An Introduction to Nichols Algebras

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Abstract Nichols algebras, Hopf algebras in braided categories with distinguished properties, were discovered several times. They appeared for the first time in the thesis of W. Nichols [72], aimed to construct new examples of Hopf algebras. In this same paper, the small quantum group $u_q(sl_3)$, with q a primitive cubic root of one, was introduced. Independently they arose in the paper [84] by Woronowicz as the invariant part of his non-commutative differential calculus. Later there were two unrelated attempts to characterize abstractly the positive part $U_q^+(\mathfrak{g})$ of the quantized enveloping algebra of a simple finite-dimensional Lie algebra $\mathfrak{g}$ at a generic parameter q. First, Lusztig showed in [64] that $U_q^+(\mathfrak{g})$ can be defined through the radical of a suitable invariant bilinear form. Second, Rosso interpreted $U_q^+(\mathfrak{g})$ in [74, 75] via quantum shuffles. This two viewpoints were conciliated later, as alternative definitions of the same notion of Nichols algebra. Other early appearances of Nichols algebras are in [65, 77]. As observed in [17, 18], Nichols algebras are basic invariants of pointed Hopf algebras, their study being crucial in the classification program of Hopf algebras; see also [10]. More recently, they are the subject of an intriguing proposal in Conformal Field Theory [79].
This is an introduction from scratch to the notion of Nichols algebra. I was invited to give a mini-course of two lessons, 90 minutes each, at the Geometric, Algebraic and Topological Methods for Quantum Field Theory, Villa de Leyva, Colombia, in July 2015. The theme was Nichols algebras, that requires several preliminaries and some experience to be appreciated; a selection of the ideas to be presented was necessary. These notes intend to preserve the spirit of the course, discussing some motivational background material in Section 1, then dealing with braided vector spaces and braided tensor categories in Section 2, arriving at last to the definition and main calculation tools of Nichols algebras in Section 3. I hope that the various Examples and Exercises scattered through the text would serve the reader to absorb the beautiful concept of Nichols algebra and its many facets. Section 4 is a survey of the main examples of, and results on, Nichols algebras that I am aware of; here

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the pace is faster and the precise formulation of some statements is referred to the literature. I apologize in advance for any possible omission. This Section has intersection with, and is an update of, the surveys [1, 19, 2], to which I refer for futher information.

1 Preliminaries

1.1 Conventions

We assume the conventions $\mathbb{N} = \{1,2,3,\dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_{\geq 2} = \mathbb{N} - \{1\}$, etc. If $k < \theta \in \mathbb{N}_0$, then we denote $\mathbb{I}_{k,\theta} = \{n \in \mathbb{N}_0 : k \leq n \leq \theta\}$ and $\mathbb{I}_\theta = \mathbb{I}_{1,\theta}$.

If $N \in \mathbb{N}$, then $\mathbb{G}_N$ denotes the group of N-roots of unity in $\Bbbk$, while $\mathbb{G}'_N$ is the subset of primitive roots of order N. Also $\mathbb{G}_\infty = \bigcup_{N\in\mathbb{N}} \mathbb{G}_N$, $\mathbb{G}'_\infty = \mathbb{G}_\infty - \{1\}$.

If V is a vector space, then $V^* := \hom_{\Bbbk}(V, \Bbbk)$ and $\langle\, , \rangle : V^* \times V \to \Bbbk$ is the evaluation.

The finite field with q elements is denoted $\mathbb{F}_q$.

We abbreviate $W \leq V$ for W *is a subobject of* V, where subobject means submodule, subgroup, subspace, subrack, according to the context.

1.2 Groups

We fix a field $\Bbbk$; later we shall assume that $\Bbbk$ is algebraically closed and has characteristic 0. We expect that the reader is familiar with the notions of group, module and representation; we use indistinctly the languages of modules and representations. As customary, we denote by $GL(V)$ the group of bijective linear transformations of a vector space V onto itself. We remind some basic definitions:

- A module is *simple* if it has exactly two submodules, 0 and itself (thus, it is different from 0). In the representation-theoretic language, one says *irreducible* instead of simple.
- A module is *semisimple* if it is a direct sum of simple submodules. In the representation theory, *completely reducible* is the translation of semisimple.

Let G be a group. We denote by $\Bbbk G$ the group algebra of G, with the canonical basis $(e_g)_{g\in G}$. Thus, there is a bijective correspondence between representations of G and of $\Bbbk G$. We observe that $\Bbbk G$ can be identified with (a subspace of) the linear dual of the vector space of functions from G to $\Bbbk$, where $e_g(f) = f(g)$, for $f : G \to \Bbbk$ and $g \in G$.

We denote by $\operatorname{Irr} G$ the set of classes of simple G-modules, up to isomorphism. For instance, $\varepsilon \in \operatorname{Irr} G$ is the class of the trivial representation, the 1-dimensional vector space where every $g \in G$ acts by 1. If $\xi \in \operatorname{Irr} G$ and V is a G-module, then

$$V_\xi := \sum_{W \leq V : W \in \xi} W$$

is the isotypical component of V of type ξ. Particularly, $V^G := V_\varepsilon$, the isotypical component of trivial type, is the submodule of G-invariants of V.

Example 1. Let U and V be G-modules. Then $\operatorname{Hom}(V,W)$ is a G-module with the action $g \cdot T = gTg^{-1}$ and $\operatorname{Hom}_G(V,W) = \operatorname{Hom}(V,W)^G$.

Theorem 1. *(Maschke). Let G be a finite group. Then the following are equivalent:*

(1) The characteristic of $\Bbbk$ does not divide $|G|$.
(2) Every finite-dimensional representation of G is completely reducible.

Assume that (1) holds. Let V be a finite-dimensional G-module. The action of

$$\int_G = \frac{1}{|G|} \sum_{g \in G} e_g \in \Bbbk G \tag{1}$$

on V is a G-morphism and a projector $V \to V^G$. (If $\Bbbk = \mathbb{C}$, then $\int_G$ is a normalized Haar measure on the discrete group G). To prove (2), it is enough to show, arguing recursively, that any $W \leq V$ admits a complement U that is also a G-submodule. So, consider $p \in \operatorname{Hom}(V,U)$ a projector onto U; then $q := \int_G \cdot p \in \operatorname{Hom}_G(V,U)$ is a projector onto U and $\ker q$ is the desired complement.

To prove (1), it is enough to assume that the representation of G on $\Bbbk G$ by left multiplication is completely reducible. Then the kernel of the projection $p : \Bbbk G \to G$, $e_g \mapsto 1$ for all $g \in G$, admits a complement U that is also a G-submodule. It turns out that U has to be the span of $x = \sum_{g \in G} e_g$; since $p(x) = |G|$, this could not be 0.

Remark 1. There is a natural notion of integral in finite-dimensional Hopf algebras that permits a generalization of the classical Maschke Theorem. This can be extended further to Hopf algebras with arbitrary dimension, but the complete reducibility in question is of comodules. See e.g. [78] for details.

Let X be a set. We denote by $\mathbb{S}_X$ the group of bijections from X onto itself, with multiplication being the composition. In particular, $\mathbb{S}_n$ is the symmetric group on n letters, i.e. $\mathbb{S}_n = \mathbb{S}_{\mathbb{I}_n}$, where $\mathbb{I}_n := \{1, \dots, n\}$. Let τ_i be the the transposition $(i\, i+1)$. Then $\mathbb{S}_n$ is generated by the τ_i, with $i \in \mathbb{I}_{n-1}$, subject to the defining relations

$$\tau_i^2 = e, \qquad i \in \mathbb{I}_{n-1}, \tag{2}$$
$$\tau_i \tau_j = \tau_j \tau_i, \qquad |i-j| \geq 2, \tag{3}$$
$$\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j, \qquad |i-j| = 1. \tag{4}$$

The group $\mathbb{S}_n$ together with $S = \{\tau_i,\, i \in \mathbb{I}_{n-1}\}$ is a Coxeter group. In particular, there is a length function $\ell : \mathbb{S}_n \to \mathbb{N}_0$, measuring the minimum of the possible expressions of an element as product of τ_i's. Thus we have the sign representation $\operatorname{sgn} : \mathbb{S}_n \to \Bbbk^\times$, $w \mapsto (-1)^{\ell(w)}$, $w \in \mathbb{S}_n$.

E. Artin introduced in 1926 the braid group $\mathbb{B}_n$, that has important applications in various areas and plays a central role in our story. Concretely, $\mathbb{B}_n$ is the group generated by σ_i, $i \in \mathbb{I}_{n-1}$, with defining relations (3) and (4) (with σ instead of τ). By definition, there is a surjective group homomorphism $\pi : \mathbb{B}_n \to \mathbb{S}_n$, $\sigma_i \mapsto \tau_i$; it admits a set-theoretical section $M : \mathbb{S}_n \to \mathbb{B}_n$ (i. e., not a group homomorphism), sometimes called the Matsumoto section, determined by

$$\begin{aligned} M(\tau_i) &= \sigma_i, & i &\in \mathbb{I}_{n-1}, \\ M(uw) &= M(u)M(w), & \text{if } \ell(uw) &= \ell(u) + \ell(w). \end{aligned} \tag{5}$$

1.3 The tensor algebra

We denote by $\tau : V \otimes W \to W \otimes V$ the usual flip $v \otimes w \mapsto w \otimes v$ between the tensor products of vector spaces V and W.

We expect that the reader is familiar with the notions of associative, commutative and Lie algebra. The ideal, respectively the subalgebra, of an algebra A generated by a subset S is denoted by $\langle S \rangle$, respectively by $\Bbbk\langle S \rangle$. A graded vector space is a vector space with a fixed grading $V = \oplus_{n\in\mathbb{N}_0} V^n$; it is locally finite if $\dim V^n < \infty$ for all $n \in \mathbb{N}_0$. In such case, its Hilbert-Poincaré series is

$$\mathscr{H}_V = \sum_{n\in\mathbb{N}_0} \dim V^n t^n \in \mathbb{Z}[[t]].$$

The *graded dual* of a locally finite graded vector space $V = \oplus_{n\in\mathbb{N}_0} V^n$ is

$$V^\star = \oplus_{n\in\mathbb{N}_0} V^{\star n}, \qquad \mathscr{V}^{\star n} = \hom_\Bbbk(V^n, \Bbbk). \tag{6}$$

A graded algebra is a graded vector space $A = \oplus_{n\in\mathbb{N}_0} A^n$ with an algebra structure such that $A^n A^m \subset A^{n+m}$.

We also assume that the reader knows the basics of the theory of categories. Let $\mathrm{Vec}_\Bbbk$, $\mathrm{Assoc}_\Bbbk$, $\mathrm{Comm}_\Bbbk$, $\mathrm{Lie}_\Bbbk$, be the categories of vector spaces, associative algebras, associative and commutative algebras, Lie algebras, over $\Bbbk$, respectively.

Let V be a vector space. As customary, we set $T^0(V) = \Bbbk$, $T^{n+1}(V) = V \otimes T^n(V)$, $n \geq 0$, and $T(V) = \oplus_{n\geq 0} T^n(V)$. We abridge

$$v_1 v_2 \dots v_n := v_1 \otimes v_2 \otimes \cdots \otimes v_n, \qquad v_1, v_2, \dots, v_n \in V.$$

The natural identifications

$$\mu_{m,n} : T^m(V) \otimes T^n(V) \simeq T^{m+n}(V)$$

patch together to an associative product $\mu : T(V) \otimes T(V) \to T(V)$, giving rise to the tensor algebra $T(V)$. This is also the free algebra on V, meaning that it satisfies the universal property:

(i) There is a linear map $\iota : V \to T(V)$, the inclusion $V = T^1(V) \hookrightarrow T(V)$.

(ii) Every linear map $\varphi : V \to A$, where A is an associative algebra, extends to a morphism of algebras $\Phi : T(V) \to A$ such that $\Phi \circ \iota = \varphi$.

In categorical terms, this means that we have a functor $T : \mathrm{Vec}_{\Bbbk} \to \mathrm{Assoc}_{\Bbbk}$ that is left adjoint to the forgetful functor $\mathrm{Assoc}_{\Bbbk} \to \mathrm{Vec}_{\Bbbk}$.

Among the plentiful applications of the tensor algebra, let us single out the construction of the enveloping algebra of a Lie algebra $\mathfrak{g}$, as the quotient

$$U(\mathfrak{g}) := T(\mathfrak{g})/\langle xy - yx - [x,y] : x,y \in \mathfrak{g}\rangle.$$

Again, this is a functor $U : \mathrm{Lie}_{\Bbbk} \to \mathrm{Assoc}_{\Bbbk}$ left adjoint to the forgetful functor $\mathrm{Assoc}_{\Bbbk} \to \mathrm{Lie}_{\Bbbk}$; indeed, every associative algebra becames a Lie algebra with the commutator $[a,b] = ab - ba$.

Remark 2. Let V be a vector space. By the universal property, the linear map $\delta : V \to T(V) \otimes T(V)$, $\delta(v) = v \otimes 1 + 1 \otimes v$, $v \in V$, extends to $\Delta : T(V) \to T(V) \otimes T(V)$; then $T(V)$ becomes a Hopf algebra. It is cocommutative, i. e. $\Delta = \tau\Delta$.

Remark 3. Let $\mathfrak{g}$ be a Lie algebra. The linear map $\delta : \mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\delta(v) = v \otimes 1 + 1 \otimes v$, $v \in \mathfrak{g}$, extends to $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, so that $U(\mathfrak{g})$ is a cocommutative Hopf algebra.

Exercise 1. Let V be a vector space and

$$L(V) := \mathrm{Prim}\, T(V) = \{x \in T(V) : \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

1. $\mathrm{Prim}\, T(V)$ is a Lie subalgebra of $T(V)$ (this is valid for every Hopf algebra).
2. $T(V) \simeq U(L(V))$.
3. $L(V)$ is the free Lie algebra on V. This provides the left adjoint to the forgetful functor $\mathrm{Lie}_{\Bbbk} \to \mathrm{Vec}_{\Bbbk}$.

1.4 The symmetric algebra

Let V be a vector space. The symmetric algebra $S(V)$ is the free *commutative* algebra on V, meaning that it satisfies the analogous universal property as above but with respect to linear maps from V to commutative algebras. Categorically, it gives a functor $S : \mathrm{Vec}_{\Bbbk} \to \mathrm{Comm}_{\Bbbk}$ left adjoint to the forgetful functor $\mathrm{Comm}_{\Bbbk} \to \mathrm{Vec}_{\Bbbk}$. Concretely,

$$S(V) := T(V)/\langle xy - yx : x,y \in V\rangle = \oplus_{n\geq 0} S^n(V),$$

so that $S(V)$ is the enveloping algebra of V with the zero bracket. In passing, we mention also the exterior algebra

$$\Lambda(V) := T(V)/\langle xy + yx : x,y \in V\rangle = \oplus_{n\geq 0} \Lambda^n(V).$$

A *quadratic* algebra is one of the form $T(W)/\langle J\rangle$, where W is a vector space and $J \leq T^2(W)$ (recall our convention in §1.1, this means that J is a suspace of $T^2(W)$).

Both $S(V)$ and $\Lambda(V)$ are quadratic algebras, of the form $T(V)/\langle J_{\mp}\rangle$ respectively, where $J_{\pm} = \{xy \pm yx : x, y \in V\}$.

The symmetric group $\mathbb{S}_n$ acts on $T^n(V)$ by $w \cdot v_1 \dots v_n = v_{w(1)} \dots v_{w(n)}$, for $n \in \mathbb{N}_0$ (where $\mathbb{S}_0 = \mathbb{S}_1$ are trivial). In particular, the isotypic components of $T^2(V)$ with respect to the action of $\mathbb{S}_2 \simeq \mathbb{Z}/2$ are J_+ for the trivial, respectively J_- for the sign, representation. It turns out that the n-th homogeneous components of the ideals $\langle J_{\pm}\rangle$ are $\mathbb{S}_n$-submodules of $T^n(V)$. Hence, $S^n(V)$ and $\Lambda^n(V)$ are $\mathbb{S}_n$-modules, and it is not difficult to see that the former is a trivial module.

Assume now that char $\Bbbk = 0$. Then the various $\mathbb{S}_n$-modules $T^n(V)$ are all completely reducible and we may consider $\widetilde{S}(V) = \oplus_{n\geq 0} T^n(V)^{\mathbb{S}_n}$.

Proposition 1. *The natural projection $T(V) \to S(V)$ induces a linear isomorphism $\widetilde{S}(V) \simeq S(V)$. Consequently, cf.* (1),

$$\ker(T(V) \to S(V)) = \langle J_-\rangle = \oplus_{n\geq 2} \ker \int_{\mathbb{S}_n}. \tag{7}$$

Similarly, the polynomial algebra $\Bbbk[X_1, \dots, X_d]$ is the free commutative algebra on the set $\mathbb{I}_d$. Thus, if $\dim V = d$, then every choice of a basis in V induces an isomorphism of algebras $S(V) \simeq \Bbbk[X_1, \dots, X_d]$.

