ K^N = 1, \quad E^N = F^N = 0\rangle. \end{aligned}$$

These Hopf algebras, neither commutative nor cocommutative, are analogues of the Frobenius kernel of $\mathfrak{sl}(2)$.

There are dual Hopf algebras, analogues of the algebra of regular functions of $SL(2)$.

$$\mathcal{O}_q(SL(2)) = \Bbbk\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} | ab = qba, \quad ac = qca, \quad bc = cb,$$
$$bd = qdb, \quad cd = qdc, \quad ad - da = (q - q^{-1})bc,$$
$$ad - qbc = 1\rangle.$$

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(Manin). If q is a root of 1 of order N odd, then

$$\mathfrak{o}_q(\mathfrak{sl}(2)) = \Bbbk\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} |\text{same relations plus}$$
$$a^N = 1 = d^N, \quad b^N = c^N = 0\rangle.$$

In 1983, Drinfeld and Jimbo introduced quantized enveloping algebras $U_q(\mathfrak{g})$, for q as above and $\mathfrak{g}$ any simple Lie algebra.

- Quantum function algebras $\mathcal{O}_q(G)$: Faddeev-Reshetikhin and Takhtajan (for $SL(N)$) and Lusztig (any simple G).

- Finite-dimensional versions when q is a root of 1.

Motivation: A braided vector space is a pair (V, c), where V is a vector space and $c : V \otimes V \to V \otimes V$ is a linear isomorphism that satisfies

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

This is called the braid equation (closely related to the quantum Yang-Baxter equation).

- Any Hopf algebra (with bijective antipode) gives a machine of solutions of the braid equation.

- The solutions associated to $U_q(\mathfrak{g})$ are very important in low dimensional topology and some areas of theoretical physics.

Braided Hopf algebra: $(R, c, \mu, u, \Delta, \varepsilon)$

- (R, c) braided vector space
- (R, μ, u) algebra, (R, Δ, ε) coalgebra
- Δ, ε algebra maps, with the multiplication μ_2 in $R \otimes R$

$$\begin{tikzcd}
R \otimes R \otimes R \otimes R \arrow[rr, "\mathrm{id} \otimes c \otimes \mathrm{id}"] \arrow[dr, "\mu_2"'] & & R \otimes R \otimes R \otimes R \arrow[dl, "\mu \otimes \mu"] \\
& R \otimes R &
\end{tikzcd}$$

- There exists $\mathcal{S} : R \to R$, the antipode.

Braided Hopf algebras appear in nature:

Let $\pi : H \to K$ be a surjective morphism of Hopf algebras that admits a section $\iota : K \to H$, also a morphism of Hopf algebras. Then

$$R = \{x \in H : (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$

is a braided Hopf algebra; it bears an action and a coaction of K. Also

$$H \simeq R\#K.$$

We say that H is the bosonization of R by K.

II. On the classification of finite-dimensional Hopf algebras
$\Bbbk = \overline{\Bbbk}$, $\operatorname{char} \Bbbk = 0$.

Let C be a coalgebra, $D, E \subset C$. Then

$D \wedge E = \{x \in C : \Delta(x) \in D \otimes C + C \otimes E\}$,

$\wedge^0 D = D$, $\wedge^{n+1} D = (\wedge^n D) \wedge D$.

More invariants of a Hopf algebra H:

- The coradical $H_0 =$ sum of all simple subcoalgebras of H.
- The *coradical filtration* is $H_n = \wedge^{n+1} H_0$.

Assume that the coradical is a Hopf subalgebra (true for $\mathfrak{u}_q(\mathfrak{sl}(2))$, false for $\mathfrak{o}_q(\mathfrak{sl}(2))$).

Example: H is pointed if $H_0 = \Bbbk G(H)$.

- The *associated graded Hopf algebra* $\operatorname{gr} H = \oplus_{n\in\mathbb{N}} H_n/H_{n-1}$.

It turns out that $\operatorname{gr} H \simeq R\#H_0$, where

- $R = \oplus_{n\in\mathbb{N}} R^n$ is a graded connected algebra and it is a braided Hopf algebra. $V := R^1 =$ *infinitesimal braiding*.

- The subalgebra of R generated by R^1 is isomorphic to the Nichols algebra $\mathfrak{B}(V)$.

Example:

$H = U_q(\mathfrak{b}) = \Bbbk\langle E, K, K^{-1} | KK^{-1} = 1 = K^{-1}K,\ KE = q^2 EK\rangle$,

$\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes 1 + K \otimes E$.

