On the automorphisms of $U_q^+(\mathfrak{g})$

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Abstract. Let $\mathfrak{g}$ be a simple complex finite dimensional Lie algebra and let $U_q^+(\mathfrak{g})$ be the positive part of the quantum enveloping algebra of $\mathfrak{g}$. If $\mathfrak{g}$ is of type A_2, the group of algebra automorphisms of $U_q^+(\mathfrak{g})$ is a semidirect product $(\mathbb{k}^\times)^2 \rtimes \operatorname{Autdiagr}(\mathfrak{g})$; any algebra automorphism is an automorphism of braided Hopf algebra, and preserves the standard grading [2]. This intriguing smallness of the group of algebra automorphisms raises questions about the extent of these phenomena. We discuss some of them in the present paper. We introduce the notion of "algebra with few automorphisms" and establish some consequences. We prove some exploratory results concerning the group of algebra automorphisms for the type B_2. We study Hopf algebra automorphisms of Nichols algebras and their bosonizations and compute in particular the group of Hopf algebra automorphisms of $U_q^+(\mathfrak{g})$.

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Introduction

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition of $\mathfrak{g}$ related to a Cartan subalgebra $\mathfrak{h}$. The structure

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of the group $\operatorname{Aut}_{\mathrm{Alg}} U_q(\mathfrak{g})$ of algebra automorphisms of the quantum enveloping algebra $U_q(\mathfrak{g})$ seems to be known only in the elementary case where $\mathfrak{g}$ is of type A_1 (see [1] or [2]). The automorphism group $\operatorname{Aut}_{\mathrm{Alg}} \check{U}_q(\mathfrak{b}^+)$ of the augmented form of the quantum enveloping algebra of the Borel subalgebra $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ is described for any $\mathfrak{g}$ in [11]. The groups of Hopf algebra automorphisms of $U_q(\mathfrak{g})$ and $\check{U}_q(\mathfrak{b}^+)$ are determined in [10] and [11] respectively. We are concerned in this paper with the group of automorphisms of the quantum enveloping algebra $U_q^+(\mathfrak{g}) = U_q(\mathfrak{n}^+)$ of the nilpotent part.

Let (V, c) be a braided vector space, see 1.1.1 below, and let $\mathfrak{B}(V)$ be the corresponding Nichols algebra. We prove that the group $\operatorname{Aut}_{\mathrm{Hopf}} \mathfrak{B}(V)$ of braided Hopf algebra automorphisms of $\mathfrak{B}(V)$ coincides with the group $GL(V, c)$ of automorphisms of braided vector space of (V, c). Thus we have

$$\operatorname{Aut}_{\mathrm{Hopf}} \mathfrak{B}(V) \subset \operatorname{Aut}_{\mathrm{GrAlg}} \mathfrak{B}(V) \subset \operatorname{Aut}_{\mathrm{Alg}} \mathfrak{B}(V),$$

where $\operatorname{Aut}_{\mathrm{GrAlg}}$ means the group of algebra automorphisms homogeneous of degree 0, in this case with respect to the standard grading of $\mathfrak{B}(V)$, and $\operatorname{Aut}_{\mathrm{Alg}}$ is the group of all algebra automorphisms.

The class of Nichols algebras includes symmetric algebras, free algebras, Grassmann algebras; thus there is no hope to have a round description of the group $\operatorname{Aut}_{\mathrm{Alg}} \mathfrak{B}(V)$. In particular the computation of this group is a well-known classical open problem in the case of the symmetric algebras.

Let $\Bbbk$ be the ground field and let (V, c) be a braided vector space of diagonal type, $n = \dim V$. Under some technical assumptions, the group $GL(V, c)$ reduces in this case to the semidirect product of the canonical action of the torus $(\Bbbk^\times)^n$ by the subgroup $\operatorname{Autdiagr}(c)$ of the symmetric group $\mathbb{S}_n$ preserving the matrix of the braiding.

A fundamental characterization due to Lusztig and Rosso (in different but equivalent formulations), says that $U_q^+(\mathfrak{g}) = \mathfrak{B}(\mathfrak{h})$ with a diagonal braiding c with matrix $(q^{d_i a_{ij}})$ built up from q and the Cartan matrix of $\mathfrak{g}$; say $n = \dim \mathfrak{h}$. Furthermore, $\operatorname{Autdiagr} c$ is in the case the group $\operatorname{Autdiagr}(\mathfrak{g})$ of automorphisms of the Dynkin diagram of $\mathfrak{g}$. Therefore,

$$\operatorname{Aut}_{\mathrm{Hopf}} U_q^+(\mathfrak{g}) \simeq (\Bbbk^\times)^n \rtimes \operatorname{Autdiagr}(\mathfrak{g}).$$

The group $\operatorname{Aut}_{\mathrm{Alg}} U_q^+(\mathfrak{g})$ was not determined yet, to the best of our knowledge. However, if $\mathfrak{g}$ of type A_2, then $\operatorname{Aut}_{\mathrm{Alg}} U_q^+(\mathfrak{g}) \simeq (\Bbbk^\times)^2 \rtimes \operatorname{Autdiagr}(\mathfrak{g})$ [2] (see also [8]). Thus, $\operatorname{Aut}_{\mathrm{Hopf}} \mathfrak{B}(V) = \operatorname{Aut}_{\mathrm{GrAlg}} \mathfrak{B}(V) = \operatorname{Aut}_{\mathrm{Alg}} \mathfrak{B}(V)$ in this case. This intriguing result motivates several questions:

Problem 1. *Is it true that* $\operatorname{Aut}_{\mathrm{Alg}} U_q^+(\mathfrak{g}) \simeq (\Bbbk^\times)^n \rtimes \operatorname{Autdiagr}(\mathfrak{g})$*?*

We conjecture that the answer is positive for any $\mathfrak{g}$. But even if the answer were negative, we would still ask: is it true that $\operatorname{Aut}_{\mathrm{Alg}} U_q^+(\mathfrak{g})$ is an algebraic group?

Problem 2. *Determine the braided vector spaces* (V, c) *such that*

$$\operatorname{Aut}_{\mathrm{Hopf}} \mathfrak{B}(V) = \operatorname{Aut}_{\mathrm{GrAlg}} \mathfrak{B}(V) = \operatorname{Aut}_{\mathrm{Alg}} \mathfrak{B}(V).$$

Graded algebras A with the property $\operatorname{Aut}_{\mathrm{GrAlg}} A = \operatorname{Aut}_{\mathrm{Alg}} A$ do not seem to abound. Thus, we dare to pose:

Problem 3. *Classify graded algebras A with the property* $\operatorname{Aut}_{\mathrm{GrAlg}} A = \operatorname{Aut}_{\mathrm{Alg}} A$.

In this paper we contribute mainly to the first Problem. Let us review the contents of the article. The first section is about Hopf algebra automorphisms of Nichols algebras and their bosonizations. We obtain from some general considerations the computation of the group $\operatorname{Aut}_{\mathrm{Hopf}} U_q(\mathfrak{b}^+) \simeq (\Bbbk^\times)^n \rtimes \operatorname{Autdiagr}(\mathfrak{g})$, recovering a result from [11].

In Section two, we briefly recall the case of type A_2. We introduce the notion of "algebras with few automorphisms"; we classify gradings of these algebras and apply to $U_q^+(\mathfrak{sl}_3)$.

Section three is an exploration of the case where $\mathfrak{g}$ is of type B_2. Since there are no nontrivial diagram automorphism in this case, the question is then to determine if $\operatorname{Aut}_{\mathrm{Alg}} U_q^+(\mathfrak{g})$ is isomorphic or not to $(\Bbbk^\times)^2$.

A basic idea to approach the group $\operatorname{Aut}_{\mathrm{Alg}} U_q^+(\mathfrak{g})$ is to study its actions on natural sets. In the paper [2], the study of the actions on the sets of central and normal elements was crucial. This method fails here because the center of $U_q^+(\mathfrak{g})$ is a polynomial algebra $\Bbbk[z, z']$ in two variables (which are homogeneous elements of degree 3 and 4 respectively for the canonical grading) and any normal element of $U_q^+(\mathfrak{g})$ is automatically central.

The next natural sets where our group acts are the various spectra; the investigation of these actions is the matter of this section. We begin by the ideals (z) and (z'); these are completely prime of height one and the factor domain $U_q^+(\mathfrak{g})/(z)$ is isomorphic to the quantum Heisenberg algebra $U_q^+(\mathfrak{sl}_3)$. Using the results of Section two, this allows to separate up to isomorphism the factor domains $U_q^+(\mathfrak{g})/(z)$ and $U_q^+(\mathfrak{g})/(z')$. We then show, first, that automorphisms of $U_q^+(\mathfrak{g})$ cannot exchange the ideals (z) and (z'); and second, that the subgroup of $\operatorname{Aut}_{\mathrm{Alg}} U_q^+(\mathfrak{g})$ of automorphisms preserving the ideal (z) reduces to the torus $(\Bbbk^\times)^2$. To progress further in this direction, we need better knowledge on the prime ideals of height one. Although the stratification of the prime spectrum is known [12], the full classification of the prime ideals is still open. We discuss the stratification for type B_2 in Subsection 3.4.

Added in proof. After acceptance of this paper, S. Launois gave a positive answer to Problem 1 for type B_2 using our Proposition 3.3. Thus our conjecture is true in this case. See [15]. Also, Problem 1 is solved for type A_3 in [16].

Convention

We denote by $\mathbb{N}$ the set of non-negative integers $\{0, 1, 2, 3, \dots\}$.

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1 Braided Hopf algebra automorphisms

In Subsections 1.3 and 1.2 the field $\Bbbk$ is arbitrary; in 1.3, $\Bbbk$ has characteristic 0 and contains an element q not algebraic over $\mathbb{Q}$.

1.1 Braided vector spaces

1.1.1 Braided vector spaces. A braided vector space is a pair (V, c) where V is a vector space V over $\Bbbk$ and $c : V \otimes V \to V \otimes V$ is a linear isomorphism that is a solution of the braid equation $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$.

A braided vector space (V, c) is *rigid* if V is finite dimensional and the map $c^\flat : V^* \otimes V \to V \otimes V^*$ is invertible, where

$$c^\flat = (\mathrm{ev}_V \otimes \mathrm{id}_{V\otimes V^*})(\mathrm{id}_{V^*} \otimes c \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*\otimes V} \otimes \mathrm{coev}_V).$$

Here $\mathrm{ev}_V : V^* \otimes V \to \Bbbk$ is the usual evaluation map, and $\mathrm{coev}_V : \Bbbk \to V \otimes V^*$ is the coevaluation (the transpose of the trace).

1.1.2 Automorphisms of a braided vector space of diagonal type. Let (V, c) be a braided vector space. The braiding $c : V \otimes V \to V \otimes V$ is said to be *diagonal* if there exists a basis $x_1, \dots, x_n$ of V and a matrix $(q_{ij})_{1\le i,j\le n}$ with entries in $\Bbbk^\times$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for any $1 \le i, j \le n$. In particular, V is rigid and has finite dimension $n \ge 1$.

Remark 1.1. If the braiding is diagonal, then the matrix $(q_{ij})_{1\le i,j\le n}$ does not depend on the basis $x_1, \dots, x_n$, up to permutation of the index set $\{1, \dots, n\}$, see [5, Lemma 1.2].

Let (V, c) be a braided vector space. A linear automorphism $g \in GL(V)$ is said to be a *braided vector space automorphism* of (V, c) if $g \otimes g$ commutes with c. We denote by $GL(V, c)$ the corresponding subgroup of $GL(V)$.