1.5 Coalgebras and Hopf algebras

We expect that the reader has acquaintance with the notions of coalgebra, bialgebra and Hopf algebra. There are several books and monographs to be initiated on these topics; some of them are [25, 66, 71, 73, 78, 80]. The reader willing to learn these matters is advised to acquire first some experience with groups and Lie algebras.

As usual, the comultiplication of a coalgebra C is denoted by Δ, for which the Sweedler notation is $\Delta(c) = c_{(1)} \otimes c_{(2)}$, and the counit by ε. If D, E are subspaces of the coalgebra C, then

$$D \wedge E := \{c \in C : \Delta(c) \in D \otimes C + C \otimes E\}.$$

Coalgebras and comodules have a distinguished feature: they are locally finite, i.e. they are union of their finite-dimensional subcoalgebras, respectively subcomodules. A coalgebra without proper subcoalgebras (remember that 0 is not a coalgebra) is called *simple*; thus a simple coalgebra is finite-dimensional. If $\Bbbk$ is algebraically closed, then every simple coalgebra is the dual of a matrix algebra.

The *coradical* of a coalgebra C is the sum of all its simple subcoalgebras, denoted C_0; it is analogous to the socle of a module (in fact it is the socle of a coalgebra as a comodule over itself). By a standard argument, the coradical is a direct sum of simple coalgebras. A coalgebra is cosemisimple if it coincides with its coradical, i.e. if it is a (direct) sum of simple subcoalgebras. A one-dimensional coalgebra is of course simple; a coalgebra is *pointed* if its coradical is a (direct) sum of one-dimensional coalgebras. Basic examples are:

- The group algebra $H = \Bbbk G$ of a group G, with $\Delta(g) = g \otimes g$, $g \in G$. Here $H_0 = H$.
- The enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$. Here $H_0 = \Bbbk$.

The study of pointed Hopf algebras started in the 70's by Taft, Wilson, Radford, Nichols and others, being those with the simplest possible coradical. Some examples beyond group algebras and enveloping algebras were discovered. In the early 80's, Reshetikhin, Kulish and Sklyanin introduced the Hopf algebra nowadays known as $U_q(sl_2)$ and soon after that, Drinfeld and Jimbo defined the quantized enveloping algebras $U_q(\mathfrak{g})$ for every finite-dimensional simple Lie algebra $\mathfrak{g}$; these are pointed Hopf algebras. Finite-dimensional pointed Hopf algebras related to $U_q(\mathfrak{g})$ appeared in the work of Lusztig [62, 63, 64]. The ICM report [28] made a deep impact in the area of Hopf algebras–and in many others. After some time the classification program of finite-dimensional pointed Hopf algebras was launched [17, 18], see the survey [19], and the classification under some hypothesis in [20]. For more references and details see [2].

The notions of filtration and grading are ubiquitous in algebra. For instance, it is useful for many purposes to filter an algebra by powers of an ideal. A coalgebra filtration of a coalgebra C is a family of subspaces $(D_n)_{n\in\mathbb{N}_0}$ such that

$$D_n \subseteq D_{n+1}, \qquad C = \bigcup_{N\in\mathbb{N}_0} D_n, \qquad \Delta(D_n) \subseteq \sum_{0\le i\le n} D_i \otimes D_{n-i}.$$

Here the first condition says that the filtration is ascending and the second that it is exhaustive. The coradical filtration is defined recursively by

$$C_0 = \text{the coradical}, \qquad C_{n+1} = C_n \wedge C_0.$$

Exercise 2. 1. Let $\mathscr{G} = \oplus_{n\in\mathbb{N}_0}\mathscr{G}^n$ be a graded coalgebra, i.e.

$$\Delta(\mathscr{G}^n) \subseteq \oplus_{0\le i\le n}\mathscr{G}^i \otimes \mathscr{G}^{n-i}.$$

Let $\mathfrak{D}_n := \oplus_{0\le i\le n}\mathscr{G}^i$. Prove that $(\mathfrak{D}_n)_{n\in\mathbb{N}_0}$ is a coalgebra filtration. We say that $\mathscr{G} = \oplus_{n\in\mathbb{N}_0}\mathscr{G}^n$ is *coradically graded* if $\mathfrak{D}_n = \mathscr{G}_n$ (in words, the coradical filtration coincides with the filtration associated to the grading).

Let A be a finite-dimensional algebra and $C = A^*$ the dual coalgebra (with the transpose of the multiplication and the unit). If $I \subseteq A$, then we set

$$I^\perp := \{c \in C : \langle c, x\rangle = 0 \text{ for all } x \in I\}.$$

2. $I \subseteq A$ is a two-sided ideal if and only if $I^\perp \subseteq C$ is a subcoalgebra.
3. If $I, J \subseteq A$, then $(IJ)^\perp = I^\perp \wedge J^\perp$.

4. Let $(I_n)_{n\in\mathbb{N}}$ be a family of subspaces of A and $D_n := I_{n+1}^{\perp}$. Then $(I_n)_{n\in\mathbb{N}}$ is a descending algebra filtration if and only if $(D_n)_{n\in\mathbb{N}_0}$ is a coalgebra filtration. Prove that $\operatorname{gr} C = \oplus_{n\in\mathbb{N}_0} D_n/D_{n-1}$ is a graded coalgebra (where $D_{-1} = 0$).
5. Let J be the Jacobson radical of A. Then $C_0 = J^{\perp}$. Conclude that $C_{n+1} = (J^n)^{\perp}$ and that the coradical filtration is a coalgebra filtration. Show that $\operatorname{gr} C$, with respect to the coradical filtration, is coradically graded.
6. Let H be a Hopf algebra with bijective antipode $\mathscr{S}$. Assume that the coradical H_0 is a subalgebra. Prove that H_0 is a Hopf subalgebra and that the coradical filtration is an ascending filtration of algebras, each term being stable under the antipode. Conclude that $\operatorname{gr} H$ is a graded Hopf algebra.
7. If C is coalgebra, then $G(C) = \{x \in C - 0 : \Delta(x) = x\otimes x\}$ is linearly independent. If H is a Hopf algebra, then $G(H)$ is a group with the multiplication of H and inverse $x^{-1} = \mathscr{S}(x)$, $x \in G(H)$.
8. The coradical of a pointed Hopf algebra H is a Hopf subalgebra: $H_0 \simeq \Bbbk G(H)$.

1.6 The tensor coalgebra

Let V be a vector space. We shall need later the tensor coalgebra $T^c(V)$; this is the vector space $T(V)$ with the comultiplication Δ given by

$$\Delta(v_1 v_2 \dots v_n) := \sum_{j\in\mathbb{I}_n} v_1 \dots v_j \otimes v_{j+1} \dots v_n, \qquad v_1,\dots,v_n \in V. \tag{8}$$

Clearly $\Delta(v) = v\otimes 1 + 1\otimes v$ for $v \in V$, but $\Delta(v_1v_2) = v_1v_2\otimes 1 + v_1\otimes v_2 + 1\otimes v_1v_2 \neq \Delta(v_1)\Delta(v_2)$, thus $\Delta \neq \Delta$ from Remark 2.

Remark 4. The coalgebra $T^c(V)$ is dual to the tensor algebra $T(V^*)$, but it is not the cofree coalgebra on V (cofree means universal with respect to maps $C \to V$, C a coalgebra). The construction of the cofree coalgebra is more delicate [80].

1.7 Gelfand-Kirillov dimension

The notion of dimension pervades all mathematics. In the dictionary affine algebraic geometry–commutative algebra, the Krull dimension is the translation of the topological dimension. A guiding principle in non-commutative algebra is to adapt ideas and tools from geometry; in this sense, there are different attempts to generalize the Krull dimension. Perhaps the best adapted is the Gelfand-Kirillov dimension, GK-dim for short; a comprehensive account is [59].

Let A be a finitely generated $\Bbbk$-algebra. Let V be a finite-dimensional subspace of A such that $A = \Bbbk\langle V\rangle$. Set

$$V^j = \underbrace{V \cdot V \cdots V}_{j \text{ times}}, \qquad A_n = \sum_{0 \le j \le n} V^j$$

The Gelfand-Kirillov dimension is defined as

$$\operatorname{GK-dim} A := \overline{\lim}_{n\to\infty} \log_n \dim A_n. \tag{9}$$

It can be shown that GK-dim$A$ does not depend on the choice of V [59, 1.1]. When A is not finitely generated, the definition is extended as follows:

$$\operatorname{GK-dim} A := \sup\{\operatorname{GK-dim} B | B \text{ finitely generated subalgebra of } A\}. \tag{10}$$

Example 2. Let V be a vector space of dimension $1 < d \in \mathbb{N}$ and $A = T(V)$. Then

$$\dim A_n = \sum_{0\le j\le n} \dim T^j(V) = \sum_{0\le j\le n} d^j = \frac{d^{n+1}-1}{d-1} \implies \log_n \dim A_n \sim \frac{n}{\log n},$$

hence GK-dim $T(V) = \infty$.

Exercise 3. Let A be a finitely generated $\Bbbk$-algebra and V a finite-dimensional subspace such that $A = \Bbbk\langle V\rangle$. Show that

$$\begin{aligned} \operatorname{GK-dim} A &= \inf\{r \in \mathbb{R} : \dim V^j \le c j^r \text{ for some } c \in \mathbb{R}, \forall j \in \mathbb{N}\} \\ &= \inf\{r \in \mathbb{R} : \dim V^j \le j^r \text{ for large } j\}. \end{aligned}$$

Exercise 4. Let V be a vector space of dimension $d \in \mathbb{N}$ and $A = S(V)$. Let $\Bbbk[X_1,\dots,X_d]^j \simeq S^j(V)$ be the subspace of homogeneous polynomials of degree j.

1. Prove that $\dim S^j(V) = \binom{d+j-1}{j}$ (for instance, argue recursively and use that $S^j(V) \simeq \Bbbk[X_1,\dots,X_{d-1}]^{j-1} \cdot X_d \oplus \Bbbk[X_1,\dots,X_{d-1}]^j$).
2. Prove that $\dim A_n = \sum_{0\le j\le n} \dim S^j(V) = \binom{d+n}{n}$ (e.g., use the linear isomorphism $\Bbbk[X_1,\dots,X_{d+1}]^n \to \oplus_{0\le j\le n} \Bbbk[X_1,\dots,X_d]^j$, $f(X_1,\dots,X_{d+1}) \mapsto f(X_1,\dots,X_d,1)$).
3. Since $\binom{d+n}{n}$ is a polynomial of degree d in n, conclude that GK-dim$S(V) = d$.
4. If $\dim V = \infty$, then GK-dim$S(V) = \infty$.

Exercise 5. Let A be a finitely generated algebra. Then GK-dim$A = 0$ if and only if A is finite-dimensional.

If A is arbitrary, then GK-dim$A = 0$ if and only if every finitely generated subalgebra is finite-dimensional. For example, if $\dim V = \infty$, then GK-dim$\Lambda(V) = 0$.

Example 3. If A is a finitely generated commutative algebra, then

$$\operatorname{GK-dim} A = \operatorname{Krull} \dim A = \dim \operatorname{Spec} A.$$

Here Spec$A$ is the Zariski spectrum of A; it could be replaced by its subset of closed points, that is the affine variety defined by A. In other words, the Gelfand-Kirillov dimension coincides with the usual dimension in the commutative case. Therefore, if A is a commutative algebra, then $\operatorname{GK-dim} A \in \mathbb{N}_0 \cup \infty$. However there are examples of non-commutative algebras A with $\operatorname{GK-dim} A = r$ for any $r \in [2, \infty)$. But there is no algebra A with $\operatorname{GK-dim} A = r$ for any $r \in (1, 2)$. See [59].

Example 4. A finitely generated group G is *virtually nilpotent* or *nilpotent-by-finite* if it has a normal nilpotent subgroup N such that G/N is finite.

- J.A. Wolf, J. Milnor and others showed that the group algebra of a virtually nilpotent group has finite Gelfand-Kirillov dimension (in an equivalent formulation).
- A celebrated Theorem of Gromov establishes the converse: if G is a finitely generated group and $\operatorname{GK-dim} \Bbbk G < \infty$ then G is virtually nilpotent.

Example 5. Let A be an algebra with an ascending algebra filtration. Then

$$\operatorname{GK-dim} A \geq \operatorname{GK-dim} \operatorname{gr} A;$$

also, the equality holds if $\operatorname{gr} A$ is finitely generated. Let $\mathfrak{g}$ be a Lie algebra; we conclude that $\operatorname{GK-dim} U(\mathfrak{g}) = \dim \mathfrak{g}$.

2 Braided tensor categories

We first discuss the notion of braided vector space, the input of the definition of Nichols algebra, and illustrate it through various examples. Then we review braided tensor categories and the example of our main interest, Yetter-Drinfeld modules.

2.1 Braided vector spaces

The Yang-Baxter equation, introduced independently by C. N. Yang in 1968, and R. J. Baxter in 1971 in statistical mechanics, has important applications in various areas of mathematics. Here we consider the equivalent braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c), \qquad c \in GL(V \otimes V), \tag{11}$$

where V is a vector space. Solutions of the braid equation (11) are the input for the definition of Nichols algebras. Following the common usage, we say that a pair (V, c), with c satisfying (11), is a *braided vector space*.

We first justify the adjective braided: we claim that the assignment

$$\sigma_j \mapsto \mathrm{id}_{T^{(j-1)}(V)} \otimes c \otimes \mathrm{id}_{T^{(n-j-1)}(V)} \tag{12}$$

gives rise to a representation $\rho_n : \mathbb{B}_n \to GL(T^n(V))$, for every $n \geq 2$. Indeed, (11) insures that (4) holds, while (3) is free from the definition. The applications of the

Yang-Baxter equation mostly arise from these representations. For us, they will useful to present Nichols algebras. But let us discuss before some classes of examples of braided vector spaces.

2.1.1 Symmetries

Here $\operatorname{char}\Bbbk \neq 2$. A *symmetry*, is a solution c of (11) such that $c^2 = \mathrm{id}$. The name alludes to the fact that ρ_n factorizes through the representation $\widetilde{\rho}_n : \mathbb{S}_n \to GL(T^n(V))$ given by

$$\tau_j \mapsto \mathrm{id}_{T^{(j-1)}(V)} \otimes c \otimes \mathrm{id}_{T^{(n-j-1)}(V)}, \qquad j \in \mathbb{I}_{n-1}.$$

Prominent examples of symmetries are:

- The transposition, i.e. the usual flip $\tau : V \otimes V \to V \otimes V$, $v \otimes w \mapsto w \otimes v$.
- The super transposition of a super vector space $V = V_0 \oplus V_1$; i.e. the linear map $s\tau : V \otimes V \to V \otimes V$, determined by $v \otimes w \mapsto (-1)^{ij} w \otimes v$ for $v \in V_i$, $w \in V_j$.

Clearly, we have the decomposition

$$T^2(V) = \ker(\mathrm{id} + c) \oplus \ker(\mathrm{id} - c).$$

2.1.2 Hecke type

Here $\operatorname{char}\Bbbk = 0$. Let $q \in \Bbbk^\times$, $q \neq -1$. The Hecke algebra of parameter q is the associative algebra $\mathtt{H}_n(q)$ generated by $(T_i)_{i \in \mathbb{I}_{n-1}}$ with relations (4) (with T instead of τ) and

$$(T_i - q\,\mathrm{id})(T_i + \mathrm{id}) = 0, \qquad i \in \mathbb{I}_{n-1}. \tag{13}$$

A braided vector space (V, c) is of *Hecke type* with label q if

$$(c - q\,\mathrm{id})(c + \mathrm{id}) = 0. \tag{14}$$

The name refers to the fact that in this case, ρ_n factorizes through the representation $\widetilde{\rho}_n : \mathtt{H}_n(q) \to GL(T^n(V))$ given by $T_j \mapsto \mathrm{id}_{T^{(j-1)}(V)} \otimes c \otimes \mathrm{id}_{T^{(n-j-1)}(V)}$, $j \in \mathbb{I}_{n-1}$.