- $H_0 = \Bbbk\langle K, K^{-1}\rangle \simeq \Bbbk\mathbb{Z}$.
- $H_n =$ subspace spanned by $K^j E^m$, $j \in \mathbb{Z}$, $m \leq n$.
- $H \simeq \operatorname{gr} H \simeq R\#\Bbbk\langle K, K^{-1}\rangle$, where
- $R = \Bbbk\langle E\rangle$, $c(E^i \otimes E^j) = q^{2ij} E^j \otimes E^i$; $\Delta(E) = E \otimes 1 + 1 \otimes E$.

Example: $H = U_q(\mathfrak{sl}(2))$

- $H_0 = \Bbbk\langle K, K^{-1}\rangle \simeq \Bbbk\mathbb{Z}$.

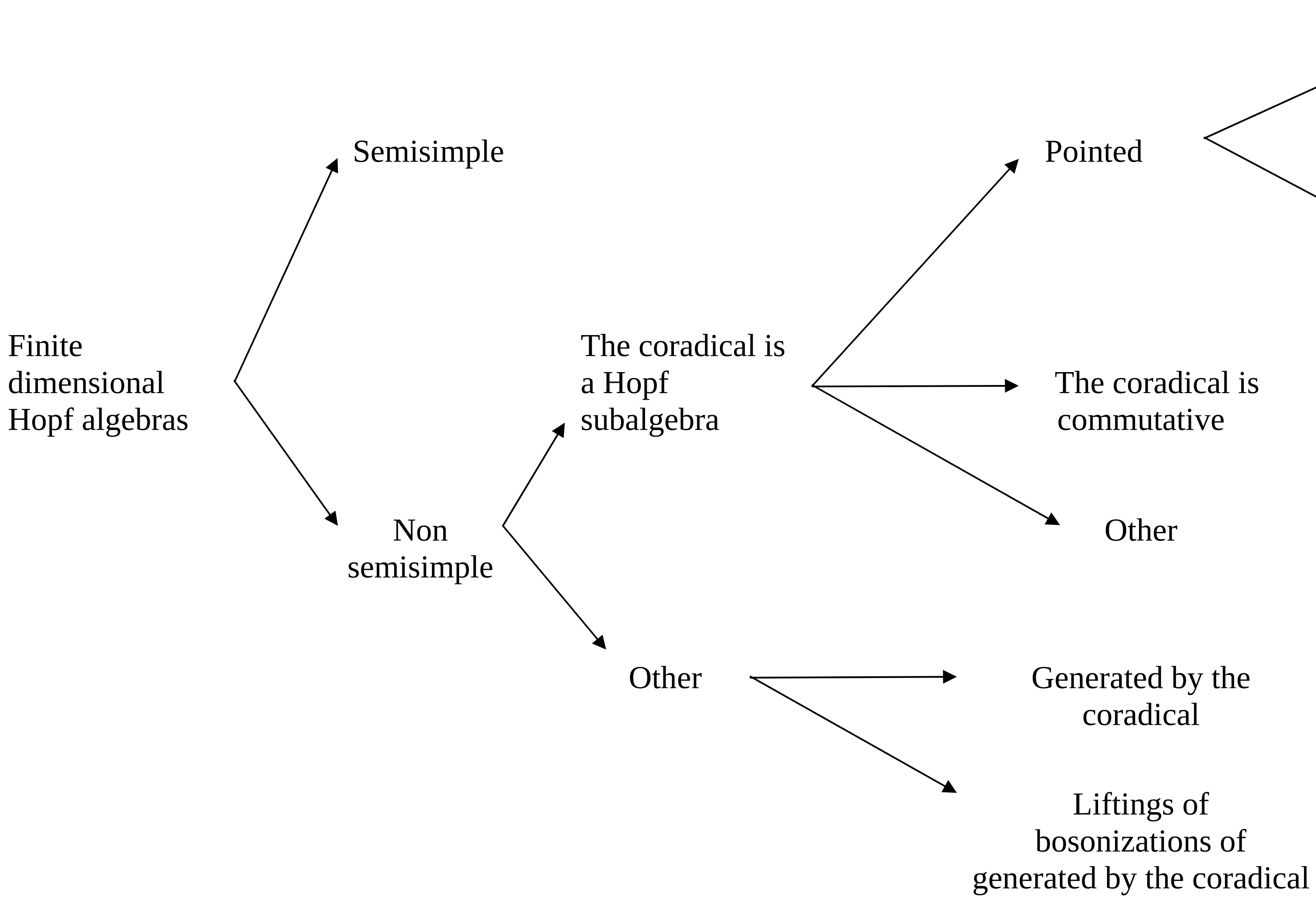
- $H_n =$ subspace spanned by $K^j E^i F^{n-i}$, $j \in \mathbb{Z}$, $i \in \mathbb{N}$.