Suppose that c is of diagonal type for some basis $x_1, \dots, x_n$ of V and some matrix $(q_{ij})_{1\le i,j\le n}$. We consider the following subgroup of the symmetric group $\mathbb{S}_n$:

$$\operatorname{Autdiagr}(c) := \{\sigma \in \mathbb{S}_n : q_{ij} = q_{\sigma(i),\sigma(j)},\ 1 \le i, j \le n\}.$$

Any σ in $\operatorname{Autdiagr}(c)$ induces naturally an automorphism $g_\sigma \in GL(V, c)$ by $g_\sigma(x_j) = x_{\sigma(j)}$ for $1 \le j \le n$. Any $g \in GL(V, c)$ of this type is called a *diagram automorphism* of (V, c). Moreover it is clear that the torus $(\Bbbk^\times)^n$ acts on V by braided vector space automorphisms. The following lemma gives necessary conditions for the group $GL(V, c)$ to be generated by these two particular subgroups.

Lemma 1.2. *Let (V, c) be a braided vector space of diagonal type, with respect to a basis $x_1, \dots, x_n$ of V and a matrix $(q_{ij})_{1\le i,j\le n}$ with entries in $\Bbbk^\times$. Assume that at least one of the following conditions is satisfied:*

(i) For any $i \ne j$, there exists h such that $q_{ih} \ne q_{jh}$.

(ii) For any $i \ne j$, there exists h such that $q_{hi} \ne q_{hj}$.

(iii) For any $i \ne j$, the matrix $\left(\begin{smallmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{smallmatrix}\right)$ is not of the form $\left(\begin{smallmatrix} q & q \\ q & q \end{smallmatrix}\right)$.

Then we have

$$GL(V, c) \simeq (\Bbbk^\times)^n \rtimes \operatorname{Autdiagr}(c).$$

Proof. Let $g \in GL(V)$; denote $g(x_i) = \sum_s \lambda_{s,i} x_s$, $1 \le i \le n$. Then

$$(g \otimes g)(c(x_i \otimes x_j)) = (g \otimes g)(q_{ij}(x_j \otimes x_i)) = \sum_{1\le r,s\le n} q_{ij} \lambda_{r,j} \lambda_{s,i}\, x_r \otimes x_s,$$

$$c(g \otimes g)(x_i \otimes x_j) = c\Big(\sum_{1\le r,s\le n} \lambda_{r,j} \lambda_{s,i}\, x_s \otimes x_r \Big) = \sum_{1\le i,j\le n} q_{sr} \lambda_{r,j} \lambda_{s,i}\, x_r \otimes x_s.$$

Therefore $g \in GL(V, c)$ is and only if

$$q_{ij} \lambda_{r,j} \lambda_{s,i} = q_{sr} \lambda_{r,j} \lambda_{s,i}, \quad \text{for all } 1 \le i, j, r, s \le n. \tag{1.1}$$

Suppose $g \in GL(V, c)$. Since g is invertible, there exists $\sigma \in \mathbb{S}_n$ such that $\lambda_{\sigma(h),h} \ne 0$ for any $1 \le h \le n$. Then we deduce from (1.1) that $q_{ij} = q_{\sigma(i),\sigma(j)}$ for all $1 \le i, j \le n$.

Assume first that (i) is satisfied, and choose $i, s \in \{1, \dots, n\}$ such that $\lambda_{s,i} \neq 0$. Apply (1.1) with any j and $r = \sigma(j)$. We obtain $q_{s,\sigma(j)} = q_{ij} = q_{\sigma(i),\sigma(j)}$, for any $1 \leq j \leq n$; then $s = \sigma(i)$. This implies that $g(x_i) = \lambda_{\sigma(i),i} x_{\sigma(i)}$ for any $1 \leq i \leq n$, and proves the result.

Assume now that (ii) is satisfied, and choose $j, r \in \{1, \dots, n\}$ such that $\lambda_{r,j} \neq 0$. Apply (1.1) with any i and $s = \sigma(i)$. We obtain $q_{\sigma(i),r} = q_{ij} = q_{\sigma(i),\sigma(j)}$, for any $1 \leq i \leq n$; then $r = \sigma(j)$ and we conclude as in the previous case.

Finally assume that we have (iii) and take $i = j$ in (1.1). If $\lambda_{s,i} \neq 0$ then $q_{ii} = q_{\sigma(i),s} = q_{s,\sigma(i)} = q_{\sigma(i),\sigma(i)}$; therefore $s = \sigma(i)$ and we conclude as above. □

1.1.3 Braided Hopf algebras. A non-categorical version of the concept of braided Hopf algebra was studied in [21]. A braided Hopf algebra is a collection (R, m, Δ, c) such that

- (R, c) is a braided vector space,
- (R, m) is an associative algebra with unit 1,
- (R, Δ) is a coassociative coalgebra with counit ε,
- $m, \Delta, 1, \varepsilon$ commute with c in the sense of [21],
- $\Delta \circ m = (m \otimes m)(\mathrm{id} \otimes c \otimes \mathrm{id})(\Delta \otimes \Delta)$,
- the identity has an inverse for the convolution product in $\operatorname{End} R$ (this inverse is called the *antipode* and denoted by $\mathcal{S}$).

Here, recall that the convolution product of $f, g \in \operatorname{End} R$ is given by $f * g = m(f \otimes g)\Delta$.

A homomorphism of braided Hopf algebras is a linear map preserving m, Δ, c.

Lemma 1.3. *Let R be a braided Hopf algebra and let $T : R \to R$ be a linear isomorphism that is an algebra and coalgebra map. Then T is a morphism of braided Hopf algebras.*

Proof. Let us define $T.f := TfT^{-1}$, $f \in \operatorname{End} R$. Then

$$T.(f * g) = T\,(m(f \otimes g)\Delta)\,T^{-1} = m(T \otimes T)(f \otimes g)(T^{-1} \otimes T^{-1})\Delta = (T.f) * (T.g);$$

hence $T.\mathcal{S} = \mathcal{S}$, or $T\mathcal{S} = \mathcal{S}T$. But it was shown in [20] that the braiding of a braided Hopf algebra can be expressed in terms of the product, coproduct and antipode. Thus T preserves also the braiding c. □

The group of braided Hopf algebra automorphisms of R is denoted by $\operatorname{Aut}_{\mathrm{Hopf}} R$.

1.1.4 Yetter-Drinfeld modules. Yetter-Drinfeld modules give rise to braided vector spaces and play a fundamental rôle in problems related to the classification of Hopf algebras.

Let us recall that a Yetter-Drinfeld module V over a Hopf algebra H with bijective antipode $\mathcal{S}$ is both a left H-module and left H-comodule, such that the action $H \otimes V \dot{\to} V$ and the coaction $\delta : V \to H \otimes V$ satisfy the compatibility condition: $\delta(h.v) = h_{(1)} v_{(-1)} \mathcal{S} h_{(3)} \otimes h_{(2)}.v_{(0)}$ for all $h \in H, v \in V$. We denote by ${}^H_H\mathcal{YD}$ the category of Yetter-Drinfeld modules over H, where morphisms respect both the action and the coaction of H.

The usual tensor product defines a structure of monoidal category on ${}^H_H\mathcal{YD}$. It is braided, with braiding $c_{V,W} : V \otimes W \to W \otimes V$ defined by $c(v \otimes w) = v_{(-1)}.w \otimes v_{(0)}$, for $V, W \in {}^H_H\mathcal{YD}$, $v \in V$, $w \in W$. Then $(V, c_{V,V})$ is a braided vector space, for any $V \in {}^H_H\mathcal{YD}$. It is known that any rigid braided vector space can be realized as a Yetter-Drinfeld module over a (non-unique) Hopf algebra, essentially by the FRT-construction; see [21] for references and details.

As in any braided monoidal category, there is the notion of Hopf algebras in ${}^H_H\mathcal{YD}$. Hopf algebras in ${}^H_H\mathcal{YD}$ are braided Hopf algebras by forgetting the action and the coaction. Conversely, let R be any braided Hopf algebra whose underlying braided vector space is rigid. Then there exists a (non-unique) Hopf algebra H such that R can be realized as a Hopf algebra in ${}^H_H\mathcal{YD}$ [21].

1.1.5 Bosonizations of a Hopf algebra in ${}^H_H\mathcal{YD}$. We recall the bosonization procedure, or Radford biproduct, found by Radford and explained in terms of braided categories by Majid.

Let H be a Hopf algebra with bijective antipode. Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. The bosonization of R by H is the (usual) Hopf algebra $A = R\#H$, with underlying vector space $R \otimes H$, whose multiplication and comultiplication are given by:

$$(r\#h)(s\#f) = r(h_{(1)}.s)\#h_{(2)}f \quad \text{and} \quad \Delta(r\#h) = r^{(1)}\#(r^{(2)})_{(-1)}h_{(1)} \otimes (r^{(2)})_{(0)}\#h_{(2)}.$$

The maps $\pi : A \to H, r\#h \mapsto \epsilon(r)h$ and $\iota : H \to A, h \mapsto 1\#h$ are Hopf algebra homomorphisms and $R = \{a \in A : (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$.

Conversely, let A, H be Hopf algebras with bijective antipode and let $\pi : A \to H$ and $\iota : H \to A$ be Hopf algebra homomorphisms such that $\pi\iota = \mathrm{id}_H$. Then $R = \{a \in A : (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$ is a Hopf algebra in ${}^H_H\mathcal{YD}$ and the multiplication induces an isomorphism of Hopf algebras $R\#H \simeq A$.

1.1.6 Nichols algebras. We recall the definition of Nichols algebra, see [4] for details and references.

Let $V \in {}^H_H\mathcal{YD}$. A graded Hopf algebra $R = \bigoplus_{n\geq 0} R(n)$ in ${}^H_H\mathcal{YD}$ is called a *Nichols algebra* of V if $\Bbbk \simeq R(0)$ and $V \simeq R(1)$ in ${}^H_H\mathcal{YD}$, and if:

$$R(1) = \mathcal{P}(R), \text{ the space of primitive elements of } R, \tag{1.2}$$

$$R \text{ is generated as an algebra by } R(1). \tag{1.3}$$

The Nichols algebra of V exists and is unique up to isomorphism; it is denoted by $\mathfrak{B}(V) = \bigoplus_{n\geq 0} \mathfrak{B}^n(V)$. The associated braided Hopf algebra (forgetting the action and the coaction) depends only on the braided vector space (V, c). We shall identify V with the subspace of homogeneous elements of degree one in $\mathfrak{B}(V)$.

Let us recall the following explicit construction of $\mathfrak{B}(V)$. For any integer $m \geq 2$, we denote by $\mathbb{B}_m$ the m-braid group. A presentation of $\mathbb{B}_m$ is given by generators $\sigma_1, \dots, \sigma_{m-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$ an $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for any $1 \leq i \leq m-2$. There is a natural projection $\pi : \mathbb{B}_m \to \mathbb{S}_m$ sending σ_i to the transposition $\tau_i := (i, i+1)$ for all i. This projection π admits a set-theoretical section $s : \mathbb{S}_m \to \mathbb{B}_m$ determined by

$$s(\tau_i) = \sigma_i, \quad 1 \leq i \leq n-1, \qquad s(\tau\omega) = s(\tau)s(\omega), \quad \text{if } \ell(\tau\omega) = \ell(\tau) + \ell(\omega).$$

Here ℓ denotes the length of an element of $\mathbb{S}_m$ with respect to the set of generators $\tau_1, \dots, \tau_{m-1}$. The map s is called the Matsumoto section. In other words, if $\omega = \tau_{i_1} \dots \tau_{i_j}$ is a reduced expression of $\omega \in \mathbb{S}_m$, then $s(\omega) = \sigma_{i_1} \dots \sigma_{i_j}$. Using the section s, the following distinguished elements of the group algebra $\Bbbk\mathbb{B}_m$ are defined:

$$\mathfrak{S}_m := \sum_{\sigma \in \mathbb{S}_m} s(\sigma).$$

By convention, we still denote by $\mathfrak{S}_m$ the images of these elements in $\operatorname{End}(T^m(V)) = \operatorname{End}(V^{\otimes m})$ via the representation $\rho_m : \mathbb{B}_m \to \operatorname{Aut}(V^{\otimes m})$ defined by $\rho_m(\sigma_i) = \operatorname{id} \otimes \dots \otimes \operatorname{id} \otimes c \otimes \operatorname{id} \otimes \dots \otimes \operatorname{id}$, with c acting on the tensor product of the copies of V indexed by i and $i+1$.