2.1.3 Diagonal type

We fix $\theta \in \mathbb{N}$ and abbreviate $\mathbb{I} = \mathbb{I}_\theta$. Let $\mathfrak{q} = (q_{ij}) \in (\Bbbk^\times)^{\mathbb{I} \times \mathbb{I}}$ and let V a vector space with a basis $(x_i)_{i \in \mathbb{I}}$. We define $c^{\mathfrak{q}} \in GL(T^2(V))$ by

$$c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad i, j \in \mathbb{I}, \tag{15}$$

Then $c^{\mathfrak{q}}$ satisfies (11). When $q_{ij} = 1$, we recover the transposition τ, and the super transposition also has this shape. By technical reasons, we say that a braided vector space (V,c) with $c = c^{\mathfrak{q}}$ as in (15) is of *diagonal type* if in addition

$$q_{ii} \neq 1, \qquad i \in \mathbb{I}. \tag{16}$$

Instead of the matrix $\mathfrak{q}$, we also give the associated Dynkin diagram[1], that has

- set of vertices $\mathbb{I}$, the i-th vertex being labelled with q_{ii};
- an edge between the vertices i and j only if $\widetilde{q}_{ij} := q_{ij}q_{ji} \neq 1$, in which case the edge is decorated by $\widetilde{q}_{ij}$.

Notice that we loose some information, but this is justified by Example 29.

We introduce the important subclass of Cartan type. Let $A = (a_{ij}) \in \mathbb{Z}^{\mathbb{I}\times\mathbb{I}}$ be a *generalized Cartan matrix*, that is, it satisfies

$$a_{ii} = 2, \qquad i \in \mathbb{I}; \tag{17}$$

$$a_{ij} \leq 0, \qquad i \neq j \in \mathbb{I}; \tag{18}$$

$$a_{ij} = 0 \iff a_{ji} = 0, \qquad i \neq j \in \mathbb{I};. \tag{19}$$

These are the input for the definition of Kac-Moody algebras [57]; among them, there are the celebrated Cartan matrices classifying finite-dimensional Lie algebras. Let (V,c) be a braided vector space of diagonal type with respect to a matrix $\mathfrak{q} = (q_{ij}) \in (\Bbbk^\times)^{\mathbb{I}\times\mathbb{I}}$. We say that (V,c) is of *Cartan type* (with matrix A) if

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \qquad i \neq j \in \mathbb{I}. \tag{20}$$

Suppose that q_{ii} is a root of 1 of order N_i, for all i. Observe that if the matrix $\mathfrak{q}$ satisfies (20) for some integers a_{ij}, then we get a generalized Cartan matrix by taking $a_{ii} = 2$ for all i and normalizing the a_{ij}'s by

$$-N_i < a_{ij} \leq 0, \qquad i \neq j \in \mathbb{I}.$$

2.1.4 Triangular type

Let (V,c) be a braided vector space with a basis $(x_i)_{i\in\mathbb{I}}$. Let V_j be the subspace generated by $(x_i)_{i\in\mathbb{I}_j}$. We say that (V,c) is of *triangular type* if there exists $\mathfrak{q} = (q_{ij}) \in (\Bbbk^\times)^{\mathbb{I}\times\mathbb{I}}$ such that

$$c(x_i \otimes x_j) \in q_{ij}\, x_j \otimes x_i + V_{j-1} \otimes V, \qquad i, j \in \mathbb{I}. \tag{21}$$

Example 6. Let $\varepsilon \in \Bbbk^\times$ and $\ell \in \mathbb{N}_{\geq 2}$. The *block* $\mathcal{V}(\varepsilon,\ell)$ is the braided vector space with a basis $(x_i)_{i\in\mathbb{I}_\ell}$ such that for $i, j \in \mathbb{I}_\ell = \{1,2,\dots,\ell\}$, $1 < j$:

[1] Actually this is called a generalized Dynkin diagram but we omit generalized.

$$c(x_i \otimes x_1) = \varepsilon x_1 \otimes x_i, \qquad c(x_i \otimes x_j) = (\varepsilon x_j + x_{j-1}) \otimes x_i. \tag{22}$$

Later on, we call $\mathscr{V}(\varepsilon, 2)$ and ε-block; this is justified by Theorem 7.

2.1.5 Rack type

To define this class of braided vector spaces, we need to discuss the notion of rack, that is an abstract version of the conjugation in a group; see [15] for more information. We start with the general notion of braided set; we leave to the reader to fill in the details of the proofs.

The braid equation (11) makes sense in any monoidal category, a basic example being the category of sets with Cartesian product as the tensor one. So, a *braided set* is a pair (X, c), where $X \neq \emptyset$ is a set and $\mathsf{c} : X \times X \to X \times X$ is a bijection such that

$$(\mathsf{c} \times \mathrm{id})(\mathrm{id} \times \mathsf{c})(\mathsf{c} \times \mathrm{id}) = (\mathrm{id} \times \mathsf{c})(\mathsf{c} \times \mathrm{id})(\mathrm{id} \times \mathsf{c}). \tag{23}$$

Also, c is called a set-theoretical solution of the quantum Yang-Baxter equation; it has been studied in many papers [30, 61, 32] etc.

Notice that any braided set (X, c) gives rise to a braided vector space $(\Bbbk X, c)$ by linearization; namely $\Bbbk X$ is the vector space with basis $(e_x)_{x \in X}$ and c extends linearly the map defined on the basis by c.

Let X be a non-empty set and $\mathsf{c} : X \times X \to X \times X$ be a bijection. If $p_1, p_2 : X \times X \to X$ are the standard projections, then we write

$$x \triangleright y = p_1 \mathsf{c}(x,y), \quad x \triangleleft y = p_2 \mathsf{c}(x,y), \quad \text{so that} \quad \mathsf{c}(x,y) = (x \triangleright y, x \triangleleft y), \quad x, y \in X.$$

Clearly, to give $\triangleright$ and $\triangleleft$ is equivalent to give c.

Exercise 6. 1. Find necessary and sufficient conditions on the pair $(\triangleright, \triangleleft)$ so that c satisfies (23).

2. Let $\mathfrak{q} : X \times X \to \Bbbk^\times$ be a function denoted $(x, y) \mapsto \mathfrak{q}_{x,y}$ and let

$$c^{\mathfrak{q}} : \Bbbk X \otimes \Bbbk X \to \Bbbk X \otimes \Bbbk X, \quad c^{\mathfrak{q}}(e_x \otimes e_y) = \mathfrak{q}_{x,y} e_{x \triangleright y} \otimes e_{x \triangleleft y}, \quad x, y \in X. \tag{24}$$

   Prove that if $c^{\mathfrak{q}}$ satisfies (11), then (X, c) is a braided set.
3. Let $\mathfrak{q}$ be as in the previous item. Assume that (X, c) is a braided set. Find necessary and sufficient conditions on $\mathfrak{q}$ so that $c^{\mathfrak{q}}$ satisfies (11).

The definition of rack arises by considering the trivial $\triangleleft$, i.e. $x \triangleleft y = x$ for all $x, y \in X$. That is, consider $\triangleright : X \times X \to X$ and correspondingly $\mathsf{c} : X \times X \to X \times X$ given by $\mathsf{c}(x, y) = (x \triangleright y, x)$, $x, y \in X$. Then c is bijective if and only if

$$\text{the map } \phi_x = x \triangleright \_\_ \text{ is bijective for any } x \in X \tag{25}$$

while c satisfies (23) if and only if

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \text{ for all } x,y,z \in X. \tag{26}$$

We say that $(X, \triangleright)$ is a *rack* if (25) and (26) hold. Morphisms of racks and subracks are defined as usual; $\operatorname{Aut} X$ denotes the group of rack automorphisms of X.

Exercise 7. Let X be a non-empty set and $\triangleright : X \times X \to X$ a function.

1. Let $\mathfrak{q} : X \times X \to \Bbbk^{\times}$ be a function, $(x,y) \mapsto \mathfrak{q}_{x,y}$, and let $c^{\mathfrak{q}} : \Bbbk X \otimes \Bbbk X \to \Bbbk X \otimes \Bbbk X$ be given by

$$c^{\mathfrak{q}}(e_x \otimes e_y) = \mathfrak{q}_{x,y} e_{x \triangleright y} \otimes e_x, \qquad x,y \in X.$$

   Prove that $(\Bbbk X, c^{\mathfrak{q}})$ is a braided vector space if and only if $(X, \triangleright)$ is a rack and

$$q_{x,y \triangleright z} q_{y,z} = q_{x \triangleright y, x \triangleright z} q_{x,z}, \qquad \forall x,y,z \in X. \tag{27}$$

2. Here is a generalization. Let W be a vector space and let $\mathfrak{q} : X \times X \to GL(W)$ be a function. Set $V = \Bbbk X \otimes W$, $e_x v := e_x \otimes v$. Let $c^{\mathfrak{q}} : V \otimes V \to V \otimes V$ be given by

$$c^{\mathfrak{q}}(e_x v \otimes e_y w) = e_{x \triangleright y} \mathfrak{q}_{x,y}(w) \otimes e_x v, \qquad x,y \in X, \qquad v,w \in W. \tag{28}$$

   Prove that $(V, c^{\mathfrak{q}})$ is a braided vector space if and only if $(X, \triangleright)$ is a rack and (27) holds.
3. Let $\mathfrak{q}, \mathfrak{p} : X \times X \to \Bbbk^{\times}$ be two functions satisfying (27) and let $\mathfrak{b} : X \to \Bbbk^{\times}$ be a function. Let $T : \Bbbk X \to \Bbbk X$ be given by $T(e_x) = \mathfrak{b}_x e_x$, $x \in X$. Find necessary and sufficient conditions such that $T : (\Bbbk X, c^{\mathfrak{q}}) \to (\Bbbk X, c^{\mathfrak{p}})$ is a morphism of braided vector spaces.

The condition (27) says that $\mathfrak{q}$ is a 2-cocycle; when $\dim W = 1$, it is part of a cohomology theory, while for $n > 1$ it is a non-abelian cocycle. Observe that any constant function $\mathfrak{q}$ is a 2-cocycle.

Braided vector spaces as in the previous Exercise are called of *rack type* and play an important role in the classification of finite-dimensional pointed Hopf algebras.

2.1.6 Racks

We discuss now examples of racks; once again the reader is encouraged to work out the details.

Example 7. Let $(X, \mathtt{c})$ be a braided set with associated $\triangleright$ and $\triangleleft$ as above. Then we say that $\mathtt{c}$ is non-degenerate if for all $x,y \in X$ the maps

$$x \triangleright \_\_ : X \to X, \qquad \_\_ \triangleleft y : X \to X$$

are both bijective. Assume that this is the case. Write $\_\_ \triangleleft y^{-1}$ for the inverse of $\_\_ \triangleleft y$. Define $\blacktriangleright : X \times X \to X$ by

$$x \blacktriangleright y = \big((x \triangleleft y^{-1}) \triangleright y\big) \triangleleft x. \tag{29}$$

Then $(X, \blacktriangleright)$ is a rack, called the derived rack of (X, c).

Example 8. Let X be a non-empty set. Given $\sigma \in \mathbb{S}_X$, the associated permutation rack $(X, \triangleright)$ is defined by $x \triangleright y = \sigma(y)$ for all $x, y \in X$.

Example 9. A group G is a rack with $x \triangleright y = xyx^{-1}$, $x, y \in G$. If $X \subset G$ is stable under conjugation by G (for example if X is a conjugacy class), then it is a subrack of G.

Example 10. A set $X \neq \emptyset$ with $\triangleright$ defined by $x \triangleright y = y$ for all $x, y \in X$ is a rack; such racks are called *abelian*. The abelian rack with elements $\{1, \dots, n\}$ is denoted $\mathbb{I}_n$.

Exercise 8. 1. If X is a rack, then $\phi : X \to \mathbb{S}_X$, $x \mapsto \phi_x$, is a morphism of racks. The subgroup of $\mathbb{S}_X$ generated by the image of ϕ is denoted by $\operatorname{Inn} X$. Thus $\operatorname{Inn} X$ acts on X. Show that $\operatorname{Inn} X$ is a normal subgroup of $\operatorname{Aut} X$ (the group of rack automorphisms).
2. When a subrack of a group is a permutation rack?
3. Let X be a subrack of a group. Then, for all $x, y \in X$, we have

$$x \triangleright x = x, \tag{30}$$

$$x \triangleright y = y \implies y \triangleright x = x, \tag{31}$$

A rack with these properties is a *crossed set*. Can a crossed set be realized always as a subrack of a group?

The following examples can be identified with subracks of groups, but they deserve a separate consideration.

Example 11. Let G be a group and $T \in \operatorname{Aut} G$. Let $\rightharpoonup_T$ be the action of G on itself given by $x \rightharpoonup_T y = x\,y\,T(x^{-1})$, $x, y \in G$. Then the orbit $\mathscr{O}_x^{G,T}$ of $x \in G$ by this action is a rack with operation

$$y \triangleright_T z = y\,T(z y^{-1}), \quad y, z \in \mathscr{O}_x^{G,u}. \tag{32}$$

The rack $(\mathscr{O}_x^{G,T}, \triangleright_T)$ is called a *twisted conjugacy class* of type (G, T).

Example 12. Let A be an abelian group and $T \in \operatorname{Aut} A$. We define the operation $\triangleright$ by

$$x \triangleright y = (1 - T)x + Ty, \qquad x, y \in A.$$

Then $(A, \triangleright)$ is a rack, denoted $\operatorname{Aff}(A, T)$. If T is multiplication by a fixed m, then the rack is denoted by $\operatorname{Aff}(A, m)$. The rack $\operatorname{Aff}(A, T)$ is isomorphic to the subrack $A \times \mathrm{id}$ of $A \rtimes \langle T \rangle$. Racks of this sort are called *affine*. For instance, the dihedral rack $\mathscr{D}_n$, $n \geq 3$, is $\operatorname{Aff}(\mathbb{Z}/n, T)$, where T is multiplication by -1.

Exercise 9. Let X be a rack; below $\dot\cup$ means disjoint union.

1. A *decomposition* of X is a pair of subracks (Y,Z) such that $X = Y \dot\cup Z$; X is *decomposable* if it admits a decomposition, *indecomposable* otherwise.
Let $\emptyset \neq Y \subsetneq X$ and $Z = X - Y$. If X is finite, then
$$(Y,Z) \text{ is a decomposition of } X \iff Y \triangleright Z \subseteq Z \text{ and } Z \triangleright Y \subseteq Y \iff X \triangleright Y \subseteq Y.$$
If X is not finite, which of the implications remain true? Find counterexamples for the rest.
2. X is indecomposable $\iff X = \mathscr{O}_x^{\operatorname{Inn} X}$ for any $x \in X$.
3. Let $n \geq 3$. Compute all subracks of $\mathscr{D}_n$. Conclude that $\mathscr{D}_n$ is indecomposable if n is odd. Prove that $\operatorname{Inn} \mathscr{D}_4 \neq \operatorname{Aut} \mathscr{D}_4$, what about $\operatorname{Inn} \mathscr{D}_n$ for $n \neq 4$?

Exercise 10. Let Y, Z be two racks and $X = Y \dot\cup Z$. The following are equivalent:

1. Structures of rack on X such that (Y,Z) is a decomposition.
2. Pairs (ς, ϖ) of morphisms of racks $\varsigma : Y \to \operatorname{Aut} Z$, $\varpi : Z \to \operatorname{Aut} Y$ such that

$$y \triangleright \varpi_z(u) = \varpi_{\varsigma_y(z)}(y \triangleright u), \quad \forall y, u \in Y,\ z \in Z, \qquad \text{i.e., } \phi_y \varpi_z = \varpi_{\varsigma_y(z)} \phi_y; \tag{33}$$
$$z \triangleright \varsigma_y(w) = \varsigma_{\varpi_z(y)}(z \triangleright w), \quad \forall y \in Y,\ z, w \in Z, \qquad \text{i.e., } \phi_z \varsigma_y = \varsigma_{\varpi_z(y)} \phi_z. \tag{34}$$

The rack X is denoted $Y\,{}_{\varsigma}\!\coprod_{\varpi} Z$, with ς omitted if $\varsigma_y = \mathrm{id}_Z$ for all $y \in Y$, idem for ϖ. Assume that Y and Z are crossed sets and that (33), (34) hold. Then X is a crossed set if and only if the following condition holds:

$$\varsigma_y(z) = z \text{ if and only if } \varpi_z(y) = y, \qquad \forall y \in Y, z \in Z. \tag{35}$$

Exercise 11. Assume that $Y = \mathbb{I}_n$. Then the previous setting reduces to a family $(\varsigma_i)_{i \in \mathbb{I}_n}$ of commuting elements in $\operatorname{Aut} Z$ and a morphism of racks $\varpi : Z \to \mathbb{S}_n$ such that

$$\varpi_z = \varpi_{\varsigma_j(z)}, \qquad z \triangleright \varsigma_j(w) = \varsigma_{\varpi_z(j)}(z \triangleright w), \qquad \forall j \in \mathbb{I}_n,\ z, w \in Z.$$

Suppose that $Y = \mathbb{I}_n$ and $Z = \mathbb{I}_m$. Then the previous setting consists of families $(\varsigma_i)_{i \in \mathbb{I}_n}$ and $(\varpi_h)_{h \in \mathbb{I}_m}$ of commuting elements in $\mathbb{S}_m$ and $\mathbb{S}_n$ respectively, such that

$$\varpi_h = \varpi_{\varsigma_j(h)}, \qquad \varsigma_j = \varsigma_{\varpi_h(j)}, \qquad \forall j \in \mathbb{I}_n,\ h \in \mathbb{I}_m. \tag{36}$$

In particular, let $\sigma \in \mathbb{S}_m$ and $\pi \in \mathbb{S}_n$ and consider the constant families $\varsigma_i = \sigma$, $i \in \mathbb{I}_n$, and $\varpi_h = \pi$, $h \in \mathbb{I}_m$. These families satisfy (36), thus we have the rack $\mathbb{I}_n\, {}_{\sigma}\!\coprod_{\pi} \mathbb{I}_m$.