- $\operatorname{gr} H = \Bbbk\langle X, Y, K, K^{-1} | KK^{-1} = 1 = K^{-1}K$,
$KX = q^2 XK$, $KY = q^{-2} YK$, $XY - qYX = 0\rangle$.

$$\Delta(X) = X \otimes 1 + K \otimes X,\ \Delta(Y) = Y \otimes 1 + K^{-1} \otimes Y.$$

- $R = \Bbbk\langle X, Y\rangle$, $c(X \otimes Y) = q^2 Y \otimes X$, $c(Y \otimes X) = q^{-2} X \otimes Y$.

$\Delta(X) = X \otimes 1 + 1 \otimes X$, $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$.


Finite dimensional Hopf algebras
Semisimple
Non semisimple
The coradical is a Hopf subalgebra
Other
Pointed
The coradical is commutative
Other
Abelian Group
Non abelian Group
Generated by the coradical
Liftings of bosonizations of generated by the coradical

Finite-dimensional pointed Hopf algebras, $\Gamma = G(H)$ abelian

• If the prime divisors of Γ are > 7, then the classification is known [AS]. The outcome is that all are variations of the Lusztig's small quantum groups.

• If the prime divisors of Γ are arbitrary, then the classification is in progress, thanks to recent results of Heckenberger and Angiono [A1, A2, H]. Besides the variations of the Lusztig's small quantum groups, there are also small quantum supergroups and a list of exceptions.

Finite-dimensional pointed Hopf algebras, $\Gamma = G(H)$ non abelian

Besides $\Bbbk\Gamma$:

- $\Gamma = \mathbb{S}_3$. [AHS], with previous work with A. Milinski, M. Graña, S. Zhang.

There are two (both dim 72): $\mathcal{A}_0 = \mathfrak{B}(V_3)\#\Bbbk\mathbb{S}_3$ and $\mathcal{A}_1$, a deformation of $\mathcal{A}_0$.

- $\Gamma = \mathbb{S}_4$. [GG], with previous work by [AHS] and A. Milinski, M. Graña.

There are (all of dim 24^3):

- $\mathfrak{B}(V)\#\Bbbk\mathbb{S}_4$, for 3 different V related to transpositions and 4-cycles.
- Two one-parameter families of deformations and a single deformation.

- $\Gamma = D_n$, n divisible by 4. [FG]. There are
- $\Lambda(V)\#\Bbbk\mathbb{S}_4$, for various V.
- Families of deformations.

For many Γ the following holds: *If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq \Gamma$, then $H \simeq \Bbbk\Gamma$.*

• [AFGV1] $\mathbb{A}_n$, $n \geq 5$.
• [FGV] $SL(2, 2^s)$,$SL(4, 2^3)$.
• [AFGV2]. Γ simple sporadic, except Fi_{22}, Baby Monster and the Monster.

Finite-dimensional copointed Hopf algebras, $G(H)$ non abelian

• $\Gamma = \mathbb{S}_3$. [AV]. There are iinfinitely many (all dim 72): $\mathcal{A}_0 = \mathfrak{B}(V_3)\#\Bbbk^{\mathbb{S}_3}$ and an infinite family of deformations.

Main open problem:

$n \geq 5$, $\mathcal{O}_2^n$ = set of transpositions in $\mathbb{S}_n$, V_n = v. s. with basis $x_{(ij)}$, $(ij) \in \mathcal{O}_2^n$. $\mathfrak{B}_n := T(V_n)$ divided by the ideal generated by

$$
\begin{aligned}
& x_{(ij)}^2, \\
& x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)}, \\
& x_{(ij)}x_{(ik)} + x_{(jk)}x_{(ij)} + x_{(jk)}x_{(jk)}, \\
& x_{(ik)}x_{(ij)} + x_{(ij)}x_{(jk)} + x_{(jk)}x_{(jk)}
\end{aligned}
$$

It is known that dim $\mathcal{B}_n$ is

- 12 for $n = 3$, [MS].
- 24^2 for $n = 4$, [MS].
- 8,294,400 for $n = 5$, [computed by Jan-Erik Roos with Bergman].
- Unknown for $n \geq 6$, even dim $\mathcal{B}_n < \infty$?