Let $\widetilde{c}$ be the canonical extension of c into a braiding of $T(V)$. Let $T(V) \underline{\otimes} T(V)$ be the algebra whose underlying vector space is $T(V) \otimes T(V)$ with the product "twisted" by $\widetilde{c}$. There is a unique algebra map $\Delta : T(V) \to T(V) \underline{\otimes} T(V)$ such that $\Delta(v) = v \otimes 1 + 1 \otimes v$, $v \in V$. Then $T(V)$ is a braided Hopf algebra and $\mathfrak{B}(V) = T(V)/J$ where $J = \bigoplus_{m\geq 0} \operatorname{Ker} \mathfrak{S}_m$ (see for instance [4], [18] or [19]).

1.2 Automorphisms of Nichols algebras and their bosonizations

1.2.1 Hopf algebra automorphisms of a Nichols algebra. We can now compute the group of Hopf algebra automorphisms of a Nichols algebra, *cf.* the notation introduced in 1.1.2.

Theorem 1.4. *There is a group isomorphism* $\mathfrak{B} : GL(V,c) \to \mathrm{Aut}_{\mathrm{Hopf}}\,\mathfrak{B}(V)$.

Proof. For any $g \in GL(V)$, we denote by $\widetilde{g}$ the canonical extension of g into an algebra automorphism of $T(V)$. If g commutes with c, then $\widetilde{g}$ commutes with $\sum_{m\geq 2}\mathfrak{S}_m$, and so induces an algebra automorphism $\mathfrak{B}(g)$ of $\mathfrak{B}(V) = T(V)/J$. In order to prove that $\mathfrak{B}(g)$ is also a coalgebra automorphism of $\mathfrak{B}(V)$, we claim that $\widetilde{g}$ is a coalgebra map: $((\widetilde{g}\otimes\widetilde{g})\circ\Delta)(v) = (\Delta\circ\widetilde{g})(v)$, for any $v \in T(V)$.

It is clear that this assertion is true when $v \in V$. So it is enough to prove that $\widetilde{g}\otimes\widetilde{g}$ is a morphism of the algebra $T(V)\underline{\otimes}T(V)$. For that, let us consider $u, v, x, y \in T(V)$. Let us set $\widetilde{c}(y\otimes u) = u'\otimes y'$ in a symbolic way. By definition of the twisted product in $T(V)$, we have $(x\otimes y)(u\otimes v) = xu'\otimes y'v$, then:

$$(\widetilde{g}\otimes\widetilde{g})((x\otimes y)(u\otimes v)) = \widetilde{g}(x)\widetilde{g}(u')\otimes\widetilde{g}(y')\widetilde{g}(v).$$

Moreover, it is easy to check that the assumption $g \in GL(V,c)$ implies $\widetilde{g} \in GL(T(V),\widetilde{c})$, so $(\widetilde{g}\otimes\widetilde{g})(u'\otimes y') = \widetilde{c}(\widetilde{g}(y)\otimes\widetilde{g}(u))$, and then:

$$(\widetilde{g}\otimes\widetilde{g})(x\otimes y)(\widetilde{g}\otimes\widetilde{g})(u\otimes v) = (\widetilde{g}(x)\otimes\widetilde{g}(y))(\widetilde{g}(u)\otimes\widetilde{g}(v)) = \widetilde{g}(x)\widetilde{g}(u')\otimes\widetilde{g}(y')\widetilde{g}(v).$$

Hence $(\widetilde{g}\otimes\widetilde{g})((x\otimes y)(u\otimes v)) = (\widetilde{g}\otimes\widetilde{g})(x\otimes y)(\widetilde{g}\otimes\widetilde{g})(u\otimes v)$, as claimed. By Lemma 1.3, $\mathfrak{B}(g)$ preserves c. Thus, we have a well-defined map $\mathfrak{B} : GL(V,c) \to \mathrm{Aut}_{\mathrm{Hopf}}\,\mathfrak{B}(V)$, which is injective by (1.3).

Conversely, let $u : \mathfrak{B}(V) \to \mathfrak{B}(V)$ be an automorphism of braided Hopf algebras. Then $u(V) = V$ since $V = P(\mathfrak{B}(V))$ and the theorem follows. □

1.2.2 Automorphisms of bosonizations. Our goal is to compute the Hopf algebra automorphisms of $A = R\#H$ when R is a Nichols algebra, under suitable hypothesis. The exposition is inspired by [3, Section 6]. We begin by a description of a natural class of such automorphisms, for general R.

Lemma 1.5. *Let H be a Hopf algebra and let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. Let $G : R \to R$ and $T : H \to H$ be linear maps. Then $G\#T := G\otimes T : R\#H \to R\#H$ is a Hopf algebra map if and only if the following conditions hold:*

$$T \text{ is a Hopf algebra automorphism of } H, \tag{1.4}$$
$$G \text{ is a Hopf algebra automorphism of } R, \tag{1.5}$$
$$G(h.s) = T(h).G(s), \qquad s \in R,\ h \in H, \tag{1.6}$$
$$\delta\circ G = (T\otimes G)\circ\delta. \tag{1.7}$$

Proof. Left to the reader. □

A pair (G,T) as in the Lemma shall be called *compatible*.

Lemma 1.6. *Let H be a Hopf algebra and V a Yetter-Drinfeld module over H.*

(i) Assume that H is cosemisimple, and that the following hypothesis holds:

(H) the types of the isotypic components of $V\#H$ under the adjoint action of H do not appear in the adjoint action of H on itself.

Then any Hopf algebra automorphism of $\mathfrak{B}(V)\#H$ is of the form $G\#T$, with (G,T) compatible.

(ii) If in addition H is commutative, then (H) *is equivalent to:*

(H′) the trivial representation does not appear as a subrepresentation of V.

Proof. (i). Let Φ be a Hopf algebra automorphism of $A = \mathfrak{B}(V)\#H$. It is known that the coradical filtration of A is $A_m = \bigoplus_{0\leq n\leq m} \mathfrak{B}^n(V)\#H$ [4, 1.7]. Since Φ is a coalgebra map, it preserves the coradical filtration. In particular, $\Phi(H) = H$ and $\Phi(H \oplus V\#H) = H \oplus V\#H$. Let $T : H \to H$ be the restriction of Φ; it is an automorphism of Hopf algebras. Also, $\Phi : H \oplus V\#H \to H \oplus V\#H$ preserves the adjoint action of H. By hypothesis (H), $\Phi(V\#H) = V\#H$. Since Φ is an algebra map, this implies that $\Phi(\mathfrak{B}^n(V)\#H) = \mathfrak{B}^n(V)\#H$, by (1.3). Let $\pi : A \to H$ be the projection with kernel $\bigoplus_{n\geq 1} \mathfrak{B}^n(V)\#H$; clearly, $\Phi\pi = \pi\Phi$. Hence $\Phi(\mathfrak{B}(V)) = \mathfrak{B}(V)$, since $\mathfrak{B}(V) = \{v \in A : (\mathrm{id} \otimes\pi)\Delta(v) = v \otimes 1\}$. Let $G : \mathfrak{B}(V) \to \mathfrak{B}(V)$ be the restriction of Φ. Since Φ is an algebra map, $\Phi = G\#T$. By Lemma 1.5, the pair (G,T) is compatible.

(ii). If H is commutative, the adjoint action of H on itself is trivial, and the isotypic components of the adjoint action of H on $V\#H$ are of the form $U\#H$, where U runs in the set of isotypic components of the adjoint action of H on V. This shows that (H) is equivalent to (H′) in this case. □

The hypothesis (H) is needed, as the following example shows. Let $A = \Bbbk[x, g, g^{-1}]$ be the tensor product of the polynomial algebra in x and the Laurent polynomial algebra in g. This is a Hopf algebra with x primitive and g group-like. The Hopf algebra automorphism $T : A \to A$, $T(g) = g$, $T(x) = x + 1 - g$, does not preserve the Nichols algebra $\Bbbk[x]$.

We now consider the following particular setting. We assume that $H = \Bbbk\Gamma$ is the group algebra of an abelian group. We also assume the existence of a basis $x_1, \dots, x_n$ of V such that, for some elements $g_1, \dots, g_n \in \Gamma$, $\chi_1, \dots, \chi_n \in \widehat{\Gamma}$, the action and coaction of Γ are given by

$$h.x_j = \chi_j(h)x_j, \qquad \delta(x_j) = g(j) \otimes x_j, \qquad 1 \leq j \leq n.$$

Theorem 1.7. *Suppose further that*

$$\chi_i \neq \varepsilon, \qquad 1 \leq i \leq n, \tag{1.8}$$

$$(g_i, \chi_i) \neq (g_j, \chi_j), \quad 1 \leq i \neq j \leq n. \tag{1.9}$$

Then there is a bijective correspondence between $\operatorname{Aut}_{\mathrm{Hopf}} \mathfrak{B}(V)\#H$*, and the set of pairs (ϕ, ψ), where ψ is a group automorphism of Γ and $\phi : V \to V$ is a* *linear isomorphism given by* $\phi(x_i) = \lambda_i x_{\sigma(i)}$, $1 \le i \le n$, *with* $\lambda_i \in \Bbbk^\times$ *and* $\sigma \in \mathbb{S}_n$, *such that*

$$\psi(g_i) = g_{\sigma(i)}, \qquad \chi_i = \chi_{\sigma(i)} \circ \psi, \qquad 1 \le i \le n. \tag{1.10}$$

Proof. Condition (1.8) guarantees that hypothesis (H′) holds. Thus any Hopf algebra automorphism of $\mathfrak{B}(V)\#H$ is of the form $G\#T$, with (G,T) compatible, by Lemma 1.6. But T is determined by a group automorphism ψ of Γ, and G is of the form $\mathfrak{B}(\phi)$ for some $\phi \in GL(V,c)$ by Theorem 1.4. Now conditions (1.6) and (1.7) imply that

$$\gamma.\phi(x_i) = (\chi_i \circ \psi^{-1})(\gamma)\phi(x_i), \qquad \delta(\phi(x_i)) = \psi(g_i) \otimes \phi(x_i), \qquad \gamma \in \Gamma,\ 1 \le i \le n,$$

and thus

$$\phi(x_i) \in \sum_{j\ :\ \psi(g_i)=g_j,\ \chi_i\circ\psi^{-1}=\chi_j} \Bbbk\, x_j, \qquad 1 \le i \le n.$$

But condition (1.9) implies that there is only one j such that $\psi(g_i) = g_j$, $\chi_i \circ \psi^{-1} = \chi_j$; set $\sigma(i) = j$. This defines σ, and clearly (1.10) holds.

Conversely, any pair (ϕ, ψ) as above gives raise to a compatible pair (G,T) with $G = \mathfrak{B}(\phi)$ and T determined by ψ. □

1.3 Hopf algebra automorphisms of Nichols algebras of Drinfeld-Jimbo type

1.3.1 Definition and notations (*cf.* [6], [14], [17]). We fix $q \in \Bbbk^\times$, q not algebraic over $\mathbb{Q}$. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of rank n over $\Bbbk$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition of $\mathfrak{g}$ related to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $C = (a_{i,j})_{1\le i,j\le n}$ the associated Cartan matrix and $(d_1, \dots, d_n)$ the relatively primes integers symmetrizing C. The quantum enveloping algebra of the nilpotent positive part $\mathfrak{n}^+$ of $\mathfrak{g}$, denoted by $U_q(\mathfrak{n}^+)$ or $U_q^+(\mathfrak{g})$, is the algebra generated over $\Bbbk$ by n generators $E_1, \dots, E_n$ satisfying the quantum Serre relations:

$$\textstyle\sum_{\nu=0}^{1-a_{i,j}} \left[{1-a_{i,j} \atop \nu} \right]_{q^{d_i}} E_i^{1-a_{i,j}-\nu} E_j (-E_i)^\nu = 0 \quad \text{for all } 1 \le i \ne j \le n.$$

The quantum enveloping algebra of the positive Borel algebra $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$, denoted by $U_q(\mathfrak{b}^+)$, is the algebra generated over $\Bbbk$ by $E_1, \dots, E_n, K_1^{\pm1}, \dots, K_n^{\pm1}$ satisfying the quantum Serre relations, the commutation between the K_i's and the q-commutation relations:

$$K_i E_j = q^{d_i a_{i,j}} E_j K_i \quad \text{for all } 1 \le i, j \le n.$$

It is well-known that $U_q(\mathfrak{b}^+)$ is a Hopf algebra for the coproduct, counit and antipode defined by

$$\begin{array}{lll} \Delta(K_i) = K_i \otimes K_i, & \varepsilon(K_i) = 1, & S(K_i) = K_i^{-1}, \\ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, & \varepsilon(E_i) = 0, & S(E_i) = -K_i E_i. \end{array}$$

We denote by H the Hopf subalgebra $H = \Bbbk[K_1^{\pm1}, \dots, K_n^{\pm1}] \simeq \Bbbk\mathbb{Z}^n$ of $U_q(\mathfrak{b}^+)$.