Here is an important notion for our purposes.

Definition 1. A finite rack X is *simple* if

- it has at least 2 elements,
- for any surjective morphism of racks $\pi : X \to Y$, either π is an isomorphism or Y has just one element.

Finite simple racks have been classified in [15, Th. 3.9, Th. 3.12], [56]. Because of its importance in recursive arguments about Nichols algebras, we state this result.

Theorem 2. *Let X be a finite simple rack with $|X|$ elements. Then either of the following holds:*

1. *$|X|$ is divisible by at least two primes. In this case, there exist*
   - *a simple non-abelian group L,*
   - *$t \in \mathbb{N}$, and*
   - *$\theta \in \operatorname{Aut} L$,*

   such that X is a twisted conjugacy class of type (G,T), where
   - *$G = L^t$ and*
   - *$T \in \operatorname{Aut}(L^t)$ acts by*

   $$T(\ell_1,\dots,\ell_t) = (\theta(\ell_t),\ell_1,\dots,\ell_{t-1}), \qquad \ell_1,\dots,\ell_t \in L.$$

   Furthermore, L and t are unique, and T only depends on its conjugacy class in $\operatorname{Out}(L^t) = \operatorname{Aut}(L^t)/Inn(L^t)$.
2. *$|X| = p^t$ where p is a prime and $t \in \mathbb{N}$. In this case, there are two possibilities:*
   a. *$t = 1$ and $X \simeq \mathbb{I}_p$ is the permutation rack of the cycle $(1,2,\dots,p)$ (this could not be realized as a conjugacy class in a group).*
   b. *X is the affine rack $(\mathbb{F}_p{}^t, T)$, where T is the companion matrix of a monic irreducible polynomial $f \in \mathbb{F}_p[\mathtt{X}]$ of degree t, different from $\mathtt{X}$ and $\mathtt{X}-\mathtt{1}$.*

Particularly, non-trivial conjugacy classes in finite simple groups are simple racks.

2.2 *Braided tensor categories*

The notion of braided vector space has a counterpart in the notion of braided tensor category, that is both technically convenient and the right formulation for applications. We briefly discuss this notion and refer to [31, 58] for extensive expositions.

2.2.1 Tensor categories

We start by the formal definitions.

A *monoidal category* is a collection $(\mathscr{C},\otimes,\mathbf{1},a,l,r)$, where

- $\mathscr{C}$ is a category;
- $\otimes : \mathscr{C}\times\mathscr{C} \to \mathscr{C}$ is a functor, called the tensor product;
- $\mathbf{1} \in \mathscr{C}$ is an object called the unit;

- $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is an invertible natural transformation, called the associativity constraint;
- $l_X : X \to X \otimes \mathbf{1}$, $r_X : X \to \mathbf{1} \otimes X$, are invertible natural transformations, called the left and right unit constraints.

These data is required to satisfy the pentagon and the triangle axioms, expressed by the commutativity of the following diagrams:

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes U \arrow[rr, "a_{(X\otimes Y),Z,U}"] \arrow[d, "a_{X,Y,Z}\otimes \mathrm{id}_U"'] & & (X \otimes Y) \otimes (Z \otimes U) \arrow[d, "a_{X,Y,Z\otimes U}"] \\
(X \otimes (Y \otimes Z)) \otimes U \arrow[dr, "a_{X,Y\otimes Z,U}"'] & & X \otimes (Y \otimes (Z \otimes U)) \\
& X \otimes ((Y \otimes Z) \otimes U) \arrow[ur, "\mathrm{id}_X \otimes a_{YZ,U}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a_{X,\mathbf{1},Y}"] & & X \otimes (\mathbf{1} \otimes Y) \\
& X \otimes Y. \arrow[ul, "l_X\otimes \mathrm{id}_Y"] \arrow[ur, "\mathrm{id}_X \otimes r_Y"'] &
\end{tikzcd}$$

The pentagon and triangle axioms guarantee that we can tensor any finite number of objects, the result being independent of the distribution of parentheses up to isomorphism, and that the unit objects can be ignored in such a product. This was shown by S. Mac Lane, who also proved any monoidal category is equivalent to a strict one (one with associative and unit constraints equal to the identity).

Let $\mathscr{C}$ be a monoidal category and $M \in \mathscr{C}$. A *left dual* of M is an object ${}^*M \in \mathscr{C}$ provided with morphisms

$$\mathbf{1} \xrightarrow{\mathrm{coev}_M} {}^*M \otimes M , \qquad M \otimes {}^*M \xrightarrow{\mathrm{ev}_M} \mathbf{1},$$

such that the composition

$$M \xrightarrow{l_M} M \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes \mathrm{coev}_M} M \otimes {}^*M \otimes M \xrightarrow{\mathrm{ev}_M \otimes \mathrm{id}} \mathbf{1} \otimes M \xrightarrow{r_M^{-1}} M \qquad (37)$$

equals id_M. Analogously, a *right dual* of M is an object $M^* \in \mathscr{C}$ provided with morphisms

$$\mathbf{1} \xrightarrow{\mathrm{coev}'_M} M \otimes M^* , \qquad M^* \otimes M \xrightarrow{\mathrm{ev}'_M} \mathbf{1},$$

such that the following composition equals id_M:

$$M \xrightarrow{r_M} \mathbf{1} \otimes M \xrightarrow{\operatorname{coev}'_M \otimes \operatorname{id}} M \otimes M^* \otimes M \xrightarrow{\operatorname{id} \otimes \operatorname{ev}_M} M \otimes \mathbf{1} \xrightarrow{l_M^{-1}} M \quad (38)$$

Clearly, if M has a right dual M^*, then M^* has itself a left dual which is M.

Exercise 12. Prove that two left duals of the same object are isomorphic.

A monoidal category $\mathscr{C}$ is *rigid* if every object in $\mathscr{C}$ has right and left duals.

Example 13. Assume that $\mathscr{C}$ is a discrete category, i.e. $\operatorname{Ob}\mathscr{C}$ is a set X and the only arrows are the identities id_x, $x \in X$. Then a structure of monoidal category on $\mathscr{C}$ is tantamount to a structure of monoid on X. Thus, a structure of rigid monoidal category on $\mathscr{C}$ is tantamount to a structure of group on X. In other words, the notion of rigid monoidal category is a categorification of the notion of group.

Definition 2. A *tensor category* (over $\mathbb{k}$) is a rigid monoidal category such that $\mathscr{C}$ is abelian $\mathbb{k}$-linear and $\otimes$ is $\mathbb{k}$-linear in each variable (that is, $\otimes$ is a bifunctor).

Example 14. Let $\operatorname{Vec}_{\mathbb{k}}$ be the category of vector spaces over $\mathbb{k}$ and let $\operatorname{vec}_{\mathbb{k}}$ be the full subcategory of the finite-dimensional ones. Then $\operatorname{Vec}_{\mathbb{k}}$ is a monoidal (abelian $\mathbb{k}$-linear) category, with $\otimes = \otimes_{\mathbb{k}}$ the usual tensor product over $\mathbb{k}$, $\mathbf{1} \simeq \mathbb{k}$, and the natural isomorphisms a, l and r from the universal property defining $\otimes_{\mathbb{k}}$. Also $\operatorname{vec}_{\mathbb{k}}$ is a tensor category; given $V \in \operatorname{vec}_{\mathbb{k}}$, we take ${}^*V = V^* = \hom_{\mathbb{k}}(V, \mathbb{k})$ and

$$\mathbb{k} \xrightarrow{\operatorname{coev}_V} {}^*V \otimes V \qquad\qquad V \otimes {}^*V \xrightarrow{\operatorname{ev}_V} \mathbb{k}$$

$$1 \longmapsto \textstyle\sum_{i \in \mathbb{I}} \alpha_i \otimes v_i, \qquad\qquad f \otimes v \longmapsto f(v),$$

where $(v_i)_{i \in \mathbb{I}}$ is a basis of V and $(\alpha_i)_{i \in \mathbb{I}}$ is its dual basis.

Exercise 13. Prove that $\operatorname{Vec}_{\mathbb{k}}$ is not rigid.

Example 15. Let H be a Hopf algebra with bijective antipode $\mathscr{S}$. Let $\operatorname{Rep} H$ be the category of representations of H and let $\operatorname{rep} H$ be the full subcategory of the finite-dimensional ones. Then $\operatorname{Rep} H$ is a monoidal subcategory of $\operatorname{Vec}_{\mathbb{k}}$ and $\operatorname{rep} H$ is a tensor subcategory of $\operatorname{vec}_{\mathbb{k}}$ (but neither is full). Indeed, if $V, W \in \operatorname{Rep} H$, then H acts on $V \otimes W$ via the comultiplication Δ; the unit is $\mathbb{k}$ with the trivial action given by the counit ε; ${}^*V$, respectively V^*, is $\hom_{\mathbb{k}}(V, \mathbb{k})$ with the action given by the transpose of the antipode, respectively its inverse.

This class of examples includes:

- The category $\operatorname{Rep} G$ of representations of a group G over $\mathbb{k}$ and the subcategory $\operatorname{rep} G$.
- The category $\operatorname{Rep} \mathfrak{g}$ of representations of a Lie algebra $\mathfrak{g}$ and the subcategory $\operatorname{rep} \mathfrak{g}$.

- The category $\mathrm{Vec}_{\Bbbk}^{G}$ of G-graded vector spaces, where G is a group, and the subcategory $\mathrm{vec}_{\Bbbk}^{G}$. Here the tensor product of $V = \oplus_{g\in G} V_g$ and $W = \oplus_{g\in G} W_g$ is graded as $V\otimes W = \oplus_{g\in G}(V\otimes W)_g$, where

$$(V\otimes W)_g = \oplus_{h\in G} V_h \otimes W_{h^{-1}g}.$$

The category of super vector spaces is the particular case $\mathrm{Svec}_{\Bbbk} = \mathrm{Vec}_{\Bbbk}^{\mathbb{Z}/2}$; as usual, $\mathrm{svec}_{\Bbbk}$ is the full subcategory of finite-dimensional objects.

Exercise 14. Assume that $\Bbbk$ is algebraically closed of characteristic 0. Let G be a finite abelian group and $\widehat{G}$ be its group of characters. Then $\mathrm{Vec}_{\Bbbk}^{G}$ is equivalent to $\mathrm{Rep}\,\widehat{G}$ as monoidal categories.

Example 16. Let H be a Hopf algebra with bijective antipode. Let $\mathscr{M}^H$, respectively ${}^H\mathscr{M}$, be the category of right, respectively left, H-comodules. Then both $\mathscr{M}^H$ and ${}^H\mathscr{M}$ are monoidal subcategories of $\mathrm{Vec}_{\Bbbk}$. Indeed, the tensor product arises via the multiplication and the unit is $\Bbbk$ with the trivial coaction. The subcategories of finite-dimensional comodules are tensor, with duals given by the the antipode, respectively its inverse.

2.2.2 Braided tensor categories

If the notion of monoidal category could be thought as an extension of the notion of monoid (or group), then it is natural to seek for the analogue of the notion of abelian group. Such an analogue was already proposed by S. Mac Lane–symmetric monoidal categories. However the weaker notion of braided category turned out to be much more flexible for applications.

A *braided monoidal category* is a monoidal category $\mathscr{C}$ provided with a natural isomorphism $c_{X,Y} : X\otimes Y \to Y\otimes X$, called the *braiding*, that is required to fulfill the hexagon axioms, meaning that the following diagrams commute:

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "c_{X,Y}\otimes \mathrm{id}"'] & X\otimes(Y\otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & (Y\otimes Z)\otimes X \arrow[d, "a_{Y,Z,X}"] \\
(Y\otimes X)\otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y\otimes(X\otimes Z) \arrow[r, "\mathrm{id}\otimes c_{X,Z}"'] & Y\otimes(Z\otimes X),
\end{tikzcd} \tag{39}$$

$$\begin{tikzcd}
X\otimes(Y\otimes Z) \arrow[r, "a^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}\otimes c_{Y,Z}"'] & (X\otimes Y)\otimes Z \arrow[r, "c_{X\otimes Y,Z}"] & Z\otimes(X\otimes Y) \arrow[d, "a^{-1}_{Z,X,Y}"] \\
X\otimes(Z\otimes Y) \arrow[r, "a^{-1}_{X,Y,Z}"'] & (X\otimes Z)\otimes Y \arrow[r, "c_{X,Z}\otimes \mathrm{id}"'] & (Z\otimes X)\otimes Y,
\end{tikzcd} \tag{40}$$

for all $X,Y,Z \in \mathscr{C}$. In addition, $\mathscr{C}$ is *symmetric* when

$$c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}, \qquad \text{for all } X,Y \in \mathscr{C}. \tag{41}$$

Loosely, (41) is abbreviated as $c^2 = \mathrm{id}$. In this case, c is called a symmetry, instead of a braiding.

Exercise 15. Assume that (41) holds. Then (39) and (40) are equivalent.

Needless to say, a braided tensor category is a tensor category that is also braided.

Example 17. The super categories $\mathrm{Vec}_{\Bbbk}$ and $\mathrm{vec}_{\Bbbk}$ are symmetric, with symmetry being the transposition τ.

Example 18. If G is a group and $\mathfrak{g}$ is a Lie algebra, then the tensor categories $\mathrm{Rep}\, G$ and $\mathrm{Rep}\,\mathfrak{g}$ are symmetric, with symmetry τ.

Example 19. The categories $\mathrm{Svec}_{\Bbbk}$ and $\mathrm{svec}_{\Bbbk}$ are symmetric, with symmetry being the super transposition $s\tau$.

Exercise 16. 1. Classify all possible braidings in the category $\mathrm{Vec}_{\Bbbk}^{\mathbb{Z}/n}$, $1 < n \in \mathbb{N}$.
2. Classify all possible braidings in the category $\mathrm{Vec}_{\Bbbk}^{G}$, where G is an abelian group; determine those that are symmetries.
3. Let G be a group. Prove that the category $\mathrm{Vec}_{\Bbbk}^{G}$ admits a braiding if and only if G is abelian.

Where the adjective braided comes from?

Proposition 2. *Let $\mathscr{C}$ be a braided monoidal category. Assume that it is strict, i.e. the associativity and unit constraints are identities. Then for all $X,Y,Z \in \mathscr{C}$,*

$$(c_{Y,Z}\otimes \mathrm{id}_X)(\mathrm{id}_Y \otimes c_{X,Z})(c_{X,Y}\otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c_{X,Y})(c_{X,Z}\otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,Z}), \tag{42}$$

equality in $\hom(X\otimes Y\otimes Z, Z\otimes Y\otimes X)$.

Thus, if $X \in \mathscr{C}$, then $c_{X,X}$ is a solution of the braid equation. If $\mathscr{C}$ is not strict, then a version of (42) with associators holds.

V. G. Drinfeld found a mechanism to construct solutions of the braid equation (11). First, he introduced the notion of quasitriangular Hopf algebra as a pair (H,R) where H is a Hopf algebra and $R \in H\otimes H$ is tailored to give $\mathrm{Rep}\, H$ a structure of braided tensor category. Second, he showed how to assign to a Hopf algebra H (say finite-dimensional to avoid technicalities), a quasitriangular Hopf algebra $D(H)$–called nowadays the Drinfeld double of H. For a better understanding of this construction, we give now the categorical version; passing from H to $D(H)$ is a particular instance of the center of a monoidal category.