III. Tensor categories.

Monoidal categories (categorical versions of groups).

A *monoidal category* is a category $\mathcal{C}$ provided with

- A bifuntor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called *tensor*;
- an object $\mathbf{1} \in \mathcal{C}$, called *unit*;
- an *associativity* constraint, i. e. a natural isomorphism

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z);$$

- left and right *unit* constraints, i. e. natural isomorphisms

$$l_X : \mathbf{1} \otimes X \simeq X, \qquad r_X : X \otimes \mathbf{1} \simeq X.$$

$(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ should satisfy the pentagon and triangle axioms, i. e. the commutativity of (1), (2), for any $X, Y, Z, W \in \operatorname{Obj} \mathcal{C}$:

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[r, "a_{X\otimes Y,Z,W}"] \arrow[d, "a_{X,Y,Z} \otimes \mathrm{id}_W"] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r, "a_{X,Y,Z\otimes W}"] & X \otimes (Y \otimes (Z \otimes W)) \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "a_{X,Y\otimes Z,W}"] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[u, "\mathrm{id}_X \otimes a_{Y,Z,W}"']
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "r_X \otimes \mathrm{id}_Y"] \arrow[dr, "a_{X,\mathbf{1},Y}"'] & & X \otimes Y \\
& X \otimes (\mathbf{1} \otimes Y) \arrow[ur, "\mathrm{id}_X \otimes l_Y"'] &
\end{tikzcd} \tag{2}$$

Let $\mathcal{C}$ be a monoidal category.

A *right dual* of $V \in \mathsf{Obj}\,\mathcal{C}$ is a collection (V^*, e_V, b_V), where

- $V^* \in \mathsf{Obj}\,\mathcal{C}$,
- $e_V : V^* \otimes V \to \mathbf{1}$ is a morphism called *evaluation*,
- $b_V : \mathbf{1} \to V \otimes V^*$ is a morphism called *coevaluation*, such that

$$\begin{tikzcd} V \arrow[r, "l_V^{-1}"] \arrow[rrrrr, bend right, "\mathsf{id}_V"'] & \mathbf{1}\otimes V \arrow[r, "b_V \otimes \mathsf{id}_V"] & (V\otimes V^*)\otimes V \arrow[r, "a_{V,V^*,V}"] & V\otimes(V^*\otimes V) \arrow[r, "\mathsf{id}_V \otimes e_V"] & V\otimes \mathbf{1} \arrow[r, "r_V"] & V \end{tikzcd},$$

$$\begin{tikzcd} V^* \arrow[r, "r_{V^*}^{-1}"] \arrow[rrrrr, bend right, "\mathsf{id}_V^*"'] & V^*\otimes \mathbf{1} \arrow[r, "\mathsf{id}_{V^*} \otimes b_V"] & V^*\otimes(V\otimes V^*) \arrow[r, "a_{V^*,V,V^*}^{-1}"] & (V^*\otimes V)\otimes V^* \arrow[r, "e_V \otimes \mathsf{id}_{V^*}"] & \mathbf{1}\otimes V^* \arrow[r, "l_{V^*}"] & V^* \end{tikzcd}.$$

A *left dual* of $V \in \operatorname{Obj}\mathcal{C}$ is a collection $({}^*V, e'_V, b'_V)$, where

- ${}^*V \in \operatorname{Obj}\mathcal{C}$,
- $e'_V : {}^*V \otimes V \to \mathbf{1}$, $b'_V : \mathbf{1} \to V \otimes {}^*V$ are morphisms such that

$$\begin{tikzcd}
V \arrow[r, "r_V^{-1}"] \arrow[rrrrr, bend right=15, "\mathrm{id}_V"'] & V \otimes \mathbf{1} \arrow[r, "\mathrm{id}_V \otimes b'_V"] & V \otimes ({}^*V \otimes V) \arrow[r, "a^{-1}_{V,{}^*V,V}"] & (V \otimes {}^*V) \otimes V \arrow[r, "e'_V \otimes \mathrm{id}_V"] & \mathbf{1} \otimes V \arrow[r, "l_V"] & V\ ,
\end{tikzcd}$$

$$\begin{tikzcd}
{}^*V \arrow[r, "l^{-1}_{{}^*V}"] \arrow[rrrrr, bend right=15, "\mathrm{id}_{{}^*V}"'] & \mathbf{1} \otimes {}^*V \arrow[r, "b'_V \otimes \mathrm{id}_{{}^*V}"] & ({}^*V \otimes V) \otimes {}^*V \arrow[r, "a_{{}^*V,V,V^*}"] & {}^*V \otimes (V \otimes {}^*V) \arrow[r, "\mathrm{id}_{{}^*V} \otimes e'_V"] & {}^*V \otimes \mathbf{1} \arrow[r, "r_{{}^*V}"] & {}^*V.
\end{tikzcd}$$

A monoidal category is *rigid* if every object admits a right and a left dual.