1.3.2 Braided Hopf algebra structure on $U_q^+(\mathfrak{g})$. From Corollary 33.1.5 of [17] or Theorem 15 of [19], we have $U_q^+(\mathfrak{g}) = \mathfrak{B}(V)$ for $V = \Bbbk E_1 \oplus \ldots \oplus \Bbbk E_n$ and c the diagonal braiding of V defined from the Cartan matrix $C = (a_{i,j})_{1\le i,j\le n}$ and the integers $(d_1, \ldots, d_n)$ by

$$c(E_i \otimes E_j) = q^{d_i a_{i,j}} E_j \otimes E_i.$$

Let $\iota : H \to U_q(\mathfrak{b}^+)$ be the inclusion and let $\pi : U_q(\mathfrak{b}^+) \to H$ be the unique Hopf algebra map such that $\pi(K_i) = K_i$, $\pi(E_i) = 0$. Then $\pi\iota = \mathrm{id}_H$ and $U_q^+(\mathfrak{g}) = \{a \in U_q(\mathfrak{b}^+) : (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$. Hence

$$U_q(\mathfrak{b}^+) \simeq U_q^+(\mathfrak{g}) \# H,$$

cf. Subsection (1.1.5). Here the coaction is determined by $\delta(E_i) = g_i \otimes E_i$, with $g_i = K_i$, $1 \le i \le n$. Also, the action is determined by $\gamma.E_i = \chi_i(\gamma)E_i$, for $\gamma \in \Gamma = \mathbb{Z}^n$, where $\chi_i \in \widehat{\Gamma}$ is defined by $\chi_i(K_j) = q^{d_i a_{i,j}}$, $1 \le i, j \le n$.

1.3.3 Automorphisms of $U_q^+(\mathfrak{g})$ and $U_q^+(\mathfrak{b})$. With the notations of 1.1.2 for $q_{i,j} = q^{d_i} a_{i,j}$, the subgroup $\mathrm{Autdiagr}\, c$ of $\mathbb{S}_n$ is the group $\mathrm{Autdiagr}\,(\mathfrak{g})$ of automorphisms of the Dynkin diagram of $\mathfrak{g}$ (see for instance [10]), which acts by automorphisms on $U_q^+(\mathfrak{g})$ and $U_q(\mathfrak{b}^+)$ by:

$$\sigma \in \mathrm{Autdiagr}\,(\mathfrak{g}) : E_i \mapsto E_{\sigma(i)},\ K_i \mapsto K_{\sigma(i)} \quad \text{for any } 1 \le i \le n.$$

The n-dimensional torus on $\Bbbk$ also acts by automorphisms on $U_q^+(\mathfrak{g})$ and $U_q(\mathfrak{b}^+)$, by:

$$(\alpha_1, \ldots, \alpha_n) \in (\Bbbk^\times)^n : E_i \mapsto \alpha_i E_i,\ K_i \mapsto K_i \quad \text{for any } 1 \le i \le n.$$

Now we can prove the following theorem.

Theorem 1.8. $\mathrm{Aut}_{\,\mathrm{Hopf}}\, U_q^+(\mathfrak{g}) \simeq (\Bbbk^\times)^n \rtimes \mathrm{Autdiagr}\,(\mathfrak{g}) \simeq \mathrm{Aut}_{\,\mathrm{Hopf}}\, U_q(\mathfrak{b}^+)$.

Proof. The first isomorphism just follows from Lemma 1.2 and Theorem 1.4.

Let $\Gamma = \mathbb{Z}^n$, g_i and χ_i as above. By Theorem 1.7, any $T \in \mathrm{Aut}_{\,\mathrm{Hopf}}\, U_q(\mathfrak{b}^+)$ is determined by a pair (ϕ, ψ), where ψ is a group automorphism of Γ and $\phi : V \to V$ is a linear isomorphism given by $\phi(x_i) = \lambda_i x_{\sigma(i)}$, $1 \le i \le n$, with $\lambda_i \in \Bbbk^\times$ and $\sigma \in \mathbb{S}_n$, such that (1.10) holds. Then

$$q^{d_i a_{i,j}} = \chi_i(K_j) = \chi_i \psi^{-1}\psi(K_j) = \chi_{\sigma(i)}(K_{\sigma(j)}) = q^{d_{\sigma(i)} a_{\sigma(i),\sigma(j)}},$$

hence $\sigma \in \mathrm{Autdiagr}\,(\mathfrak{g})$. Furthermore, ψ is uniquely determined by σ. This implies the second isomorphism. □

Remark 1.9. The second isomorphism in the Theorem was proved previously in [11] as a corollary of the description of the group of all algebra automorphisms $\mathrm{Aut}_{\,\mathrm{Alg}} U_q(\mathfrak{b}^+)$ (in fact for the slightly different augmented algebra $\check{U}_q(\mathfrak{b}^+)$). Effectively there exist automorphisms of the algebra $\check{U}_q(\mathfrak{b}^+)$ which are not Hopf algebra automorphisms; in particular some combinatorial infinite subgroups of $n \times n$ matrices with coefficients in $\mathbb{Z}$, as well as the natural action of the $2n$-dimensional torus on the E_i's and K_j's.

2 The case where $\mathfrak{g}$ is of type A_2

In this section the field $\Bbbk$ has characteristic 0; in Subsection 2.2, $q \in \Bbbk^\times$ is not a root of one.

2.1 Graded algebras with few automorphisms

Here we consider graded algebras with few automorphisms. To begin with, we recall the well-known equivalence between gradings and rational actions, see [6, p. 150 ff.] and references therein.

Let $\mathcal{H}$ be an algebraic group. A representation $\rho : \mathcal{H} \to \operatorname{Aut}_{\Bbbk} V$ of $\mathcal{H}$ on a vector space V is *rational* if V is union of finite dimensional rational $\mathcal{H}$-modules.

If $\mathcal{H} = (\Bbbk^\times)^r$ is a torus, then there is a bijective correspondence between

- rational actions of $\mathcal{H}$ on V, and
- gradings $V = \bigoplus_{m \in \mathbb{Z}^r} V_m$.

In this correspondence, V_m is the isotypic component of type m, where $\mathbb{Z}^r$ is identified with the group of rational characters of $\mathcal{H}$. Thus, $\mathcal{H}$-submodules of V are rational, and they are exactly the graded subspaces of V.

Let A be an associative algebra over $\Bbbk$. A rational action of $\mathcal{H}$ on A is one induced by a rational representation $\rho : \mathcal{H} \to \operatorname{Aut}_{\mathrm{Alg}} A$ by algebra automorphisms. If $\mathcal{H} = (\Bbbk^\times)^r$ is a torus, then there is a bijective correspondence between

- rational actions of $\mathcal{H}$ on A, and
- algebra gradings $A = \bigoplus_{m \in \mathbb{Z}^r} A_m$.

Definition 2.1. *An associative algebra A has* few automorphisms *if the following conditions hold.*

(i) There exists a finite dimensional $\operatorname{Aut}_{\mathrm{Alg}} A$*-invariant subspace V such that the restriction* $\operatorname{Aut}_{\mathrm{Alg}} A \to GL(V)$ *is injective; we identify* $\operatorname{Aut}_{\mathrm{Alg}} A$ *with its image in $GL(V)$;*

(ii) $\operatorname{Aut}_{\mathrm{Alg}} A$ *is an algebraic subgroup of $GL(V)$;*

(iii) the action of $\operatorname{Aut}_{\mathrm{Alg}} A$ *on A is rational;*

(iv) the connected component $(\operatorname{Aut}_{\mathrm{Alg}} A)_0$ *of the identity of* $\operatorname{Aut}_{\mathrm{Alg}} A$ *is isomorphic to a torus* $(\Bbbk^\times)^r$*.*

Let A be an algebra with few isomorphisms. Then A has a *canonical* grading induced by the rational action of $(\operatorname{Aut}_{\mathrm{Alg}} A)_0 \simeq (\Bbbk^{\times})^r$:

$$A = \bigoplus_{m \in \mathbb{Z}^r} A_{(m)}.$$

Let $|\,| : \mathbb{Z}^r \to \mathbb{Z}$ be the function $|m| = \sum_{1 \le j \le r} m_j$, if $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$. The $\mathbb{Z}$-grading induced by the canonical grading via $|\,|$ shall be called the *standard grading* and denoted $A = \bigoplus_{M \in \mathbb{Z}} A_{[M]}$. Thus

$$A_{[M]} = \bigoplus_{m \in \mathbb{Z}^r : |m| = M} A_{(m)}, \qquad M \in \mathbb{Z}.$$

Lemma 2.2. *Let A be an algebra with few automorphisms.*

(i) Any algebra automorphism preserves the canonical grading, and those in $(\operatorname{Aut}_{\mathrm{Alg}} A)_0$ are homogeneous of degree 0.

(ii) Assume that the canonical grading is nonnegative: $A = \bigoplus_{m \in \mathbb{N}^r} A_{(m)}$. Then any algebra automorphism is homogeneous of degree 0 with respect to the standard grading.

Proof. (i). Let $\theta \in \operatorname{Aut}_{\mathrm{Alg}} A$, let $\operatorname{inn}_\theta : \operatorname{Aut}_{\mathrm{Alg}} A \to \operatorname{Aut}_{\mathrm{Alg}} A$ be the inner automorphism defined by θ and let $\widehat{\operatorname{inn}_\theta} : \mathbb{Z}^r \to \mathbb{Z}^r$ be the induced group homomorphism. Then $\theta\left(A_{(m)}\right) = A_{(\widehat{\operatorname{inn}_\theta}(m))}$ for any $m \in \mathbb{Z}^r$, and this implies the claim.

(ii). Under this hypothesis, the matrix of $\widehat{\operatorname{inn}_\theta}$ in the canonical basis has nonnegative entries. Thus $\theta\left(A_{[M]}\right) \subseteq \sum_{s \ge 0} A_{[M]+s}$; but some power of θ belongs to $(\operatorname{Aut}_{\mathrm{Alg}} A)_0$, hence only $s = 0$ survives. $\square$

Starting from the canonical grading, new gradings of A can be constructed by means of morphisms of groups $\mathbb{Z}^r \to \mathbb{Z}^t$; we show now that no other grading arises in this case.

Theorem 2.3. *Let A be an algebra with few automorphisms and let*

$$A = \bigoplus_{n \in \mathbb{Z}^t} A_n \tag{2.1}$$

be any algebra grading of A. Then there is a morphism of groups $\varphi : \mathbb{Z}^r \to \mathbb{Z}^t$ such that

$$A_n = \bigoplus_{m \in \mathbb{Z}^r : \varphi(m) = n} A_{(m)}, \tag{2.2}$$

for all $n \in \mathbb{Z}^t$.