Exercise 17. Let $\mathscr{C}$ be a monoidal category. Prove that $\mathscr{Z}(\mathscr{C})$ (the center of $\mathscr{C}$) defined as follows is a braided monoidal category:

- The objects are pairs (Z,γ) where $Z\in\mathscr{C}$ and γ is a natural isomorphism

$$\gamma_X : X\otimes Z\to Z\otimes X, \qquad\qquad X\in\mathscr{C},$$

  such that the following diagram commutes:

$$\begin{tikzcd}
X\otimes(Y\otimes Z) \arrow[r, "a^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}\otimes\gamma_Y"'] & (X\otimes Y)\otimes Z \arrow[r, "\gamma_{X\otimes Y}"] & Z\otimes(X\otimes Y) \arrow[d, "a^{-1}_{Z,X,Y}"] \\
X\otimes(Z\otimes Y) \arrow[r, "a^{-1}_{X,Y,Z}"'] & (X\otimes Z)\otimes Y \arrow[r, "\gamma_X\otimes\mathrm{id}"'] & (Z\otimes X)\otimes Y.
\end{tikzcd} \tag{43}$$

  By the similarity of (43) with (40), γ is called a half-braiding.
- The morphisms between pairs (Z,γ) and (Z',γ') are maps $f:Z\to Z'$ in $\mathscr{C}$ such that

$$(f\otimes\mathrm{id}_X)\gamma_X=\gamma'_X(\mathrm{id}_X\otimes f): X\otimes Z\to Z\otimes X, \qquad \text{for all } X\in\mathscr{C}.$$

- The tensor product of (Z,γ) and (Z',γ') is $(Z\otimes Z',\widetilde{\gamma})$, where $\widetilde{\gamma}$ is defined by the commutativity of the diagram

$$\begin{tikzcd}
X\otimes(Z\otimes Z') \arrow[r, "a^{-1}_{X,Z,Z'}"] \arrow[d, "\widetilde{\gamma}"'] & (X\otimes Z)\otimes Z' \arrow[r, "\gamma_X\otimes\mathrm{id}"] & (Z\otimes X)\otimes Z' \arrow[d, "a_{Z,X,Z'}"] \\
(Z\otimes Z')\otimes X & Z\otimes(Z'\otimes X) \arrow[l, "a^{-1}_{Z,Z',X}"'] & Z\otimes(X\otimes Z') \arrow[l, "\mathrm{id}\otimes\gamma'_X"']
\end{tikzcd} \tag{44}$$

- The unit object is $(\mathbf{1},r^{-1}l)$.
- The braiding between (Z,γ) and (Z',γ') is

$$c_{(Z,\gamma),(Z',\gamma')}=\gamma'_Z.$$

  If $(Z,\gamma)\in\mathscr{Z}(\mathscr{C})$ and Z has a left dual in $\mathscr{C}$, then (Z,γ) has a left dual in $\mathscr{Z}(\mathscr{C})$.

Exercise 18. Compute explicitly $\mathscr{Z}(\operatorname{Rep} G)$ and $\mathscr{Z}(\operatorname{Vec}_{\Bbbk} G)$.

2.2.3 Yetter-Drinfeld modules

Let H be a Hopf algebra with bijective antipode $\mathscr{S}$. Let $G(H)$ be the group of group-like elements. This is the point we wanted to reach:

Definition 3. A *Yetter Drinfeld module* over H is a vector space V provided with

- a structure of left H-module $\cdot : H\otimes V\to V$ and
- a structure of left H-comodule $\delta: V\to H\otimes V$, such that

- for all $h \in H$ and $v \in V$, the following compatibility condition holds:

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} \mathscr{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}. \tag{45}$$

Thus we have the category ${}^H_H\mathscr{YD}$ of Yetter-Drinfeld modules, with morphisms being linear maps that preserve both the action and the coaction.

Exercise 19. Prove that ${}^H_H\mathscr{YD}$ is a braided tensor category, with the tensor product of modules and comodules and braiding

$$c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}, \qquad V, W \in {}^H_H\mathscr{YD}, \qquad v \in V, w \in W. \tag{46}$$

Here $c_{V,W}$ is bijective because $\mathscr{S}$ is so; indeed

$$c^{-1}_{W,V}(v \otimes w) = w_{(0)} \otimes \mathscr{S}^{-1}(w_{(-1)}) \cdot v, \quad V, W \in {}^H_H\mathscr{YD}, \quad v \in V, w \in W. \tag{47}$$

That is, the assignment $H \rightsquigarrow {}^H_H\mathscr{YD}$ is the categorical version of $H \rightsquigarrow D(H)$; indeed, when H is finite-dimensional, ${}^H_H\mathscr{YD}$ is equivalent to $\operatorname{Rep} D(H)$.

Exercise 20. Show that ${}^H_H\mathscr{YD}$ is equivalent as tensor category to $\mathscr{Z}({}^H\mathscr{M})$.

Notice that there are four versions of Yetter-Drinfeld categories, the other three being ${}^H\mathscr{YD}_H$ (left comodules and right modules), $\mathscr{YD}^H_H$ and ${}_H\mathscr{YD}^H$.

Summarizing, given H as above, every $V \in {}^H_H\mathscr{YD}$ provides a braided vector space, namely $(V, c_{V,V})$. Two questions come up naturally: Does every braided vector space (V, c) arise as a Yetter-Drinfeld module over some H? (For short, we say that (V, c) is *realized* over H). If yes, then in so many ways? The answer to the first is affirmative, up to a technical hypothesis:

Definition 4. A finite-dimensional braided vector space (V, c) is *rigid* if the map $c^\flat : V^* \otimes V \to V \otimes V^*$ given by

$$f \otimes v \longmapsto \sum_i (\mathrm{ev} \otimes \mathrm{id} \otimes \mathrm{id})(f \otimes c(v \otimes v_i) \otimes \alpha^i)$$

is invertible, where (v_i) is a basis of V and (α^i) its dual basis.

Proposition 3. *[33, 81, 77, 43] Let (V, c) be a rigid braided vector space. Then there is a Hopf algebra $H(V)$ such that $V \in {}^{H(V)}_{H(V)}\mathscr{YD}$ and $c = c_{V,V}$.*

The construction of $H(V)$ is done in two steps: first, one attaches a bialgebra $A(V)$ such that $V \in {}^{A(V)}_{A(V)}\mathscr{YD}$–this is the celebrated FRT-construction. Second, and here rigidity is needed, one passes from $A(V)$ to $H(V)$.

However, $H(V)$ does not provide, by far, the unique realization and the problem of classifying or even characterizing all of them contains some subtleties.

Example 20. A pair $(g,\chi)\in G(H)\times \hom_{\mathrm{alg}}(H,\Bbbk)$ is called a *YD-pair* for H provided that

$$\chi(h)\,g=\chi(h_{(2)})h_{(1)}\,g\,\mathscr{S}(h_{(3)}), \qquad h\in H. \tag{48}$$

If (g,χ) is a YD-pair, then $g\in Z(G(K))$.

YD-pairs classify the $V\in {}^H_H\mathscr{YD}$ with $\dim V=1$. Indeed, if (g,χ) is a YD-pair, then $\Bbbk_g^\chi=\Bbbk$ with action and coaction given by χ and g respectively, is in ${}^H_H\mathscr{YD}$. In fact, (48) is just (45). Clearly, the braiding of $\Bbbk_g^\chi$ is multiplication by $q=\chi(g)$.

Example 21. Let $\mathfrak{q}=(q_{ij})\in(\Bbbk^\times)^{\mathbb{I}\times\mathbb{I}}$ satisfying (16) and let V the corresponding braided vector space of diagonal type with respect to a basis $(x_i)_{i\in\mathbb{I}}$. A principal realization of (V,c) is a collection $(g_i,\chi_i)_{i\in\mathbb{I}}$ of YD-pairs such that $q_{ij}=\chi_j(g_i)$ for all $i,j\in\mathbb{I}$. But there might be realizations different from these.

Example 22. Assume that $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk=0$. If $H=\Bbbk\Gamma$, where Γ is a finite abelian, then ${}^H_H\mathscr{YD}$ is semisimple and its simple objects have dimension 1. Now (48) always holds. In conclusion, every $V\in{}^H_H\mathscr{YD}$ of dimension $\theta\in\mathbb{N}$ is determined by families $(g_i)_{i\in\mathbb{I}_\theta}$ and $(\chi_i)_{i\in\mathbb{I}_\theta}$; the braiding of V is of diagonal type with matrix $\mathfrak{q}=(q_{ij})$, $q_{ij}=\chi_j(g_i)$, for all $i,j\in\mathbb{I}_\theta$.

Example 23. We now explain how to realize blocks of dimension 2, cf. Example 6. A *YD-triple* for H is a collection (g,χ,η) where (g,χ), is a YD-pair for H, $\eta\in\operatorname{Der}_{\chi,\chi}(H,\Bbbk)$, $\eta(g)=1$ and

$$\eta(h)g=\eta(h_{(2)})h_{(1)}g\mathscr{S}(h_{(3)}), \qquad h\in H. \tag{49}$$

Let (g,χ,η) be a YD-triple. Let $\mathscr{V}_g(\chi,\eta)$ be a vector space with a basis $(x_i)_{i\in\mathbb{I}_2}$, where action and coaction are given by

$$h\cdot x_1=\chi(h)x_1, \qquad h\cdot x_2=\chi(h)x_2+\eta(h)x_1, \qquad \delta(x_i)=g\otimes x_i,$$

$h\in H$, $i\in\mathbb{I}_2$. Then $\mathscr{V}_g(\chi,\eta)\in{}^H_H\mathscr{YD}$, the compatibility being granted by (48), (49). Since $\eta(g)\neq 0$, then $\mathscr{V}_g(\chi,\eta)$ is indecomposable in ${}^H_H\mathscr{YD}$. As braided vector space, $\mathscr{V}_g(\chi,\eta)$ is the block $\mathscr{V}(\varepsilon,2)$, where $\varepsilon:=\chi(g)$.

Exercise 21. Find a realization of the block $\mathscr{V}(\varepsilon,2)$ over $\Bbbk\mathbb{Z}$.

Exercise 22. Let G be a group. Prove that $M\in{}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$ if and only if M is a G-module with a G-grading $M=\oplus_{\gamma\in G}M_\gamma$ such that $g\cdot M_\gamma=M_{g\gamma g^{-1}}$. Consequently, if $N\leq M$ is a Yetter-Drinfeld submodule, then N inherits the grading; in particular $N\neq 0$ implies $N_\gamma\neq 0$ for some $\gamma\in G$.

Example 24. Let G be a finite group. Let $\mathscr{O}$ be a conjugacy class in G, pick $x\in\mathscr{O}$ and (W,ρ) an irreducible representation of $G^x=\{g\in G: gx=xg\}$, i,.e. the centralizer (or the isotropy subgroup) of x. Let

$$M(\mathscr{O},\rho)=\operatorname{Ind}_{G^x}^G\rho=\Bbbk G\otimes_{\Bbbk G^x}W. \tag{50}$$

We want to show that $M(\mathscr{O},\rho) \in {}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$, for which we need to define the coaction. Let $(x_i)_{i\in\mathbb{I}_m}$ be a numeration of $\mathscr{O}$, $m = |\mathscr{O}|$. Then there are $(z_i)_{i\in\mathbb{I}_m}$ in G such that

$$z_i \triangleright x = z_i x z_i^{-1} = x_i, \qquad\qquad i \in \mathbb{I}_m.$$

Thus $G = \coprod_{i\in\mathbb{I}_m} z_i G^x$. We may normalize the choice by $x_1 = x$ and $z_1 = e$. Now

$$M(\mathscr{O},\rho) = \oplus_{i\in\mathbb{I}_m} \Bbbk z_i \otimes W; \tag{51}$$

the action of G is explicitly given by

$$g\cdot(z_i \otimes w) = z_j \otimes \rho(y)(w), \qquad \text{if } gz_i = z_j y, \qquad g \in G,\, i \in \mathbb{I},\, w \in W.$$

We define $\delta : M(\mathscr{O},\rho) \to \Bbbk G \otimes M(\mathscr{O},\rho)$ by

$$\delta(z_i \otimes w) = x_i \otimes (z_i \otimes w), \qquad\qquad i \in \mathbb{I},\, w \in W.$$

(In the formulation of Exercise 22, the grading is (51) with $\Bbbk z_i \otimes W$ in degree x_i). We prove the compatibility condition (45). Let $g \in G$, $i \in \mathbb{I}$, $w \in W$ and suppose that $gz_i = z_j y$ with $y \in G^x$. Then

$$\delta(g\cdot(z_i \otimes w)) = x_j \otimes (z_j \otimes \rho(y)(w)),$$
$$g(z_i \otimes w)_{(-1)} g^{-1} \otimes g\cdot(z_i \otimes w)_{(0)} = g x_i g^{-1} \otimes z_j(\otimes \rho(y)(w)).$$

and the first line equals the second because

$$g x_i g^{-1} = g z_i x z_i^{-1} g^{-1} = z_j y x y^{-1} z_j^{-1} = z_j x z_j^{-1} = x_j.$$

Using Exercise 22, we check that $M(\mathscr{O},\rho)$ is a simple Yetter-Drinfeld module. Clearly $\dim M(\mathscr{O},\rho) = |\mathscr{O}| \dim W$. Also it is easy to see that $M(\mathscr{O},\rho) \simeq M(\mathscr{O}',\rho')$ implies $\mathscr{O} = \mathscr{O}'$ and $\rho = \rho'$ (we have picked one element in each conjugacy class). Since ${}^{\Bbbk G}_{\Bbbk G}\mathscr{YD} \simeq \operatorname{Rep} D(\Bbbk G)$, we conclude that

$$\bigoplus_{\substack{\mathscr{O} \text{ conjugacy class} \\ \rho\in \operatorname{Irr} G^x}} \operatorname{End} M(\mathscr{O},\rho) \leq D(\Bbbk G). \tag{52}$$

Assume that $\Bbbk$ is algebraically closed and that $\operatorname{char}\Bbbk$ does not divide $|G|$. Then By a counting argument, see [15, p. 63], we see that the equality holds in (52); hence

- the category ${}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$ is semisimple and
- any simple object in ${}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$ is isomorphic to $M(\mathscr{O},\rho)$ for a unique $(\mathscr{O},\rho)$.

Finally, the braiding in $M(\mathscr{O},\rho)$ is given by

$$c((z_k \otimes v)\cdot(z_i \otimes w)) = x_k \cdot (z_i \otimes w) \otimes (z_k \otimes v) = z_j \otimes \rho(y)(w) \otimes (z_k \otimes v)$$

where $x_k z_i = z_j y$. Now, $x_j = z_j y \triangleright x = x_k z_i \triangleright x = x_k \triangleright x_i$. In other words, $M(\mathscr{O},\rho)$ is isomorphic to a braided vector space of rack type. Namely consider the rack $\mathscr{O}$; then the map $M(\mathscr{O},\rho) \to \Bbbk X \otimes W$, $z_i \otimes w \mapsto e_{x_i} \otimes w$ is an isomorphism of braided vector spaces, where $\mathfrak{q} : X \times X \to GL(W)$, $\mathfrak{q}_{x_i,x_j} = \rho(y)$.

3 Nichols algebras

Nichols algebras are a special kind of Hopf algebras in braided tensor categories. Our main interest is in Nichols algebras in the category ${}^H_H\mathscr{YD}$. We start with the definition of Hopf algebra in a braided tensor category; then we discuss the concept of Nichols algebra. Finally we overview several techniques to compute Nichols algebras. Throughout we refrain from using parentheses and the associativity constraints, as justified by Mac Lane coherence theorem.

3.1 Hopf algebras in braided tensor categories

Let $\mathscr{C}$ be a monoidal category. A *monoid* in $\mathscr{C}$ is a triple (M,μ,u), where $M \in \mathscr{C}$, $\mu : M \otimes M \to M$ and $u : \mathbf{1} \to M$ are morphisms in $\mathscr{C}$, such that the following diagrams commute:

$$\begin{tikzcd} M\otimes M\otimes M \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & M\otimes M \arrow[d, "\mu"] \\ M\otimes M \arrow[r, "\mu"] & M, \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1}\otimes M \arrow[r, "u\otimes\mathrm{id}"] \arrow[d, "l"'] & M\otimes M \arrow[dl, "\mu"'] \\ M & M\otimes\mathbf{1} \arrow[l, "r"] \arrow[u, "\mathrm{id}\otimes u"'] \end{tikzcd} \tag{53}$$

When $\mathscr{C}$ is actually a tensor category, it is also customary to say *algebra* in $\mathscr{C}$ instead of monoid in $\mathscr{C}$. Indeed, a monoid in $\mathrm{Vec}_\Bbbk$ is just an associative algebra over $\Bbbk$.

Example 25. Let G be a group. An algebra in $\mathrm{Vec}_\Bbbk^G$ is just a G-graded algebra.

The dual notion of *comonoid* in $\mathscr{C}$ is obtained reversing the arrows. That is, a comonoid is a triple (C,δ,ε), where $C \in \mathscr{C}$, $\delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbf{1}$ are morphisms in $\mathscr{C}$, such that the following diagrams commute:

$$\begin{tikzcd} C\otimes C\otimes C & C\otimes C \arrow[l, "\mathrm{id}\otimes\delta"'] \\ C\otimes C \arrow[u, "\delta\otimes\mathrm{id}"] & C, \arrow[l, "\delta"] \arrow[u, "\delta"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1}\otimes C \arrow[d, "l"'] & C\otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\ C \arrow[ur, "\delta"] & C\otimes\mathbf{1}. \arrow[l, "r"] \end{tikzcd} \tag{54}$$

When $\mathscr{C}$ is a tensor category, we say *coalgebra* instead of comonoid.