Examples:

- $\mathcal{C}$ discrete category (only arrows are the identities)

monoidal $\leftrightsquigarrow$ monoid rigid monoidal $\leftrightsquigarrow$ group

- $\mathsf{Vec}_{\Bbbk}$ = category of vector spaces over $\Bbbk$, $\otimes = \otimes_{\Bbbk}$

$V \in \mathsf{Vec}_{\Bbbk}$ has duals $(V^* = \mathsf{Hom}(V, \Bbbk) = {}^*V) \iff \dim V < \infty \rightsquigarrow$
$\mathsf{vec}_{\Bbbk}$ = category of fin. dim. vector spaces is rigid

- R a $\Bbbk$-algebra, Bimod_R = category of R-bimodules, $\otimes = \otimes_R$

- G a group, Rep_G, $\otimes = \otimes_{\Bbbk}$; rep_G = fin. dim. reps. is rigid

$V, W \in \mathsf{Rep}_G$, $v \in V, w \in W, g \in G$: $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$.

• $\mathfrak{g}$ a Lie algebra, $\mathsf{Rep}_{\mathfrak{g}}$, $\otimes = \otimes_{\Bbbk}$; $\mathsf{rep}_{\mathfrak{g}}$ = fin. dim. reps. is rigid
$V, W \in \mathsf{Rep}_{\mathfrak{g}}$, $v \in V, w \in W, X \in \mathfrak{g}$: $X \cdot (v \otimes w) = X \cdot v \otimes w + v \otimes X \cdot w$.

• H a Hopf algebra with bijective antipode $\mathcal{S}$, Rep_H, $\otimes = \otimes_{\Bbbk}$;
rep_H = fin. dim. reps. is rigid

• $V, W \in \mathsf{Rep}_H$, $v \in V, w \in W, X \in H$: set $\Delta(X) = \sum_i X_i \otimes X^i$, then

$$X \cdot (v \otimes w) = X \cdot v \otimes w + v \otimes X \cdot w.$$

• $\mathbf{1} := \Bbbk \in \mathsf{Rep}_H$: if $X \in H$, then $X \cdot 1 = \varepsilon(X)1$.

• $V \in \mathsf{rep}_H \rightsquigarrow V^* = \mathrm{Hom}(V, \Bbbk) = {}^*V$ as v. sp. but with **different** actions: $v \in V, X \in H, \alpha \in \mathbf{V}^* = \mathrm{Hom}(V, \Bbbk), \beta \in {}^*\mathbf{V} = \mathrm{Hom}(V, \Bbbk)$

$$\langle X \cdot \alpha, v \rangle = \langle \alpha, \mathcal{S}(X) \cdot v \rangle, \qquad \langle X \cdot \beta, v \rangle = \langle \beta, \mathcal{S}^{-1}(X) \cdot v \rangle.$$

Tensor categories

A *tensor category* is a monoidal category $\mathcal{C}$ such that

- $\mathcal{C}$ is abelian (good kernels and cokernels);
- $\mathcal{C}$ is $\Bbbk$-linear ($\mathrm{Hom}(V, W)$ is a $\Bbbk$-v. sp., composition is bilinear);
- the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is $\Bbbk$-bilinear;
- the unit $\mathbf{1} \in \mathcal{C}$ is simple and $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) \simeq \Bbbk$;

Example: H a Hopf algebra with bijective antipode $\mathcal{S}$

$\rightsquigarrow$ $\mathrm{Rep}\, H$ is a tensor category

Another construction: (H, ω) a *spherical* Hopf algebra.

$\Longrightarrow$ Rep H has a factor tensor category $\underline{Rep}\, H$ that is semisimple but not Rep K for any K.

Problem: compute finite tensor subcategories of $\underline{Rep}\, H$.

Example: If q is a root of 1 and $H = \mathfrak{u}_q(g)$, then the category of tilting modules is a finite tensor subcategory of $\underline{Rep}\, H$ (Andersen-Padarowski).

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