Proof. Let $\mathcal{T}$ be the torus $\mathbb{Z}^t$ and let ρ be the representation of $\mathcal{T}$ induced by the grading (2.1). Let V be the vector subspace as in Definition 2.1; since V is stable under $\mathrm{Aut}_{\mathrm{Alg}}A$, it is also clearly stable under $\mathcal{T}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, "\rho"] \arrow[dr, "\rho_{|V}"'] & & \mathrm{Aut}_{\mathrm{Alg}}A \arrow[dl, "\mathrm{res}"] \\
& GL(V). &
\end{tikzcd}$$

The map $\rho_{|V}$ is a homomorphism of algebraic groups; then ρ is homomorphism of algebraic groups, say by [13, Ex. 3.10, p. 21]. Since $\mathcal{T}$ is connected, $\rho(\mathcal{T}) \subseteq (\mathrm{Aut}_{\mathrm{Alg}}A)_0$. Thus, the transpose of ρ induces a morphism of groups $\varphi : \mathbb{Z}^r \to \mathbb{Z}^t$. But then $A_{(m)} \subset A_{\varphi(m)}$. Since

$$A = \sum_{m\in\mathbb{Z}^r} A_{(m)} = \sum_{n\in\mathbb{Z}^t}\Big(\sum_{m\in\mathbb{Z}^r:\varphi(m)=n} A_{(m)}\Big) \subseteq \bigoplus_{n\in\mathbb{Z}^t} A_n = A,$$

we get the equality (2.2). $\square$

2.2 Algebra automorphisms of $U_q^+(\mathfrak{sl}_3)$

2.2.1 Notations. We suppose here that $\mathfrak{g} = \mathfrak{sl}_3$. Then we have $n = 2$, $C = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, $d_1 = d_2 = 1$, and $U_q^+(\mathfrak{g})$ is the algebra generated over $\Bbbk$ by E_1 and E_2 satisfying the relations: $E_1^2E_2 - (q^2+q^{-2})E_1E_2E_1 + E_2E_1^2 = E_2^2E_1 - (q^2 + q^{-2})E_2E_1E_2 + E_1E_2^2 = 0$. The algebra $U_q^+(\mathfrak{sl}_3)$ is usually named the *quantum Heisenberg algebra*. In the following we will denote it by $\mathbb{H}$. Setting $E_3 = E_1E_2 - q^2E_2E_1$, it is easy to check (see for instance [2]) that $\mathbb{H}$ is the iterated Ore extension generated over $\Bbbk$ by the three generators E_1, E_2, E_3 with relations:

$$E_1E_3 = q^{-2}E_3E_1, \qquad E_2E_3 = q^2E_3E_2, \qquad E_2E_1 = q^{-2}E_1E_2 - q^{-2}E_3.$$

The center of $\mathbb{H}$ is the a polynomial algebra in one indeterminate $Z(\mathbb{H}) = \Bbbk[\Omega]$ where the quantum Casimir element Ω is given by:

$$\Omega = (1-q^{-4})E_3E_1E_2 + q^{-4}E_3^2 = E_3\overline{E_3}, \quad \text{with } \overline{E_3} = E_1E_2 - q^{-2}E_2E_1.$$

Let $\mathbb{H} = \bigoplus \mathbb{H}_{m,n}$ be the canonical $\mathbb{N}^2$-grading of $\mathbb{H}$, defined putting E_1 on degree $(1,0)$ and E_2 on degree $(0,1)$. In particular $E_3, \overline{E_3} \in \mathbb{H}_{1,1}$ and $\Omega \in \mathbb{H}_{2,2}$.

2.2.2 Automorphisms and gradings of $U_q^+(\mathfrak{sl}_3)$. For all $\alpha, \beta \in \Bbbk^\times$, there exists one automorphism $\widetilde{\psi}_{\alpha,\beta}$ of $\mathbb{H}$ such that $\widetilde{\psi}_{\alpha,\beta}(E_1) = \alpha E_1$ et $\widetilde{\psi}_{\alpha,\beta}(E_2) = \beta E_2$. We introduce $\widetilde{G} := \{\widetilde{\psi}_{\alpha,\beta}\,;\, \alpha,\beta \in \Bbbk^\times\} \simeq (\Bbbk^\times)^2$, and the diagram automorphism ω of $\mathbb{H}$ defined by $\omega(E_1) = E_2$ and $\omega(E_2) = E_1$. Studying the action of any algebra automorphism of $\mathbb{H}$ on the center and on the set of normal elements of $\mathbb{H}$, Proposition 2.3 of [2] proves that $\mathrm{Aut}_{\mathrm{Alg}}\mathbb{H}$ is the semi-direct product of $\widetilde{G}$ by the subgroup of order 2 generated by ω (see Proposition 4.4 of [8] for another proof). So we have for the type A_2 the following positive answer to Problem 1.

Theorem 2.4. *For $\mathfrak{g}$ of type A_2, the algebra $\mathbb{H} = U_q^+(\mathfrak{g})$ satisfies* $\mathrm{Aut}_{\mathrm{Alg}}\mathbb{H} \simeq (\Bbbk^\times)^2 \rtimes \mathbb{S}_2$. □

Here is a consequence of this result which will be useful in the next section.

Corollary 2.5. *Let $\mathbb{H} = \bigoplus \mathbb{H}_{m,n}$ be the canonical $\mathbb{N}^2$-grading of $\mathbb{H}$. Let $\mathbb{H} = \bigoplus T_{m,n}$ be another $\mathbb{N}^2$-algebra grading of $\mathbb{H}$. Then there exists a matrix $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in M(2,\mathbb{Z})$, with non-negative entries, such that*

$$\mathbb{H}_{m,n} \subset T_{pm+qn,rm+sn}, \qquad \text{for all } (m,n) \in \mathbb{N}^2.$$

Proof. This follows from Theorem 2.3, since $\mathbb{H}$ has few automorphisms by Theorem 2.4. □

3 Partial results on the case where $\mathfrak{g}$ is of type B_2

A natural step in the study of Problem 1 would be to consider the other Lie algebras $\mathfrak{g}$ of rank two. We summarize in this section some partial results concerning the case B_2.

In this section, $\Bbbk$ is an algebraically closed field of characteristic zero, and $q \in \Bbbk^\times$ a quantization parameter not a root of one.

3.1 Some ring-theoretical properties of the algebra U^+

3.1.1 Notations. Let $\mathfrak{g}$ be the complex simple Lie algebra over $\Bbbk$ of type B_2. We denote $U^+ = U_q^+(\mathfrak{g})$. We recall all notations of 1.3.1 but denote now by e_i the generators E_i; we have here $n = 2$, $C = \left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$, and $(d_1, d_2) = (2, 1)$. Then the quantum Serre relations are:

$$\sum_{\nu=0}^{2} \left[\begin{smallmatrix} 2 \\ \nu \end{smallmatrix}\right]_{q^2} e_i^{2-\nu} e_j (-e_i)^\nu = 0 \qquad \text{for } i = 1, j = 2, a_{i,j} = -1, d_i = 2,$$

$$\sum_{\nu=0}^{3} \left[\begin{smallmatrix} 3 \\ \nu \end{smallmatrix}\right]_{q} e_i^{3-\nu} e_j (-e_i)^\nu = 0 \qquad \text{for } i = 2, j = 1, a_{i,j} = -2, d_i = 1.$$

We compute the quantum binomial coefficients: $\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]_{q^2} = \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]_{q^2} = 1$, $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]_{q^2} = q^2 + q^{-2}$, $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]_{q} = \left[\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right]_{q} = 1$, $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]_{q} = \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]_{q} = q^2 + 1 + q^{-2}$. We conclude that U^+ is the algebra generated over $\Bbbk$ by two generators e_1 and e_2 with commutation relations:

$$e_1^2 e_2 - (q^2 + q^{-2}) e_1 e_2 e_1 + e_2 e_1^2 = 0, \qquad \text{(S1)}$$
$$e_2^3 e_1 - (q^2 + 1 + q^{-2}) e_2^2 e_1 e_2 + (q^2 + 1 + q^{-2}) e_2 e_1 e_2^2 - e_1 e_2^3 = 0. \qquad \text{(S2)}$$

3.1.2 U^+ as an iterated Ore extension. From the natural generators e_1 and e_2 of U^+, we introduce following [22] the q-brackets:

$$e_3 = e_1e_2 - q^2e_2e_1 \qquad \text{and} \qquad z = e_2e_3 - q^2e_3e_2,$$

Relations (S1) and (S2) imply: $e_1e_3 = q^{-2}e_3e_1$, $e_1z = ze_1$ and $e_2z = ze_2$. In particular z is central in U^+. From [22], the monomials $(z^ie_3^je_1^ke_2^l)_{(i,j,k,l)\in\mathbb{N}^4}$ form a PBW basis of U^+. So U^+ is the algebra generated over $\Bbbk$ by e_1, e_2, e_3, z with relations:

$$\begin{array}{lll} e_3z = ze_3, & & \\ e_1z = ze_1, & e_1e_3 = q^{-2}e_3e_1, & \\ e_2z = ze_2, & e_2e_3 = q^2e_3e_2 + z, & e_2e_1 = q^{-2}e_1e_2 - q^{-2}e_3. \end{array}$$

In other words U^+ is the iterated Ore extension (*cf.* [6]):

$$U^+ = \Bbbk[e_3, z][e_1\,;\,\sigma][e_2\,;\,\tau,\delta] = S[e_2\,;\,\tau,\delta], \quad \text{with notation } S = \Bbbk[e_3, z][e_1\,;\,\sigma],$$

where σ is the automorphism of $\Bbbk[e_3, z]$ defined by $\sigma(z) = z$, $\sigma(e_3) = q^{-2}e_3$, τ is the automorphism of $\Bbbk[e_3, z][e_1\,;\,\sigma]$ defined by $\tau(z) = z$, $\tau(e_3) = q^2e_3$, $\tau(e_1) = q^{-2}e_1$, δ is the τ-derivation of $\Bbbk[e_3, z][e_1\,;\,\sigma]$ defined by $\delta(z) = 0$, $\delta(e_3) = z$, $\delta(e_1) = -q^{-2}e_3$, and S is the subalgebra of U^+ generated by e_3, z and e_1.

3.1.3 Grading of U^+. We consider the canonical grading $U^+ = \bigoplus_{n\geq 0} U_n$ putting the natural generators e_1 and e_2 in degree one (and then e_3 and z are of degree 2 and 3 respectively) defined from the basis $(z^ie_3^je_1^ke_2^l)_{(i,j,k,l)\in\mathbb{N}^4}$ de U^+ by: $U_n = \bigoplus_{3i+2j+k+l=n} \Bbbk z^ie_3^je_1^ke_2^l$ for any $n \geq 0$. We denote by $I = \bigoplus_{n\geq 1} U_n$ the ideal generated by e_1, e_2, e_3, z.