There are straightforward definitions of morphisms of monoids, and thus of the category of monoids in $\mathscr{C}$, and also of actions of monoids on objects of $\mathscr{C}$, and thus of the category of objects in $\mathscr{C}$ with action of a fixed monoid. However extra structure is needed to define the tensor product of two monoids.

Definition 5. Let $\mathscr{C}$ be a *braided* monoidal category. The tensor product of two monoids $M = (M, \mu_M, u_M)$ and $N = (N, \mu_N, u_N)$ in $\mathscr{C}$ is the monoid

$$M\underline{\otimes}N = (M \otimes N, \mu_{M\otimes N}, u_{M\otimes N}),$$

where $\mu_{M\otimes N}$ and $u_{M\otimes N}$ are defined by the following compositions:

$$\begin{tikzcd} M\otimes N\otimes M\otimes N \arrow[rr, "\mu_{M\otimes N}"] \arrow[dr, "\mathrm{id}_M \otimes c_{N,M}"'] & & M\otimes N, \\ & M\otimes M\otimes N\otimes N \arrow[ur, "\mu_M\otimes\mu_N"'] & \end{tikzcd}$$

$$\begin{tikzcd} M\otimes N \arrow[rr, "u_{M\otimes N}"] \arrow[dr, "u_M\otimes u_N"'] & & \mathbf{1}. \\ & \mathbf{1}\otimes\mathbf{1} \arrow[ur, "l"'] & \end{tikzcd}$$

Exercise 23. 1. Prove that the unit of a monoid is unique. Idem for the counit of a comonoid.
2. Prove that $M\underline{\otimes}N$ is a monoid, i.e. it satisfies (53).
3. Define the tensor product comonoid of two comonoids; show that it satisfies (54).
4. Let M be a monoid and C a comonoid in $\mathscr{C}$. Define the convolution product

$$* : \hom_{\mathscr{C}}(C,M) \times \hom_{\mathscr{C}}(C,M) \to \hom_{\mathscr{C}}(C,M), \qquad f * g = \mu(f \otimes g)\delta.$$

Prove that $*$ is associative and has unit $u\varepsilon$.

Definition 6. Let $\mathscr{C}$ be a braided tensor category. A bialgebra in $\mathscr{C}$ is a collection $(B, \mu, u, \Delta, \varepsilon)$ such that

- (B, μ, u) is an algebra (a monoid) in $\mathscr{C}$;
- (B, Δ, ε) is a coalgebra in $\mathscr{C}$;
- $\Delta : B \to B\underline{\otimes}B$ is a morphism of algebras.

A Hopf algebra in $\mathscr{C}$ is a bialgebra B such that the identity $\mathrm{id}_B \in \hom_{\mathscr{C}}(B,B)$ admits an inverse $\mathscr{S}$ for the convolution product $*$; i.e. there exists $\mathscr{S} \in \hom_{\mathscr{C}}(B,B)$ such that

$$\mathscr{S} * \mathrm{id}_B = \mathrm{id}_B * \mathscr{S} = u\varepsilon.$$

Example 26. Let H be a Hopf algebra with bijective antipode and $\mathscr{C} = {}^H_H\mathscr{Y}\mathscr{D}$. Let $V \in {}^H_H\mathscr{Y}\mathscr{D}$. Then the tensor algebra $T(V)$ is an algebra in ${}^H_H\mathscr{Y}\mathscr{D}$. Thus we may consider the algebra $T(V)\underline{\otimes}T(V)$, which is not the same as the algebra $T(V) \otimes T(V)$. For instance, if $y, u \in T(V)$, then the product in $T(V)\underline{\otimes}T(V)$ gives

$$(1\otimes y)(u\otimes 1) = (y_{(-1)}\cdot u)\otimes y_{(0)}.$$

By the universal property, since $T(V)\underline{\otimes}T(V)$ is in particular an associative algebra, there is a unique map $\underline{\Delta}: T(V)\to T(V)\underline{\otimes}T(V)$ such that $\underline{\Delta}(v) = v\otimes 1 + 1\otimes v$, $v\in V$. Notice that $\underline{\Delta}$ neither coincides with the Δ in Remark 2 nor with the Δ in §1.6.

Exercise 24. The tensor algebra $T(V)$ with the map $\underline{\Delta}: T(V)\to T(V)\underline{\otimes}T(V)$ defines a Hopf algebra in ${}^H_H\mathscr{YD}$ (define the antipode on $T^n(V)$ recursively on n).

Exercise 25. There is a coalgebra map $\mu: T^c(V)\underline{\otimes}T^c(V)\to T^c(V)$ determined by $\mu(v\otimes 1) = v = \mu(1\otimes v)$, $v\in V$; with this, $T^c(V)$ is a graded Hopf algebra in ${}^H_H\mathscr{YD}$.

We shall see plenty of examples of Hopf algebras in braided tensor categories, *aka* braided Hopf algebras. But before that, let us show how they appear in nature.

3.2 Bosonization

A basic result in group theory establishes an equivalence between the following two situations:

(a) $\pi: G\to L$ and $\iota: L\to G$ are morphisms of groups such that $\pi\iota = \mathrm{id}_L$.
(b) L and N are groups with L acting on N by group homomorphisms.

Namely, if (a) holds, then one takes $N = \ker\pi$; while if (b) holds, then $G\simeq N\rtimes L$. The situation is slightly more complicated when we consider the parallel setting for Hopf algebras; as we shall see, braided Hopf algebras appear in a natural way.

We start considering the situation:

(A) $\pi: A\to H$ and $\iota: H\to A$ are morphisms of Hopf algebras such that $\pi\iota = \mathrm{id}_H$.

It turns out that the right analogue of $\ker\pi$ in this setting is

$$R = A^{\mathrm{co}H} = \{a\in A: (\mathrm{id}\otimes\pi_H)\Delta(a) = a\otimes 1\}$$

(Un)fortunately, this is not a Hopf algebra, but, following Radford and Majid, see [73, 66], we claim that R is a braided Hopf algebra in ${}^H_H\mathscr{YD}$; explicitly, via

$$\begin{aligned} &h\cdot r = h_{(1)}r\mathscr{S}(h_{(2)}),\\ &r_{(-1)}\otimes r_{(0)} = \pi(r_{(1)})\otimes r_{(2)},\\ &R \text{ is a subalgebra of } A,\\ &\Delta_R(r) = r^{(1)}\otimes r^{(2)} = \vartheta_R(r_{(1)})\otimes r_{(2)}, \quad r\in R, h\in H. \end{aligned} \tag{55}$$

We leave the proof to the reader, who may find useful the map $\vartheta_R: A\to R$ given by

$$\vartheta_R(a) = a_{(1)}\iota\pi(\mathscr{S}(a_{(2)})), \qquad a\in A; \tag{56}$$

it satisfies $\quad \vartheta_R(rh) = r\varepsilon(h), \qquad \vartheta_R(hr) = h\cdot r, \quad r\in R, h\in H. \quad (57)$

It is tempting to guess that the situation (A) would be equivalent to

(B) H is a Hopf algebra and R is a braided Hopf algebra.

This is indeed the case; it remains to produce a Hopf algebra $R\#H$ from R and H, and this is done by a construction proposed by Radford, and interpreted in terms of braided categories by Majid; see [66, 73]. Concretely, $R\#H = R\otimes H$ as a vector space, so we use the notation $r\#h = r\otimes h$, $r\in R$, $h\in H$. This is a Hopf algebra by

$$\begin{aligned}(r\#h)(s\#f) &= r(h_{(1)}\cdot s)\#h_{(2)}f,\\ \Delta(r\#h) &= r^{(1)}\#(r^{(2)})_{(-1)}h_{(1)}\otimes(r^{(2)})_{(0)}\#h_{(2)}.\end{aligned} \tag{58}$$

We call $R\#H$ the bosonization of R (some authors say the Radford biproduct instead). We are back in situation (A) by the maps $\pi: R\#H\to H$ and $\iota: H\to R\#H$,

$$\pi(r\#h) = \varepsilon(r)h, \qquad \iota(h) = 1\#h, \qquad r\in R,\ h\in H.$$

Exercise 26. 1. Prove that $R\#H$ is a Hopf algebra with the structure (55), with antipode $\mathscr{S}_{R\#H}$ determined by

$$\mathscr{S}_{R\#H}(r) = \mathscr{S}(r_{(-1)})\mathscr{S}_R(r_{(0)}), \qquad r\in R. \tag{59}$$

Then $p_R: R\#H\to R$, $p_R(r\#h) = r\varepsilon(h)$, $r\in R$, $h\in H$, is a morphism of coalgebras.

2. Prove that $R = A^{\operatorname{co}H}$ is a Hopf algebra in ${}^H_H\mathscr{YD}$ with the structure (58). Along the way, prove that the antipode $\mathscr{S}_R$ of R is given by

$$\mathscr{S}_R(r) = r_{(-1)}\mathscr{S}(r_{(0)}), \qquad r\in R, \tag{60}$$

is a morphism in ${}^H_H\mathscr{YD}$ and is anti-multiplicative and anti-comultiplicative in the following sense:

$$\begin{aligned}\mathscr{S}_R\mu &= \mu(\mathscr{S}_R\otimes\mathscr{S}_R)c = \mu c(\mathscr{S}_R\otimes\mathscr{S}_R),\\ \Delta_R\mathscr{S}_R &= (\mathscr{S}_R\otimes\mathscr{S}_R)c\Delta_R = c(\mathscr{S}_R\otimes\mathscr{S}_R)\Delta_R,\end{aligned} \tag{61}$$

3. Let R be a Hopf algebra in ${}^H_H\mathscr{YD}$. The adjoint representation of R on itself is the linear map $\operatorname{ad}_c: R\to \operatorname{End}R$ given by

$$\operatorname{ad}_c x(y) = \mu(\mu\otimes\mathscr{S})(\operatorname{id}\otimes c)(\Delta\otimes\operatorname{id})(x\otimes y), \qquad x,y\in R.$$

Show that this is indeed an algebra map. Explicitly,

$$\operatorname{ad}_c x(y) = x^{(1)}[(x^{(2)})_{(-1)}\cdot y]\mathscr{S}((x^{(2)})_{(0)}) = \operatorname{ad}x(y), \qquad x,y\in R. \tag{62}$$

Show the the second equality (use (55) and the expression of the antipode). Let $\mathscr{P}(R) = \{x\in R: \Delta_R(x) = x\otimes 1 + 1\otimes x\}$, the space of primitive elements. Then

$$\operatorname{ad}_c x(y) = xy - (x_{(-1)}\cdot y)x_{(0)}, x\in\mathscr{P}(R), y \qquad \in R. \tag{63}$$

Hence $\mathscr{P}(R)$ is a Yetter-Drinfeld submodule of R. Using (61), show that

$$\operatorname{ad}_c x(\mathscr{S}_R(y)) = \mathscr{S}_R(\operatorname{ad}_{c^{-1}} x(y)), \qquad x \in \mathscr{P}(R), y \in R. \tag{64}$$

4. Let X be a Yetter-Drinfeld submodule of R. Then $\mathscr{S}_R(\Bbbk\langle X\rangle) = \Bbbk\langle \mathscr{S}_R(X)\rangle$.

3.3 Nichols algebras: definitions

We are now ready to address the main objective of this paper. Let H be a Hopf algebra with bijective antipode.

Let $V \in {}^H_H\mathscr{YD}$; for simplicity of the exposition we assume that $\dim V < \infty$, although this is not needed in most places. Recall that the tensor algebra $T(V)$ and the tensor coalgebra $T^c(V)$ are Hopf algebras in ${}^H_H\mathscr{YD}$, see Exercises 24 and 25. By the universal property of the tensor algebra, there is a morphism of algebras

$$\Omega : T(V) \to T^c(V) \qquad \text{such that} \qquad \Omega(v) = v, \qquad \text{for all } v \in V. \tag{65}$$

It is not difficult to see that Ω

- is a morphism in ${}^H_H\mathscr{YD}$,
- it preserves the coalgebra structure,
- it preserves the grading.

Indeed all properties follow because they hold at the level of V. In short, Ω is a morphism of graded Hopf algebras in ${}^H_H\mathscr{YD}$. We denote

$$\Omega_n = \Omega_{|T^n(V)}; \qquad \text{hence} \qquad \Omega = \sum_n \Omega_n.$$

Definition 7. The Nichols algebra $\mathscr{B}(V)$ is the image of the map Ω.

Let $\mathscr{J}(V) := \ker \Omega$; then $\mathscr{J}(V) = \oplus_{n\geq 2} \mathscr{J}^n(V)$, where $\mathscr{J}^n(V) = \ker \Omega^n$. Then

$$\mathscr{B}(V) = \oplus_{n\geq 0} \mathscr{B}^n(V) \simeq T(V)/\mathscr{J}(V), \qquad \mathscr{B}^n(V) \simeq T^n(V)/\mathscr{J}^n(V).$$

We give now a first alternative description of $\mathscr{J}(V)$. Recall the representation $\rho_n : \mathbb{B}_n \to GL(V^{\otimes n})$ of the braid group $\mathbb{B}_n$, cf. (12). Recall also the Matsumoto section $M : \mathbb{S}_n \to \mathbb{B}_n$, cf. (5).

Proposition 4. *If $n \geq 2$, then*

$$\Omega_n = \sum_{\sigma \in \mathbb{S}_n} \rho_n(M(\sigma)) \in End(V^{\otimes n}). \tag{66}$$

In particular, the algebra and the coalgebra structures of $\mathscr{B}(V)$ depend on the braided vector space (V, c) but not on the specific realization in ${}^H_H\mathscr{YD}$.

For instance, write $c_1 = c\otimes \mathrm{id}$, $c_2 = \mathrm{id}\otimes c$. Then

$$\Omega_2 = \mathrm{id} + c, \qquad\qquad \Omega_3 = \mathrm{id} + c_1 + c_2 + c_1c_2 + c_2c_1 + c_1c_2c_1.$$

Here is an abstract characterization of Nichols algebras.

Proposition 5. *[19] The ideal $\mathscr{J}(V)$ is maximal in the set*

$$\mathfrak{C} := \{J = \oplus_{n\geq 2} J^n \text{ is a graded Hopf ideal and Yetter-Drinfeld submodule of } T(V)\}.$$

Let $\mathscr{B} = \oplus_{n\geq 0}\mathscr{B}^n$, $\mathscr{E} = \oplus_{n\geq 0}\mathscr{E}^n$ be graded Hopf algebras in ${}^H_H\mathscr{YD}$ such that

$$\mathscr{B}^1 \simeq \mathscr{E}^1 \simeq V \text{ in } {}^H_H\mathscr{YD}.$$

Assume that $\mathscr{B}$ satisfies $\dim \mathscr{B}^n < \infty$. Then the graded dual of $\mathscr{B}$, denoted by $\mathscr{B}^\star$, is again a graded Hopf algebra in ${}^H_H\mathscr{YD}$, see (6).

Definition 8. 1. If $\mathscr{B} = \Bbbk\langle V\rangle$, then we say that $\mathscr{B}$ is a *pre-Nichols algebra*. By definition, there is a surjective map $T(V) \to \mathscr{B}$ of graded Hopf algebras in ${}^H_H\mathscr{YD}$; but the kernel of this map is contained in $\mathscr{J}(V)$ by Proposition 5, so that there is also a surjective map $\mathscr{B} \to \mathscr{B}(V)$ of graded Hopf algebras in ${}^H_H\mathscr{YD}$.

2. We say $\mathscr{E}$ is a *post-Nichols algebra* if it is coradically graded. Dually, there are injective maps $\mathscr{B}(V) \to \mathscr{E} \to T^c(V)$ of graded Hopf algebras in ${}^H_H\mathscr{YD}$.

Indeed, $\mathscr{B}$ is a pre-Nichols algebra (of V) if and only if $\mathscr{B}^\star$ is a post-Nichols algebra (of V^*); here we need that $\dim V < \infty$. Thus, for $\mathscr{B}$ a pre-Nichols, and $\mathscr{E}$ a post-Nichols, algebra, the situation can be summarized by the following commutative diagram:

$$\begin{tikzcd}
T(V) \arrow[rr, two heads] \arrow[rrrr, bend left, "\Omega"] \arrow[dr, two heads] & & \mathscr{B}(V) \arrow[rr, hook] \arrow[dr, hook] & & T^c(V) \\
& \mathscr{B} \arrow[ur, two heads, "\pi"'] & & \mathscr{E} \arrow[ur, hook] &
\end{tikzcd}$$

The next characterization is a natural consequence of this discussion, see [19].