3.1.4 A localization of U^+. The subalgebra of U^+ generated over $\Bbbk$ by e_1 and e_3 is a quantum plane (with $e_3e_1 = q^2e_1e_3$); we will denote it by $\Bbbk_{q^2}[e_3, e_1]$. Its localization at the powers of e_3 and e_1 is the quantum torus $\Bbbk_{q^2}[e_3^{\pm}, e_1^{\pm}]$. The automorphism τ and the τ-derivation δ extend to $\Bbbk_{q^2}[e_3^{\pm}, e_1^{\pm}]$, and denoting by V the algebra $\Bbbk_{q^2}[e_3^{\pm}, e_1^{\pm}][z][e_2\,;\,\tau,\delta]$, we obtain the embedding:

$$U^+ = \Bbbk[e_3, z][e_1\,;\,\sigma][e_2\,;\,\tau,\delta] = \Bbbk_{q^2}[e_3, e_1][z][e_2\,;\,\tau,\delta] \ \subset\ V = \Bbbk_{q^2}[e_3^{\pm}, e_1^{\pm}][z][e_2\,;\,\tau,\delta],$$

Let us introduce in U^+ the bracket: $w = e_2e_3 - e_3e_2 = z + (q^2-1)e_3e_2 \in U_3$. It follows from commutation relations in 3.1.2 that: $e_1w = we_1 + (1-q^{-2})e_3^2$, $e_2w = q^2we_2$, et $e_3w = q^{-2}we_3$. Then the element:

$$z' = e_1w - q^{-4}we_1 \in U_4$$

satisfies $z'e_1 = e_1z'$ and $z'e_2 = e_2z'$, and so is central in U^+. A straightforward computation shows that its development in the PBW basis of 3.1.2 is:

$$z' = (1-q^{-4})(1-q^{-2})e_3e_1e_2 + q^{-4}(1-q^{-2})e_3^2 + (1-q^{-4})ze_1.$$

In particular, $z' = s_1e_2 + s_0$, with $s_0 = q^{-4}(1-q^{-2})e_3^2 + (1-q^{-4})ze_1 \in \Bbbk_{q^2}[e_3, e_1][z]$, and $s_1 = (1-q^{-4})(q^2-1)e_1e_3$ non-zero in $\Bbbk_{q^2}[e_3, e_1][z]$. So we have in V the identity $e_2 = s_1^{-1}z' - s_1^{-1}s_0$, with s_1^{-1} and $s_1^{-1}s_0$ in $\Bbbk_{q^2}[e_3^{\pm}, e_1^{\pm}][z]$. Explicitly:

$$e_2 = \tfrac{1}{(1-q^{-4})(q^2-1)}e_3^{-1}e_1^{-1}z' + \tfrac{1}{q^4-1}e_3^{-1}z - \tfrac{1}{q^2-1}e_1^{-1}e_3 \qquad (1)$$

We conclude:

$$U^+ \subset V = \Bbbk_{q^2}[e_3^{\pm}, e_1^{\pm}][z, z'].$$

Observe that z and z' being central in V, the only relation between the generators of V which is not a commutation is the q^2-commutation $e_3e_1 = q^2e_1e_3$.

3.1.5 Conjugation in U^+. We have introduced in 3.1.2 the homogeneous element e_3 of degree two defined from natural generators e_1 et e_2 by $e_3 = e_1e_2 - q^2e_2e_1$, which satisfy $e_1e_3 = q^{-2}e_3e_1$ and $e_2e_3 - q^2e_3e_2 = z$. Conjugating q in q^{-1}, we can also consider:

$$\overline{e_3} = e_1e_2 - q^{-2}e_2e_1 = (1-q^{-4})e_1e_2 + q^{-4}e_3,$$

and prove that $e_1\overline{e_3} = q^2\overline{e_3}e_1$ and $e_2\overline{e_3} - q^{-2}\overline{e_3}e_2 = q^{-4}z$. We obtain in particular the relations $e_3\overline{e_3} = (1-q^{-4})q^2e_1e_3e_2 + q^{-4}e_3^2$ and $z' = (1-q^{-2})(e_3\overline{e_3} + (1+q^{-2})ze_1)$, which will be used further in the paper.

3.1.6 Center and normalizing elements of U^+. Denote by $Z(U^+)$ the center and $N(U^+)$ the set of normalizing elements in U^+.

Lemma 3.1. $N(U^+) = Z(U^+) = \Bbbk[z, z']$.

Proof. The calculation of $Z(U^+)$ can be deduced from general results of [7]. The equality $N(U^+) = Z(U^+)$ for the type B_2 was observed in [8], Remark 2.2 (iii). We give here a short direct proof using the embedding $U^+ \subset V$. Take $f \in N(U^+)$ non-zero. From Proposition 2.1 of [8], f is q-central in U^+, that is there exist $m, n \in \mathbb{Z}$ such that $fe_1 = q^me_1f$ and $fe_2 = q^ne_2f$, and so $fe_3 = q^{m+n}e_3f$. In V, the element f is a finite sum:

$$f = \textstyle\sum_{i,j\in\mathbb{Z}} f_{i,j}(z, z')e_1^ie_3^j \quad \text{with } f_{i,j}(z, z') \in \Bbbk[z, z'].$$

Since the polynomials $f_{i,j}(z, z')$ are central in V, the identities $fe_1 = q^me_1f$ and $fe_3 = q^{m+n}e_3f$ give by identification $2j = m$ and $2i = -m-n$ for all i, j such that $f_{i,j}(z, z') \neq 0$. Then $f = f_{i,j}(z, z')e_1^ie_3^j$, where $i = -\frac{m+n}{2}$ and $j = \frac{m}{2}$.

Its follows from relation (1) of 3.1.4 and from q-commutation $fe_2 = q^n e_2 f$ that $-2j + 2i = 2i = -2j - 2i = n$. So $i = j = n = 0$, and $f = f_{0,0}(z, z') \in \Bbbk[z, z']$. We have proved that $N(U^+) \subseteq \Bbbk[z, z']$. The inverse inclusions $\Bbbk[z, z'] \subseteq Z(U^+) \subseteq N(U^+)$ are clear. □

Remark 3.2. For $\mathfrak{g}$ of type B_2, Autdiagr $(\mathfrak{g})$ is trivial and so the main Problem 1 must be here formulated as: do we have $\mathrm{Aut}_{\mathrm{Alg}} U^+ \simeq (\Bbbk^\times)^2$? The method used in [2] for the case A_2 was based on the facts that any automorphism of $\mathbb{H}$ preserves the center $Z(\mathbb{H})$ which is a polynomial algebra in one variable and the non-empty set of normal but non central elements of $\mathbb{H}$. It follows from Lemma 3.1 that the second argument fails for the case B_2, and that the first one is much more complicated to use in view of the structure of the automorphism group of a commutative polynomial algebra in two variables. A natural idea to determine the group $\mathrm{Aut}_{\mathrm{Alg}} U^+$ is to study the action of an automorphism on the prime spectrum of U^+. The structure of this spectrum remains widely unknown as far as we know (see further final remark of 3.3) and we only present some partial results in the following. In particular the two central generators z and z' do not play symmetric rôles (see Proposition 3.4) and we conjecture that any automorphism of U^+ stabilizes the prime ideal (z), which would be sufficient to solve the problem because of Proposition 3.3.

3.2 Automorphisms stabilizing the prime ideal (z)

3.2.1 The factor algebra $U^+/(z)$. It is clear that the ideal (z) generated in U^+ by the central element z is completely prime, and the factor domain $U^+/(z)$ is the quantum enveloping algebra $\mathbb{H} = U_q^+(\mathfrak{sl}_3)$ considered in 2.2.1. The canonical map $\pi : U^+ \to U^+/(z)$ induces an isomorphism between $U^+/(z)$ and $\mathbb{H}$ defined by $\pi(e_1) = E_1$ and $\pi(e_2) = E_2$, where E_1 and E_2 are the natural generators of $\mathbb{H}$ introduced in 2.2.1. Observe that $\pi(e_3) = E_3$ and $\pi(z') = (1 - q^{-2})\Omega$.

3.2.2 The group $\mathrm{Aut}_z(U^+)$. We introduce the subgroup $\mathrm{Aut}_z(U^+)$ of all automorphisms of the algebra U^+ which stabilize the ideal (z). In particular $\mathrm{Aut}_z(U^+)$ contains the subgroup $G := \{\psi_{\alpha,\beta}\,;\, \alpha, \beta \in \Bbbk^\times\} \simeq (\Bbbk^\times)^2$ where $\psi_{\alpha,\beta}$ is the automorphism defined for any $\alpha, \beta \in \Bbbk^\times$ by:

$$\psi_{\alpha,\beta}(e_1) = \alpha e_1 \quad \text{et} \quad \psi_{\alpha,\beta}(e_2) = \beta e_2.$$

It is clear that any $\theta \in \mathrm{Aut}_z U^+$ induces an automorphism $\widetilde{\theta} \in \mathrm{Aut}\,\mathbb{H}$ defined by $\widetilde{\theta}(E_1) = \pi(\theta(e_1))$ et $\widetilde{\theta}(E_2) = \pi(\theta(e_2))$. We denote by Φ the group morphism $\Phi : \mathrm{Aut}\, U^+ \to \mathrm{Aut}\,\mathbb{H}\,;\, \theta \mapsto \widetilde{\theta}$. The notation is coherent since $\Phi(G) = \widetilde{G}$ for the group $\widetilde{G} \simeq (\Bbbk^\times)^2$ introduced in 2.2.1. The next proposition proves that Φ defines an isomorphism between $\mathrm{Aut}_z U^+$ and $\widetilde{G}$.

Proposition 3.3. *The subgroup* $\mathrm{Aut}_z(U^+)$ *of algebra automorphisms of* U^+ *stabilizing the ideal* (z) *is isomorphic to* $(\Bbbk^\times)^2$.

Proof. Step 1: we prove that, for any $\theta \in \mathrm{Aut}_z U^+$, there exist $\lambda, \mu \in \Bbbk^\times$ and $p(z) \in \Bbbk[z]$ such that $\theta(z) = \lambda z$ and $\theta(z') = \mu z' + p(z)$.

Take $\theta \in \mathrm{Aut}_z U^+$. There exists $u \in U^+$, $u \neq 0$ such that $\theta(z) = uz$. But $\theta(z) \in Z(U^+)$ because $z \in Z(U^+)$, and then $u \in Z(U^+)$. Similarly $\theta^{-1}(z) = vz$ for some $v \in Z(U^+)$, $v \neq 0$. So $z = \theta(\theta^{-1}(z)) = \theta(v)uz$ in $Z(U^+) = \Bbbk[z, z']$ (see 3.1.6), which implies $u \in \Bbbk^\times$. Denoting $u = \lambda$, we conclude that $\theta(z) = \lambda z$ with $\lambda \in \Bbbk^\times$. The restriction of θ to $Z(U^+)$ is a $\Bbbk$-automorphism of $\Bbbk[z, z']$ such that $\theta(z) = \lambda z$. By surjectivity, the z'-degree of $\theta(z')$ is necessarily 1. Denote $\theta(z') = r(z)z' + p(z)$ with $r(z), p(z) \in \Bbbk[z]$, $r(z) \neq 0$. Using an analogue expression for $\theta^{-1}(z')$ in the equality $z' = \theta(\theta^{-1}(z'))$ we obtain that $r(z) = \mu \in \Bbbk^\times$.

Step 2: we prove that $\mathrm{Im}\,\Phi = \widetilde{G}$.

Let $\theta \in \mathrm{Aut}\, U^+$ and $\widetilde{\theta} = \Phi(\theta)$. From Theorem 2.4, there exist $\alpha, \beta \in \Bbbk^\times$ and $i \in \{0, 1\}$ such that $\widetilde{\theta} = \widetilde{\psi}_{\alpha,\beta}\omega^i$. Suppose that $i = 1$. Then $\omega = \Phi(\theta')$ for $\theta' = \psi_{\alpha^{-1},\beta^{-1}}\theta$, which satisfies $\pi(\theta'(e_1)) = \omega(E_1) = E_2$ and $\pi(\theta'(e_2)) = \omega(E_2) = E_1$. In other words, there exist $a, b \in U^+$ such that $\theta'(e_1) = e_2 + za$ and $\theta'(e_2) = e_1 + zb$. Applying θ' to the first Serre relation (S1) in U^+, we obtain:

$$(e_2 + za)^2(e_1 + bz) - (q^2 + q^{-2})(e_2 + za)(e_1 + bz)(e_2 + za) + (e_1 + bz)(e_2 + za)^2 = 0.$$

Using the grading of 3.1.3, this identity develops into an expression $s + t = 0$ with $s = e_2^2 e_1 - (q^2 + q^{-2})e_2 e_1 e_2 + e_1 e_2^2 \in U_3$ and the rest t in $\bigoplus_{n \geq 5} U_n$. Then $s = 0$. But the relations of 3.1.5 allow to compute: $s = e_2(e_2 e_1 - q^2 e_1 e_2) - q^{-2}(e_2 e_1 - q^2 e_1 e_2)e_2 = -q^2 e_2 \overline{e_3} + \overline{e_3} e_2 = -q^{-2} z \neq 0$. So a contradiction. We conclude that $i = 0$, and then $\widetilde{\theta} = \widetilde{\psi}_{\alpha,\beta} \in \widetilde{G}$.

Step 3: we prove that Φ is injective.