Proposition 6. *The graded Hopf algebra $\mathscr{B}$ is isomorphic to $\mathscr{B}(V)$ if and only if*

1. it is generated as an algebra by V, $\mathscr{B} = \Bbbk\langle V\rangle$,

2. it is coradically graded.

We summarize the characterizations, or alternative definitions of the Nichols algebra $\mathscr{B}(V)$, or equivalently the defining ideal $\mathscr{J}(V) = \oplus_{n\geq 2}\mathscr{J}^n(V)$:

- $\mathscr{B}(V)$ =image of $\Omega : T(V) \to T^c(V)$. Thus, $\mathscr{J}(V) = \ker\Omega$.

- $\Omega = \sum_{n\geq 2}\Omega_n$, $\Omega_n = \sum_{\sigma\in\mathbb{S}_n}\rho_n(M(\sigma))$. Thus $\mathscr{J}^n(V) = \ker\sum_{\sigma\in\mathbb{S}_n}\rho_n(M(\sigma))$.
- $\mathscr{J}(V)$ is maximal in the class $\mathfrak{C}$ of graded Hopf ideals as in Proposition 5.
- $\mathscr{B}(V)$ is the only graded Hopf algebra both coradically graded and generated in degree 1 (by V). That is the only pre- and simultaneously post-Nichols algebra of V (up to isomorphisms).

There is a useful criterion with skew derivations to find relations of $\mathscr{B}(V)$, $V \in {}^H_H\mathscr{YD}$; see e.g. [16] for details. Let $f \in V^*$. Let $\partial_f \in \operatorname{End} T(V)$ be given by

$$\partial_f(1) = 0, \qquad \partial_f(v) = f(v), \quad v \in V, \tag{67}$$

$$\partial_f(xy) = x\partial_f(y) + \sum_j \partial_{f_j}(x)y_j, \qquad \text{where } c^{-1}(y\otimes f) = \sum_j f_j \otimes y_j. \tag{68}$$

Here is the criterion:

- Let $x \in T^n(V)$, $n \geq 2$. If $\partial_f(x) = 0$ for all $f \in V^*$, then $x \in \mathscr{J}^n(V)$.

Suppose that there are a basis $(x_i)_{i\in\mathbb{I}}$ of V, with dual basis $(f_i)_{i\in\mathbb{I}}$, and a family $(g_i)_{i\in\mathbb{I}}$ in $G(H)$ such that $\delta(x_i) = g_i \otimes x_i$, for $i \in \mathbb{I}$. Set $\partial_i = \partial_{f_i}$, $i \in \mathbb{I}$. Then (68) for all f is equivalent to

$$\partial_i(xy) = x\partial_i(y) + \partial_i(x)\, g_i \cdot y, \qquad x, y \in T(V), \qquad i \in \mathbb{I}. \tag{69}$$

The preceding arguments are the gate to the applications of Nichols algebras to the classification of pointed Hopf algebras [19, 20, 4], see also [10]. Indeed, let A be a pointed Hopf algebra and let $\operatorname{gr} A$ be the graded coalgebra associated to the coradical filtration. Then

$$\operatorname{gr} A \simeq \mathscr{R}\#\Bbbk G(A), \quad \text{where } \mathscr{R} = \oplus_{n\geq 0}\mathscr{R}^n \text{ is a graded Hopf algebra in } {}^H_H\mathscr{YD}.$$

Set $V = \mathscr{R}^1$. Now $\mathscr{R}$ is coradically graded, as it arises from the coradical filtration, in short it is a post-Nichols algebra of V; while its subalgebra generated by V is isomorphic to $\mathscr{B}(V)$. This leads to following problems:

- When is $\dim\mathscr{B}(V) < \infty$? For such V, classify its finite-dimensional post-Nichols algebras.
- When is the Gelfand-Kirillov dimension of $\mathscr{B}(V)$ finite? For such V, classify its post-Nichols algebras with finite Gelfand-Kirillov dimension.

We do not need only the list of all V positively answering these questions, but also we need to compute the Nichols algebras explicitly. By this we mean:

- Find a basis of $\mathscr{B}(V)$, hence the dimension or the Gelfand-Kirillov dimension, and
- describe the defining relations, i.e. a minimal set of generators of the ideal $\mathscr{J}(V)$.

Analogously for the mentioned post-Nichols algebras.

Needless to say, there is no hope presently to solve in full generality these problems. Towards the first question above, it was proposed in [18]:

Conjecture 1. Assume that $\operatorname{char}\Bbbk = 0$ and that H is semisimple. Let $V \in {}^H_H\mathscr{YD}$ such that $\dim \mathscr{B}(V) < \infty$. Then there is no finite-dimensional post-Nichols algebra except $\mathscr{B}(V)$ itself.

The conjecture contains the following particular case:

Conjecture 2. Assume that $\operatorname{char}\Bbbk = 0$. Then every finite-dimensional pointed Hopf algebra is generated by group-like and skew-primitive elements.

The last Conjecture is definitely false if either $\operatorname{char}\Bbbk > 0$ or else the finite-dimensional requirement is dropped.

Theorem 3. *[22] Assume that* $\operatorname{char}\Bbbk = 0$*. Then every finite-dimensional pointed Hopf algebra* with abelian group of group-likes *is generated by group-like and skew-primitive elements.*

3.4 Nichols algebras: techniques

Here we discuss approaches to compute Nichols algebras.

3.4.1 Direct computation

First, let $m \in \mathbb{N}_{\geq 2}$. The m-th approximation of $\mathscr{B}(V)$ is

$$\widehat{\mathscr{B}}_m(V) = T(V)/\langle \oplus_{2\leq n\leq m} \mathscr{J}^n(V)\rangle = T(V)/\Big\langle \oplus_{2\leq n\leq m} \ker\big(\sum_{\sigma\in\mathbb{S}_n} \rho_n(M(\sigma))\big)\Big\rangle,$$

cf. (66). By definition, there is an epimorphism of graded Hopf algebras

$$\pi_m : \widehat{\mathscr{B}}_m(V) \to \mathscr{B}(V).$$

A brutal approach would be to compute $\widehat{\mathscr{B}}_m(V)$ for $m = 2,3,\dots$ and at each step try to figure out whether π_m is an isomorphism, using some of the characterizations of $\mathscr{B}(V)$. In principle, $\mathscr{J}^2(V) = \ker(\mathrm{id} + c)$ is effectively computable, but the difficulties mount with m, as Ω_n is the sum of $n!$ terms acting on a vector space of dimension $(\dim V)^n$. Other drawbacks are that the ideal $\mathscr{J}(V)$ need not be finitely generated, nor have quadratic relations at all; even to predict the lowest degree relations is not within reach. A variation of this approach would be:

- Find a set R_1 of relations in $\mathscr{J}(V)$; i.e. $\mathscr{J}^2(V)$ or some relations of small order.
- Compute the pre-Nichols algebra $\mathscr{B}_1 = T(V)/\langle R_1\rangle$, i.e. find a basis B_1 of $\mathscr{B}_1$.
- Decide whether the image of B_1 in $\mathscr{B}(V)$ is linearly independent (here derivations are the best option). If yes, then $\mathscr{B}(V) \simeq \mathscr{B}_1$.
- If no, then we would have found a new set of relations R_2; set $\mathscr{B}_2 = \mathscr{B}_1/\langle R_1\rangle$ and start again.

Of course the success of this approach depends on great doses of intuition–and luck.

Exercise 27. Let $q \in \Bbbk^\times$ and let $\mathscr{A} = \mathbb{Z}[\mathbf{q}]$ be the polynomial algebra.

1. Let $(n)_{\mathbf{q}} = 1 + \mathbf{q} + \cdots + \mathbf{q}^{n-1}$ and $(n)!_{\mathbf{q}} = (n)_{\mathbf{q}} \dots (2)_{\mathbf{q}} (1)_{\mathbf{q}} \in \mathscr{A}$. The $\mathbf{q}$-binomial numbers are
$$\binom{n}{i}_{\mathbf{q}} := \frac{(n)!_{\mathbf{q}}}{(n-i)!_{\mathbf{q}} (i)!_{\mathbf{q}}}, \qquad n \in \mathbb{N}, i \in \mathbb{I}_{0,n}.$$
Prove that
$$\mathbf{q}^k \binom{n}{k}_{\mathbf{q}} + \binom{n}{k-1}_{\mathbf{q}} = \binom{n+1}{k}_{\mathbf{q}}, \qquad k \in \mathbb{I}_n. \tag{70}$$
Conclude that $\binom{n}{i}_{\mathbf{q}} \in \mathscr{A}$. Let $\binom{n}{i}_{q} \in \Bbbk$ be the specialization of $\binom{n}{i}_{\mathbf{q}}$ at q.
2. Let A be an associative algebra; let $u, v \in A$ be q-commuting elements, i.e. $uv = qvu$. Then the *quantum binomial formula* holds:
$$(u+v)^n = \sum_{i=0}^{n} \binom{n}{i}_q v^i u^{n-i}, \qquad \text{for every } n \in \mathbb{N}. \tag{71}$$
3. Let V be a braided vector space of dimension 1 with braiding $c(x \otimes x) = qx \otimes x$ for all $x \in V$. Fix $x \in V - 0$. Let
$$N := \begin{cases} \operatorname{ord} q, & \text{if } q \in \mathbb{G}'_\infty, \\ 1, & \text{otherwise.} \end{cases}$$
Prove that $x^N \in \mathscr{P}(T(V))$. Conclude that
$$\mathscr{B}(V) \simeq \begin{cases} T(V)/\langle x^N \rangle, & \text{if } q \in \mathbb{G}'_\infty, \\ T(V), & \text{otherwise.} \end{cases}$$

Example 27. [17] Let $\mathfrak{q} = (q_{ij}) \in (\Bbbk^\times)^{\mathbb{I} \times \mathbb{I}}$. Let V be a braided vector space with basis $(x_i)_{i \in \mathbb{I}}$ and braiding (15); (16) is not assumed. Let $N_k := \begin{cases} \operatorname{ord} q_{kk}, & \text{if } q_{kk} \in \mathbb{G}'_\infty, \\ 1, & \text{otherwise,} \end{cases}$ for $k \in \mathbb{I}$. Suppose that
$$q_{ij} q_{ji} = 1, \qquad \text{for all } i \neq j \in \mathbb{I}.$$
It is easy to check that
$$x_i x_j - q_{ij} x_j x_i \in \mathscr{J}^2(V), \qquad \text{for all } i \neq j \in \mathbb{I}. \tag{72}$$

Then

$$\mathscr{B} = T(V)/\langle x_i x_j - q_{ij} x_j x_i,\, i \neq j \in \mathbb{I}; \quad x_k^{N_k},\, q_{kk} \in \mathbb{G}'_\infty \rangle \tag{73}$$

is a pre-Nichols algebra of V. Using linear algebra arguments, one may check that $\mathscr{B} \simeq \mathscr{B}(V)$ and that

$$\{x_1^{a_1} x_2^{a_2} \dots x_\theta^{a_\theta} : 0 \leq a_k \leq N_k, \text{ if } q_{kk} \in \mathbb{G}'_\infty; \quad 0 \leq a_k \text{ otherwise } \} \text{ is a basis of } \mathscr{B}(V).$$

Definition 9. The algebra presented by generators and relations as in the right-hand side of (73) is called a *quantum linear space*.

Notice that there are examples of quantum planes (quantum linear spaces with $\theta = 2$) that are braided Hopf algebras with respect to braidings not of diagonal type; this was first noticed in [38].

Example 28. [14] Let (V, c) be a braided vector space. Assume that

- $\dim V = 2$,
- $\mathscr{J}^2(V) \neq 0$,
- c is not of diagonal type.

Then $\mathscr{B}(V)$ is known. The starting point is the classification of braided vector spaces of dimension 2 [55]. The outcome is that, as algebras, the examples arising are variations of quantum planes, variations of the Jordan and super Jordan algebras (see §4.3) and some strange examples.

3.4.2 Dual

Let $V \in {}^H_H\mathscr{YD}$ finite dimensional. As we observed after Definition 8, the graded dual of a pre-Nichols of V is a post-Nichols algebra of V^* and vice versa, thus we have:

Remark 5. The graded dual of $\mathscr{B}(V)$ is isomorphic to $\mathscr{B}(V^*)$.

This gives some new information without extra effort, as V^* need not be isomorphic to V as braided vector space. For instance, let $X = (X, \triangleright)$ be a rack and $\mathfrak{q}$ a 2-cocycle as in (27). Let $X^{-1} = (X, \triangleright^{-1})$, where $x \triangleright^{-1} y = \phi_x^{-1}(y)$, cf. Exercise 8. Let also $\mathfrak{q}^* : X^{-1} \times X^{-1} \to \mathbb{k}^\times$ given by

$$\mathfrak{q}^*_{xy} = \mathfrak{q}_{x, x \triangleright^{-1} y}, \qquad x, y \in X.$$

Then the braided vector space dual to $(\mathbb{k}X, c^{\mathfrak{q}})$, is $(\mathbb{k}X^{-1}, c^{\mathfrak{q}^*})$. See [39] for details.

3.4.3 Twisting

V. G. Drinfeld introduced in [29] the twisting of quasi-Hopf algebras, meaning conjugation of the comultiplication by a suitable element, to keep account of equivalences of tensor categories. This was specialized to Hopf algebras in [76], with the definition of multiparametric quantum groups as application. The dual version, called twisting of the multiplication, appeared in [27]. We recall this last one. Let H be a Hopf algebra.

Definition 10. A linear map $\phi : H \otimes H \to \Bbbk$ is a unitary 2-cocycle if

$$\phi \text{ is invertible with respect to the convolution product } *, \text{ see Exercise 23;} \tag{74}$$

$$\phi(x_{(1)} \otimes y_{(1)})\, \phi(x_{(2)} y_{(2)} \otimes z) = \phi(y_{(1)} \otimes z_{(1)})\, \phi(x \otimes y_{(2)} z_{(2)}), \tag{75}$$

$$\phi(x \otimes 1) = \phi(1 \otimes x) = \varepsilon(x), \tag{76}$$

for all $x, y, z \in H$. Let ϕ be a unitary 2-cocycle and define a new multiplication $\cdot_\phi$ in the vector space H by

$$x \cdot_\phi y = \phi(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} \phi^{-1}(x_{(3)}, y_{(3)}), \qquad x, y \in H.$$

Then $H_\phi = (H, \cdot_\phi, \Delta)$ is a Hopf algebra.

Exercise 28. Let G be a group. A unitary 2-cocycle on $\Bbbk G$ is equivalent to a 2-cocycle $\phi \in Z^2(G, \Bbbk^\times)$, i. e. a map $\phi : G \times G \to \Bbbk^\times$ such that

$$\phi(g,h)\phi(gh,t) = \phi(h,t)\phi(g,ht), \quad \phi(g,e) = \phi(e,g) = 1, \quad g,h,t \in G. \tag{77}$$

The relation with bosonization was established in [68].

Theorem 4. *[68, Theorem 2.7, Corollary 3.4] Let $\phi : H \otimes H \to \Bbbk$ be an invertible unitary 2-cocycle.*

(a) There exists an equivalence of braided categories $\mathscr{T}_\phi : {}^H_H\mathscr{YD} \to {}^{H_\phi}_{H_\phi}\mathscr{YD}$, $V \mapsto V_\phi$, which is the identity on the underlying vector spaces, morphisms and coactions, and transforms the action of H on V to $\cdot_\phi : H_\phi \otimes V_\phi \to V_\phi$,

$$h \cdot_\phi v = \phi(h_{(1)}, v_{(-1)})(h_{(2)} \cdot v_{(0)})\_0\, \phi^{-1}((h_{(2)} \cdot v_{(0)})\_{-1}, h_{(3)}),$$

$h \in H_\phi$, $v \in V_\phi$. The monoidal structure on $\mathscr{T}_\phi$ is given by the natural transformation $b_{V,W} : (V \otimes W)_\phi \to V_\phi \otimes W_\phi$

$$b_{V,W}(v \otimes w) = \phi(v_{(-1)}, w_{(-1)}) v\_0 \otimes w\_0, \quad v \in V, w \in W.$$

(b) $\mathscr{T}_\phi$ preserves Nichols algebras: $\mathscr{B}(V)_\phi \simeq \mathscr{B}(V_\phi)$ as objects in ${}^{H_\phi}_{H_\phi}\mathscr{YD}$. In particular, the Hilbert-Poincaré series of $\mathscr{B}(V)$ and $\mathscr{B}(V_\phi)$ are the same.