We fix an automorphism $\theta \in \mathrm{Aut}_z(U^+)$ such that $\theta \in \mathrm{Ker}\,\Phi$. By definition of $\mathrm{Ker}\,\Phi$, there exist $a, b \in U^+$ such that:

$$\theta(e_1) = e_1 + za \quad \text{and} \quad \theta(e_2) = e_2 + zb.$$

The elements e_3 and $\overline{e_3}$ being defined as q-brackets of e_1 and e_2, we deduce:

$$\begin{aligned} \theta(e_3) &= e_3 + zc, \quad \text{where } c = (e_1 b - q^2 b e_1) + (a e_2 - q^2 e_2 a) + z(ab - q^2 ba) \\ \theta(\overline{e_3}) &= \overline{e_3} + z\overline{c}, \quad \text{where } \overline{c} = (e_1 b - q^{-2} b e_1) + (a e_2 - q^{-2} e_2 a) + z(ab - q^{-2} ba) \end{aligned}$$

Applying θ to the relations $e_1 e_3 = q^{-2} e_3 e_1$ (see 3.1.2) and $\overline{e_3} = (1 - q^{-4})e_1 e_2 + q^{-4} e_3$ (see 3.1.5) we obtain by identification:

$$(e_1c - q^{-2}ce_1) + (ae_3 - q^{-2}e_3a) + z(ac - q^{-2}ca) = 0$$
$$(1 - q^{-4})ae_2 + (1 - q^{-4})e_1b + q^{-4}c + (1 - q^{-4})zab = \overline{c}$$

Consider now $z = e_2e_3 - q^2e_3e_2$. By step one, there exists some $\lambda \in \Bbbk^\times$ such that: $\lambda z = z + z(e_2c - q^2ce_2) + z(be_3 - q^2e_3b) + z^2(bc - q^2cb)$. Simplifying by z, we obtain $\lambda - 1 = (e_2c - q^2ce_2) + (be_3 - q^2e_3b) + z(bc - q^2cb)$. The right member lies in the ideal I defined in 3.1.3 and the left member is a scalar in $\Bbbk = U_0$. Therefore $\lambda = 1$. We conclude that:

$$\theta(z) = z \quad \text{and} \quad (e_2c - q^2ce_2) + (be_3 - q^2e_3b) + z(bc - q^2cb) = 0.$$

Similar calculations for the other central generator $z' = (1 - q^{-2})(e_3\overline{e_3} + (1 + q^{-2})ze_1)$ (see 3.1.5), using the equality $\theta(z') = \mu z' + p(z)$ from step one, give $\mu = 1$ and $p(z) = (1 - q^{-2})zs(z)$ for some $s(z) \in \Bbbk[z]$ which satisfies:

$$s(z) = (1 + q^{-2})za + zc\overline{c} + e_3\overline{c} + q^{-4}ce_3 + (1 - q^{-4})ce_1e_2.$$

From 3.1.2, we can consider the degree function deg at the indeterminate e_2 in the polynomial algebra $U^+ = S[e_2\,;\,\tau,\delta]$. Introduce in particular the three positive integers $d = \deg\theta(e_1)$, $d' = \deg\theta(e_2)$ and $d'' = \deg\theta(e_3)$. Comparing the degree of the two members of all the equalities obtained above, we obtain by very technical considerations whose details are left to the reader that we have necessarily $d = d'' = 0$ et $d' = 1$. In other words there exist $a, b_1, b_0, c \in S$ such that:

$$\theta(e_1) = e_1 + za, \quad \theta(e_2) = e_2 + z(b_1e_2 + b_0), \quad \theta(e_3) = e_3 + zc, \quad \theta(z) = z.$$

In particular the restriction θ_S of θ to S is an automorphism of S fixing z. Consider the field of fractions $K = \Bbbk(z)$, the algebra $T = K[e_3][e_1\,;\,\sigma] \supseteq S$ and the extension θ_T of θ_S to T. Since T is a quantum plane over $\Bbbk$ (with $e_1e_3 = q^{-2}e_3e_1$), we can apply proposition 1.4.4 of[1] and deduce that the K-automorphism θ_T satisfies $\theta(e_1) = fe_1$ and $\theta(e_3) = ge_3$ for some $f, g \in \Bbbk^\times$. Because $\theta(e_1)$ and $\theta(e_3)$ are in the subalgebra $S = \Bbbk[z][e_3][e_1\,;\,\sigma]$ of $T = \Bbbk(z)[e_3][e_1\,;\,\sigma]$, we have in fact f, g non-zero in $\Bbbk[z]$. By the same argument for the automorphism θ^{-1}, there exist f', g' non-zero in $\Bbbk[z]$ such that $\theta^{-1}(e_1) = f'e_1$ and $\theta^{-1}(e_3) = g'e_3$. By composition of θ and θ^{-1} it follows that $ff' = gg' = 1$, and then $f, g \in \Bbbk^\times$. Denoting $f = \alpha$ and $g = \gamma$, we obtain $\alpha e_1 = e_1 + za$ and $\gamma e_3 = e_3 + zc$. These equalities in S imply $(\alpha - 1)e_1 \in zS$ and $(\gamma - 1)e_3 \in zS$ with $\alpha, \gamma \in \Bbbk^\times$, and so $\alpha = \gamma = 1$ and $a = c = 0$. To sum up, we have $\theta(e_1) = e_1$, $\theta(e_3) = e_3$ and $\theta(z) = z$. It is then easy to check that we have also $\theta(e_2) = e_2$. We conclude that $\theta = \mathrm{id}_{U^+}$. □

3.3 Non permutability of the central generators z and z'

3.3.1 The prime ideal (z'). The ideal (z') generated in U^+ by the central element z' is completely prime. A direct proof consists in checking by computations in the iterated Ore extension U^+ and its localization $V = \Bbbk_{q^2}[e_3^{\pm 1}, e_1^{\pm 1}][z, z']$ that any element $v \in V$ satisfying $z'v \in U^+$ is necessarily an element of U^+; then the ideal $(z') = z'U^+$ is no more than the contraction $z'V \cap U^+$ of the completely prime ideal $z'V$ of V. The complete primeness of (z') can also be proved by the algorithmic method of [9], or can be deduced from the description of the $\mathcal{H}$-prime spectrum of U^+ (see further 3.4) following the method of [12].

Proposition 3.4. *There are no algebra automorphisms of U^+ sending (z) to (z') or (z') to (z).*

Proof. Recall that $\mathbb{H}$ denotes the factor algebra $U^+/(z)$ and π the canonical map $U^+ \to \mathbb{H}$ (see 2.2.1 and 3.2.1). Denote $\mathbb{H}' = U^+/(z')$ and π' the canonical map $U^+ \to \mathbb{H}'$. We have observed that the center of $\mathbb{H}$ is the polynomial algebra $Z(\mathbb{H}) = \Bbbk[\Omega]$; note that Ω equals $\pi(z')$ up to a multiplication by a non-zero scalar. By direct calculations in $\mathbb{H}'$ one can prove similarly that the center of $\mathbb{H}'$ is the polynomial algebra $Z(\mathbb{H}') = \Bbbk[\pi'(z)]$. Consider the standard $\mathbb{N}^2$-grading $U^+ = \bigoplus U_{m,n}$ putting e_1 on degree $(1,0)$ and e_2 on degree $(0,1)$. The central generator z is homogeneous of degree $(1,2)$ and the $\mathbb{N}^2$-grading induced on $\mathbb{H} = U^+/(z)$ is just the $\mathbb{N}^2$-grading $\mathbb{H} = \bigoplus \mathbb{H}_{n,m}$ considered in 2.2.1. The central generator z' is homogeneous of degree $(2,2)$ and the $\mathbb{N}^2$-grading induced on $\mathbb{H}' = U^+/(z')$ will be denoted by $\mathbb{H}' = \bigoplus \mathbb{H}'_{n,m}$.

Now suppose that there exists some algebra automorphism θ of U^+ such that $\theta(z') = (z)$. Then θ induces an isomorphism $\widehat{\theta} : \mathbb{H}' \to \mathbb{H}$ defined by $\widehat{\theta}\pi' = \pi\theta$. In particular $\mathbb{H}$ can be graded by the $\mathbb{N}^2$-grading $\mathbb{H} = \bigoplus T_{n,m}$ defined by $T_{n,m} = \widehat{\theta}(\mathbb{H}'_{m,n})$. Since $E_3, \overline{E_3} \in \mathbb{H}_{1,1}$ and $\Omega \in \mathbb{H}_{2,2}$, it follows from corollary 2.5 there exist two integers $r, s \geq 1$ such that $E_3, \overline{E_3} \in T_{r,s}$ and $\Omega \in T_{2r,2s}$. Set $t_1 = \widehat{\theta}^{-1}(E_3)$, $t_2 = \widehat{\theta}^{-1}(\overline{E_3})$, and $t = \widehat{\theta}^{-1}(\Omega) = \widehat{\theta}^{-1}(E_3\overline{E_3}) = t_1t_2$. By construction we have $t_1, t_2 \in \mathbb{H}'_{r,s}$ and $t \in \mathbb{H}'_{2r,2s}$. Because $Z(\mathbb{H}) = \Bbbk[\Omega]$ and $\widehat{\theta}$ is an isomorphism of algebras, t is a generator of the polynomial algebra $Z(\mathbb{H}') = \Bbbk[\pi'(z)]$. Then there exist $\lambda \in \Bbbk^\times, \mu \in \Bbbk$ such that $t = \lambda\pi'(z) + \mu$. Since $t \in \mathbb{H}'_{2r,2s}$, we have necessarily $\mu = 0$. We obtain in $\mathbb{H}'$ the equality $\pi'(z) = \lambda^{-1}t_1t_2$ where $\lambda \in \Bbbk^\times$ and $t_1, t_2 \in \mathbb{H}'_{r,s}$. Choosing $u_1, u_2 \in U_{r,s}$ such that $\pi'(u_1) = \lambda^{-1}t_1$ and $\pi'(u_2) = t_2$, we deduce in U^+ an equality $z = u_1u_2 + z'u$ for some $u \in U^+$. Back to the $\mathbb{N}$-grading $U^+ = \bigoplus U_n$ introduced in 3.1.3, we have clearly $u_1u_2 \in U_{2(r+s)}$ whereas $z \in U_3$ and $z' \in U_4$. It follows that $u = 0$, and then $z = u_1u_2$ gives the contradiction. □

Remark 3.5. Suppose that we can prove similarly that U^+/I is not isomorphic to $\mathbb{H} \simeq U^+/(z)$ for any height one prime ideal I of U^+, then we could deduce that any algebra automorphism of U^+ necessarily stabilizes (z) and then give by Proposition 3.3 a positive answer to Problem 1 (see 3.2). For example, if $I = (z-\alpha)$ with $\alpha \in \Bbbk^\times$, it is possible to separate U^+/I from $\mathbb{H}$ up to isomorphism (by technical considerations on the q-brackets in the both factor algebras which are not developed here). Unfortunately the complete description of height one prime ideals in U^+ is not known as far as we know. We restrict in the last subsection to the graded prime ideals.

3.4 Action on the $\mathcal{H}$-spectrum

3.4.1 $\mathcal{H}$-spectrum of $U_q^+(\mathfrak{g})$. In this subsection, $\mathfrak{g}$ is an arbitrary simple finite-dimensional Lie algebra. We consider all data and notations of 1.3.1. Denote by $\mathcal{H}$ the n-dimensional torus $(\Bbbk^\times)^n$. The canonical $\mathbb{Z}^n$-grading on $U_q^+(\mathfrak{g})$, given by $\deg E_i = \epsilon_i$, the i-th term of the canonical basis of $\mathbb{Z}^n$, induces a rational action of $\mathcal{H}$ on $U_q^+(\mathfrak{g})$. The $\mathcal{H}$-spectrum of $U_q^+(\mathfrak{g})$ is the set of graded prime ideals. We denote it by $\mathcal{H}$-Spec $U_q^+(\mathfrak{g})$. It determines a stratification of the whole prime spectrum Spec $U_q^+(\mathfrak{g})$. We refer to [6, Section II] for more details on stratifications of iterated Ore extensions, and summarize here some results obtained in [12] concerning the case of $U_q^+(\mathfrak{g})$.

More generally, [12] determines a stratification of Spec S^w for a family of algebras S^w (denoted also by R_0^w, see [12, p. 217, l. 7] for the equivalence between the two notations), where w is an element of the Weyl group W of $\mathfrak{g}$. The space of graded prime ideals of S^w is indexed by the set

$$W \diamondsuit^w W := \{(x, y) \in W \times W : x \leq w \leq y\},$$

where $\leq$ is the Bruhat order (see for example [14, App. 1]). The algebra $U_q^+(\mathfrak{g})$ is the particular case $w = e$, and thus the index set of $\mathcal{H}$-Spec $U_q^+(\mathfrak{g})$ is just W. The author introduces in [12] an ideal $Q(y)$ for any $y \in W$ (this is the ideal $Q(e, y)_e$ with the notations of [12, p. 231 and p. 236]), and proves (see [12, 7.1.2, (ii), p. 239] that

$$\mathcal{H} - \operatorname{Spec} U_q^+(\mathfrak{g}) = \{Q(y)\}_{y \in W}.$$

By [12, 6.11, p. 236], we have $Q(y) \subset Q(y')$ if and only if $y' \leq y$. Furthermore, [12, 5.3.3, p. 225] (with notation $Y(y) = Y_e(e, y)$) asserts that

$$\operatorname{Spec} U_q^+(\mathfrak{g}) = \coprod_{y \in W} Y(y), \tag{3.1}$$

where $Y(y)$ denotes the set of prime ideals containing $Q(y)$. By [12, 6.12, p. 237], each subset $Y(y)$ has a unique minimal element, namely $Q(y)$, and coincides with the $\mathcal{H}$-strata of Spec $U_q^+(\mathfrak{g})$ corresponding to the graded prime ideal $Q(y)$, that is:

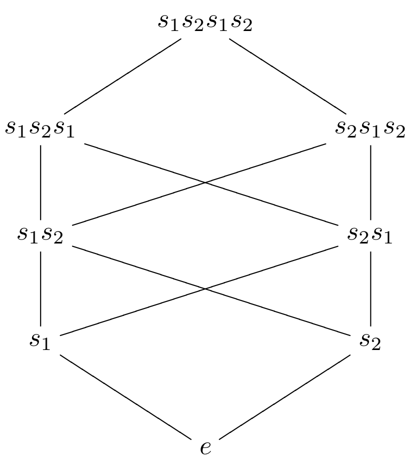
$$Y(y) = \operatorname{Spec}_{Q(y)} U_q^+(\mathfrak{g}) = \{P \in \operatorname{Spec} U_q^+(\mathfrak{g}) \,:\, Q(y) = \bigcap_{h \in \mathcal{H}} h.P\}.$$

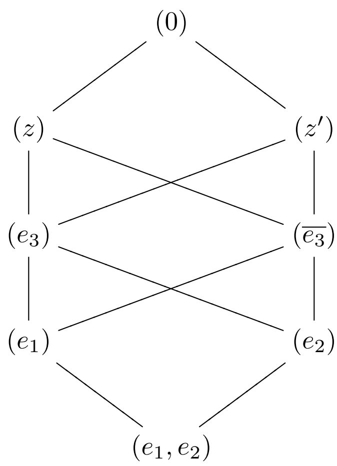
Applying [12, 6.13, p. 238], the closure of each $Y(y)$ is a union of other $Y(y')$'s, namely $\overline{Y(y)} = \coprod_{y' \in W, y' \leq y} Y(y')$, and the disjoint union (3.1) is then a stratification of Spec $U_q^+(\mathfrak{g})$.

3.4.2 $\mathcal{H}$-spectrum of U^+ for the type B_2. Suppose now that $\mathfrak{g}$ is of type B_2 and keep all notations of 3.1. The Weyl group W is of order 8. Its elements can be described by their action on the roots $\{\varepsilon_1, \varepsilon_2\}$ and as words on the two generators s_1 and s_2 as follows:

$$\begin{array}{rllrll} s_1: & \varepsilon_1 \mapsto \varepsilon_2, & \varepsilon_2 \mapsto \varepsilon_1, & s_2: & \varepsilon_1 \mapsto \varepsilon_1, & \varepsilon_2 \mapsto -\varepsilon_2, \\ s_1s_2: & \varepsilon_1 \mapsto \varepsilon_2, & \varepsilon_2 \mapsto -\varepsilon_1, & s_2s_1: & \varepsilon_1 \mapsto -\varepsilon_2, & \varepsilon_2 \mapsto \varepsilon_1, \\ s_1s_2s_1s_2 = s_2s_1s_2s_1: & \varepsilon_1 \mapsto -\varepsilon_1, & \varepsilon_2 \mapsto \varepsilon_2, & e: & \varepsilon_1 \mapsto \varepsilon_1, & \varepsilon_2 \mapsto \varepsilon_2. \end{array}$$

Using the results of [12] recalled in 3.4.1, the $\mathcal{H}$-prime spectrum of $U^+ = U_q^+(\mathfrak{g})$ has exactly 8 elements. The ideals (0), (e_1), (e_2) and (e_1, e_2) are clearly graded prime ideals of U^+. This is also the case for the ideals (e_3) and $(\overline{e_3})$, the factor algebras being in both cases domains isomorphic to a quantum plane. Finally the prime ideals (z) and (z') considered in 3.2 and 3.3 are also graded prime ideals of U^+. So the poset $(W, \leq)$ and the $\mathcal{H}$-spectrum of U^+ are:





Proposition 3.6. *The subgroup of algebra automorphisms of U^+ stabilizing the $\mathcal{H}$-spectrum of U^+ is isomorphic to $(\mathbb{k}^\times)^2$.*

Proof. Any automorphism of U^+ stabilizing the $\mathcal{H}$-spectrum of U^+ preserves the set of height one prime graded ideals of U^+, that is $\{(z), (z')\}$. Then the result follows from Propositions 3.3 and 3.4. $\square$

3.4.3 Automorphisms of some subalgebras of U^+ indexed by W. Take all data and notations of 1.3.1 and consider for any w in the Weyl group W of $\mathfrak{g}$ and for $e = \pm 1$ the subalgebras $U^+(w, e)$ of $U_q^+(\mathfrak{g})$ defined in [17]. For any reduced expression $w = s_{i_1}s_{i_2}\dots s_{i_n}$ of an element $w \in W$ the elements

$$E_{i_1}^{(c_1)} T'_{i_1,e}(E_{i_2}^{(c_2)}) \cdots T'_{i_1,e} T'_{i_2,e} \cdots T'_{i_{n-1},e}(E_{i_n}^{(c_n)})$$

for various $(c_1, c_2, \dots, c_n) \in \mathbf{N}^n$ form a basis of a subspace $U^+(w, e)$ of $U_q^+(\mathfrak{g})$ which does not depend of the reduced expression of w ([17] 40.2.1 p. 321). The

Lusztig automorphisms $T'_{i,e}$ appearing in this definition are the symmetries of $U_q^+(\mathfrak{g})$ related to the braid group action defined in [17] 37.1.2. From [17] 40.2.1 (d), we have: $\ell(s_iw) = \ell(w) - 1 \Rightarrow E_iU^+(w,e) \subset U^+(w,e)$. In particular, for w_0 the element of maximal length in W, we have $U^+(w_0,e) = U_q^+(\mathfrak{g})$ (see the remark after 40.2.2 of [17]). The subspaces $U^+(w,e)$ can be identified with the subalgebras $U_q(n_w)$ of lemma 1.5 of [8] and also appear in [14] p. 123.

Suppose now that $\mathfrak{g}$ is of type B_2, take all notations of 3.1 and denote by A_w the subalgebra $U^+(w,1)$ of U^+, for any $w \in W$ expressed as a word into the generators s_1 and s_2 of 3.4.2. Using the above definition of the basis of A_w and the properties of the automorphisms $T'_{j,1}$ (in particular [17] 39.2.3), straightforward calculations allow to obtain the following description by generators of the eight subalgebras A_w of U^+:

$$\begin{tikzcd}
 & U^+ = A_{s_1s_2s_1s_2} = A_{s_2s_1s_2s_1} \arrow[dl, dash] \arrow[dr, dash] & \\
\Bbbk\langle e_2, w, e_3\rangle = A_{s_1s_2s_1} \arrow[d, dash] & & A_{s_2s_1s_2} = \Bbbk\langle e_1, \overline{e_3}, \overline{w}\rangle \arrow[d, dash] \\
\Bbbk\langle e_2, w\rangle = A_{s_1s_2} \arrow[d, dash] & & A_{s_2s_1} = \Bbbk\langle e_1, \overline{e_3}\rangle \arrow[d, dash] \\
\Bbbk\langle e_2\rangle = A_{s_1} \arrow[dr, dash] & & A_{s_2} = \Bbbk\langle e_1\rangle \arrow[dl, dash] \\
 & A_e = \Bbbk &
\end{tikzcd}$$

with the commutation relations:

$$\text{left side} \quad \begin{cases} e_2w = q^2we_2, & \\ e_3w = q^{-2}we_3, & e_3e_2 = e_2e_3 - w, \\ e_1w = we_1 + (1-q^{-2})e_3^2, & e_1e_2 = q^2e_2e_1 + e_3, \; e_1e_3 = q^{-2}e_3e_1. \end{cases}$$

$$\text{right side} \quad \begin{cases} e_1\overline{e_3} = q^2\overline{e_3}e_1, & & \\ \overline{w}\,\overline{e_3} = q^{-2}\overline{e_3}\,\overline{w}, & \overline{w}e_1 = e_1\overline{w} + (q^2-1)\overline{e_3}^2, & \\ e_2\overline{e_3} = \overline{e_3}e_2 + \overline{w}, & e_2e_1 = q^2e_1e_2 - q^2\overline{e_3}, & e_2\overline{w} = q^{-2}\overline{w}e_2 \end{cases}$$

The eight algebras are iterated Ore extensions. At level one, A_{s_1} and A_{s_2} are just commutative polynomial algebras in one variable. At level two, $A_{s_1s_2}$ and $A_{s_2s_1}$ are isomorphic to a same quantum plane (with parameter q^2) and so by [1] their group of algebra automorphisms is isomorphic to $(\Bbbk^\times)^2$. The third level introduces some asymmetry in the diagram. One can prove by direct calculations (which are left to the reader) similar to the proof of [2, Lemme 2.2 and Proposition 2.3] that:

(i) the center of the algebra $A_{s_1s_2s_1}$ is $Z = \Bbbk[z]$ with $z = (1-q^2)e_3e_2 + w$;

its set of normal elements is $N = \bigcup_{n\geq 0} \Bbbk[z]w^n$;

its automorphism group is $(\Bbbk^\times)^2$ acting by $(\alpha,\beta) : e_2 \mapsto \alpha e_2,\ e_3 \mapsto \beta e_3,\ w \mapsto \alpha\beta w$.

(ii) the center of the algebra $A_{s_2s_1s_2}$ is $Z' = \Bbbk[u]$ with $u = (1-q^{-4})e_1\overline{w} + (q^2 - 1)\overline{e_3}^2$;

its set of normal elements is $N' = \bigcup_{n\geq} \Bbbk[u]\overline{e_3}^n$;

its automorphism group is $(\Bbbk^\times)^2$ acting by $(\alpha,\gamma) : e_1 \mapsto \alpha e_1,\ \overline{e_3} \mapsto \gamma\overline{e_3},\ \overline{w} \mapsto \alpha^{-1}\gamma^2\overline{w}$.

Using the fact that an isomorphism from $A_{s_1s_2s_1}$to $A_{s_2s_1s_2}$ must map N to N' and Z to Z', one can prove by direct computations and identifications using the basis of monomials into the natural generators, that the algebras $A_{s_1s_2s_1}$ and $A_{s_2s_1s_2}$ are not isomorphic.

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