Example 29. Let $\mathfrak{q} = (q_{ij}), \mathfrak{q}' = (q'_{ij}) \in (\Bbbk^\times)^{\mathbb{I} \times \mathbb{I}}$ satisfying (16). We say that $\mathfrak{q}$ and $\mathfrak{q}'$ are twist-equivalent if

$$q_{ii} = q'_{ii}, \qquad i \in \mathbb{I} \qquad \text{and} \qquad q_{ij} q_{ji} = q'_{ij} q'_{ji}, \qquad i \neq j \in \mathbb{I}.$$

In other words, twist-equivalent means that the matrices $\mathfrak{q}$ and $\mathfrak{q}'$ hace the same Dynkin diagram, cf. §2.1.3. Let V and V' be the braided vector spaces of diagonal type associated to $\mathfrak{q}$ and $\mathfrak{q}'$, respectively. If $\mathfrak{q}$ and $\mathfrak{q}'$ are twist-equivalent, then the Hilbert-Poincaré series of $\mathscr{B}(V)$ and $\mathscr{B}(V')$ coincide; this consequence of Theorem 4 was observed in [19, Proposition 3.9].

Example 30. Let X be a rack (isomorphic to a conjugacy class in a finite group) and let $\mathfrak{q}$ and $\mathfrak{q}'$ be 2-cocycles on X. We say that $\mathfrak{q}$ and $\mathfrak{q}'$ are twist-equivalent if there exists $\phi : X \times X \to \Bbbk^\times$ such that $\mathfrak{q}' = \mathfrak{q}^\phi$, which is

$$q^\phi_{xy} = \phi(x,y)\phi^{-1}(x \triangleright y, x)\, q_{xy}, \quad x,y \in X. \tag{78}$$

If $\mathfrak{q}$ and $\mathfrak{q}'$ are twist-equivalent, then the Hilbert-Poincaré series of $\mathscr{B}(X,\mathfrak{q})$ and $\mathscr{B}(X,\mathfrak{q}')$ coincide [13, §3.4].

Exercise 29. Let X be a rack, $\mathfrak{q}$ a 2-cocycle on X and $\phi : X \times X \to \Bbbk^\times$. Show that $\mathfrak{q}^\phi$ defined by (78) is a 2-cocycle iff for any $x,y,z \in X$, we have

$$\begin{aligned}\phi(x,z)\phi(x \triangleright y, x \triangleright z)\phi(x \triangleright (y \triangleright z), x)\phi(y \triangleright z, y)&\\ = \phi(y,z)\phi(x, y \triangleright z)\phi(x \triangleright (y \triangleright z), x \triangleright y)\phi(x \triangleright z, x)&\end{aligned} \tag{79}$$

Hence, if X is a subrack of a group G and $\phi \in Z^2(G, \Bbbk^\times)$, then $\phi|_{X\times X}$ satisfies (79).

3.4.4 Discard

There are techniques to prove that a Nichols algebra has infinite dimension, or GK-dimension. Various of them are related to decompositions, as explained below. Let (V,c) be a braided vector space. We mention in this line of thought:

- If $W \hookrightarrow V$ is a braided subspace, respectively $V \twoheadrightarrow W$ is a quotient braided space, then $\mathscr{B}(W) \hookrightarrow \mathscr{B}(V)$, respectively $\mathscr{B}(V) \twoheadrightarrow \mathscr{B}(W)$. Thus, if V has a braided subspace or a braided quotient whose Nichols algebra has infinite dimension, (or GK-dimension), then so has $\mathscr{B}(V)$. There are elaborations of these arguments specific to rack type that are evoked below.
- Assume that V has a filtration of braided subspaces: $0 = V_0 \subsetneq V_1 \cdots \subsetneq V_d = V$. Then this filtration propagates to $\mathscr{B}(V)$ and the associated $\operatorname{gr}\mathscr{B}(V)$ turns out to be a pre-Nichols algebra of $\operatorname{gr} V$. Thus, if $\mathscr{B}(\operatorname{gr} V)$ has infinite dimension, (or GK-dimension), then so has $\mathscr{B}(V)$.

3.4.5 Decomposition

Let $\theta \geq 2$. Assume that (V,c) satisfies

$$V = V_1 \oplus \cdots \oplus V_\theta, \qquad c(V_i \otimes V_j) = V_j \otimes V_i, \qquad i,j \in \mathbb{I}_\theta. \tag{80}$$

Here we suppose that the $\mathscr{B}(V_i)$'s are known and seek to infer $\mathscr{B}(V)$. This idea, mentioned in passing in [1, p. 41], is a roundabout approach, where instead of computing the relations or the basis, one looks for combinatorial invariants reminiscent of the Weyl group. The principal actors are the maps

$$c_{ij} := c_{|V_i\otimes V_j} : V_i \otimes V_j \to V_j \otimes V_i, , \qquad\qquad i, j \in \mathbb{I}_\theta.$$

Exercise 30. [41] If $\theta = 2$ and $c_{21}c_{12} = \mathrm{id}_{V_1\otimes V_2}$, then $\mathscr{B}(V) \simeq \mathscr{B}(V_1)\underline{\otimes}\mathscr{B}(V_2)$.

Here are some particular instances of this situation:

- Assume that $\dim V_i = 1$ for all i. Then V is of diagonal type, up to (16); cf. §4.2.
- Assume that there exists a Hopf algebra H such that $V_i \in {}^H_H\mathscr{YD}$ and is irreducible in this category. This setting was considered in [16, 50].
- See §4.5 for the case $H = \Bbbk G$, G a finite group.
- Assume that either $\dim V_i = 1$ or V_i is a block. Then the classification of all V such that GK-dim $\mathscr{B}(V) < \infty$ was obtained in [4], see §4.4. Here is a crucial remark that should be useful in other settings:

Assume that $\theta = 2$ but that either V_1 or V_2 is not irreducible, or both. The combinatorial invariants from [16, 49] are not available but we may proceed as follows. There are natural morphisms of braided Hopf algebras

$$\pi : \mathscr{B}(V) \to \mathscr{B}(V_1), \qquad \iota : \mathscr{B}(V_1) \to \mathscr{B}(V), \qquad \text{such that } \pi\iota = \mathrm{id}_{\mathscr{B}(V_1)}.$$

As in §3.2, we consider $K = \mathscr{B}(V)^{\mathrm{co}\,\mathscr{B}(V_1)}$, but now in ${}^H_H\mathscr{YD}$. Remarkably,

$$\mathscr{B}(V) \simeq K\#\mathscr{B}(V_1) \qquad \text{and} \qquad K \simeq \mathscr{B}\left(\mathrm{ad}_c\,\mathscr{B}(V_1)(V_2)\right),$$

[50, Proposition 8.6], cf. also [16, Lemma 3.2]. Ingenuously, one may try to compute the Nichols algebra of $\mathrm{ad}_c\,\mathscr{B}(V_1)(V_2)$; at a first glance this appears more complicated, but sometimes this works.

4 Classes of Nichols algebras

We discuss in the last Section of this paper several classes of Nichols algebras. From now on $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk = 0$.

4.1 Symmetries and Hecke type

Here the situation is quite simple:

Proposition 7. *Let (V, c) be a braided vector space such that c is either a symmetry or of Hecke type with label $q \notin \mathbb{G}_\infty$. Then $\mathscr{B}(V) \simeq T(V)/\langle\ker(c+\mathrm{id})\rangle$.*

See [19, Proposition 3.4]; the argument is taken from a paper by Andrés Abella and the author. By [42], it follows that $\mathscr{B}(V)$ is a Koszul algebra, see *loc. cit.*

4.2 Diagonal type

Nichols algebras of diagonal type were studied in depth. In the finite-dimensional setting, there are two main results:

- The classification of all finite-dimensional Nichols algebras of diagonal type appears in [45], using the Weyl groupoid introduced in [44].
- The defining relations of the finite-dimensional Nichols algebras of diagonal type appear in [21, 22].

We refer to the survey [3] for details, since both answers are very long and require a careful preparation. One of the outcomes is that the theory of Nichols algebras of diagonal type embeds into Lie theory. Here is a remarkable instance of this affirmation:

Theorem 5. *[18, 44] Let V be a braided vector space of Cartan type with Cartan matrix A, see §2.1.3. Then* $\dim \mathscr{B}(V) < \infty$ *if and only if A is a finite Cartan matrix (i.e. corresponds to a finite dimensional simple Lie algebra).*

This result was proved in [18] under some restrictions on the matrix $\mathfrak{q}$ of the braiding, by reduction to the theory of quantum groups. A proof valid without restrictions appears in [44] based on the beautiful theory of the Weyl groupoid.

As for finite Gelfand-Kirillov dimension, the validity of the following Conjecture would say that the classification follows from [45]. Let (V,c) be a braided vector space of diagonal type.

Conjecture 3. [4] If GK-dim $\mathscr{B}(V) < \infty$, then its Weyl groupoid is finite.

The following partial results support the Conjecture.

Theorem 6. *[5] If either its Weyl groupoid is infinite and* $\dim V = 2$*, or else V is of affine Cartan type, then GK-dim $\mathscr{B}(V) = \infty$.*

Let us finally discuss an example with many applications.

Example 31. Let V be a braided vector space of dimension 2, of diagonal type with braiding matrix $\mathfrak{q} = \begin{pmatrix} q & q \\ q & q \end{pmatrix}$.

- The case $q = 1$ is not of diagonal type, strictly speaking, by our requirement (16). Nevertheless, $\mathscr{B}(V) \simeq S(V)$.
- If $q = -1$, then $\mathscr{B}(V) \simeq \Lambda(V)$.
- If $q \in \mathbb{G}'_N$, then V is of Cartan type $\begin{pmatrix} 2 & 2-N \\ 2-N & 2 \end{pmatrix}$. Thus, if $N = 3$, then is of Cartan type A_2 and $\dim \mathscr{B}(V) = 27$.
- If $N > 3$, then GK-dim $\mathscr{B}(V) = \infty = \dim \mathscr{B}(V)$ by Theorems 6, respectively 5.

4.3 Triangular type

Here we give a glimpse to the main results in [4] on Nichols algebras with finite GK-dimension over an abelian group. Succinctly, these results consist of

- The classification of braided vector spaces whose Nichols algebras have finite GK-dim, that admit a decomposition (80) whose components are ± 1-blocks or points, i.e. $V = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} \oplus \cdots \oplus V_\theta$ where

  $$V_h \quad \varepsilon_h\text{-block}, \quad \varepsilon_h^2 = 1,\ h \in \mathbb{I}_t; \qquad V_i \quad q_{ii}\text{-point}, \quad q_{ii} \in \Bbbk^\times,\ i \in \mathbb{I}_{t+1,\theta},$$

  with $c(V_i \otimes V_j) = V_j \otimes V_i$, $i, j \in \mathbb{I}_\theta$. Set as usual $c_{ij} = c_{|V_i \otimes V_j}$. We assume that

  – V is not of diagonal type, i.e. $t > 0$;
  – the braiding c_{ij} between a block $i \in \mathbb{I}_t$ and a point $j \in \mathbb{I}_{t+1,\theta}$ has the form (83).

- The classification of Yetter-Drinfeld modules over abelian groups whose Nichols algebras have finite GK-dim, that admit a decomposition $V = V_1 \oplus V_2$ like (80) where V_1 is a ± 1-block and V_2 is a point, but c_{12} does not have the form (83).

We point out that the first classification mentioned assumes the validity of Conjecture 3. The explicit formulation requires some preparation, so we refer to [5] for full details. Instead we discuss here two relevant steps of the proof–steps that do not require Conjecture 3.

To start with, recall the block $\mathscr{V}(\varepsilon, \ell)$, $\varepsilon \in \Bbbk^\times$ and $\ell \in \mathbb{N}_{\geq 2}$, cf. Example 6.

Theorem 7. *[4, Theorem 1.2, Propositions 3.4, 3.5] The Gelfand-Kirillov dimension of the Nichols algebra $\mathscr{B}(\mathscr{V}(\varepsilon, \ell))$ is finite if and only if $\ell = 2$ and $\varepsilon^2 = 1$.*

The algebras $\mathscr{B}(\mathscr{V}(\varepsilon, 2))$ have GK-dim 2 and are presented by generators x_1 and x_2 with defining relations

$$x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, \qquad \text{if } \varepsilon = 1; \tag{81}$$

$$x_2x_{21} - x_{21}x_2 - x_1x_{21}, \qquad x_1^2, \qquad \text{if } \varepsilon = -1, \tag{82}$$

where $x_{21} = x_2x_1 + x_1x_2$.

This result explains why we restrict to ± 1-blocks (recall that these means also dimension 2).

Next we turn our attention to the setting *one block plus one point*, i.e. braided vector spaces $V = V_1 \oplus V_2$, where V_1 has a basis $(x_i)_{i \in \mathbb{I}_2}$, V_2 has a basis (x_3). Our key hypothesis is that the braiding has the shape

$$(c(x_i \otimes x_j))_{i,j \in \mathbb{I}_3} = \begin{pmatrix} \varepsilon x_1 \otimes x_1 & (\varepsilon x_2 + x_1) \otimes x_1 & q_{12}x_3 \otimes x_1 \\ \varepsilon x_1 \otimes x_2 & (\varepsilon x_2 + x_1) \otimes x_2 & q_{12}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + ax_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}, \tag{83}$$

with $\varepsilon^2 = 1$ and $q_{ij} \in \Bbbk^{\times}$, $i, j \in \mathbb{I}_2$. We do not want to have $c^2_{|V_1 \otimes V_2} \overset{\star}{=} \mathrm{id}$ because we know the answer, see Exercise 30; here $\star$ is equivalent to

$$q_{12}q_{21} = 1 \text{ and } a = 0.$$

So $c^2_{|V_1 \otimes V_2}$ is determined by $q_{12}q_{21}$, that we call the *interaction*, and a, of which we consider a normalized version, that we call the *ghost*: $\mathscr{G} = \begin{cases} -2a, & \varepsilon = 1, \\ a, & \varepsilon = -1. \end{cases}$ If $\mathscr{G} \in \mathbb{N}$, then we say that the ghost is *discrete*.

Theorem 8. *[4] Let V be a braided vector space with braiding* (83). *Then GK-dim $\mathscr{B}(V) < \infty$, if and only if the ghost is discrete and V is as in Table 1.*

The meaning of the diagrams are:

- $\boxplus$, respectively $\boxminus$, says that V_1 is a 1-block, respectively a -1-block.
- The label over the point is q_{22}.
- The edge $\overset{\mathscr{G}}{\rule{2em}{0.4pt}}$ says that $q_{12}q_{21} = 1$; $\mathscr{G}$ is discrete but arbitrary unless explicitly given.
- The edge $\overset{(-1,1)}{\rule{2em}{0.4pt}}$ says that $q_{12}q_{21} = -1$ and $\mathscr{G} = 1$.

Table 1 Nichols algebras of a block and a point with finite GK-dim

V	diagram	GK-dim	generators and relations
$\mathfrak{L}(1,\mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\rule{2em}{0.4pt}} \overset{1}{\bullet}$	$\mathscr{G}+3$	$\Bbbk\langle x_1, x_2, x_3 \mid x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, x_1x_3 - q_{12}x_3x_1,$ $z_{1+\mathscr{G}}, z_tz_{t+1} - q_{21}q_{22}z_{t+1}z_t, 0 \le t < \mathscr{G}\rangle$
$\mathfrak{L}(-1,\mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\rule{2em}{0.4pt}} \overset{-1}{\bullet}$	2	$\Bbbk\langle x_1, x_2, x_3 \mid x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,$ $x_1x_3 - q_{12}x_3x_1, z_{1+\mathscr{G}}, z_t^2, 0 \le t \le \mathscr{G}\rangle$
$\mathfrak{L}_-(1,\mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\rule{2em}{0.4pt}} \overset{1}{\bullet}$	$\mathscr{G}+3$	$\Bbbk\langle x_1, x_2, x_3 \mid x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21},$ $x_1x_3 - q_{12}x_3x_1, x_{21}x_3 - q_{12}^2x_3x_{21}, z_{1+2\mathscr{G}},$ $z_{2k+1}^2, z_{2k}z_{2k+1} - q_{21}q_{22}z_{2k+1}z_{2k}, 0 \le k < \mathscr{G}\rangle$
$\mathfrak{L}_-(-1,\mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\rule{2em}{0.4pt}} \overset{-1}{\bullet}$	$\mathscr{G}+2$	$\Bbbk\langle x_1, x_2, x_3 \mid x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21}, x_3^2,$ $x_1x_3 - q_{12}x_3x_1, x_{21}x_3 - q_{12}^2x_3x_{21}, z_{1+2\mathscr{G}},$ $z_{2k}^2, z_{2k-1}z_{2k} - q_{21}q_{22}z_{2k}z_{2k-1}, 0 < k \le \mathscr{G}\rangle$
$\mathfrak{L}(\omega,1)$	$\boxplus \overset{1}{\rule{2em}{0.4pt}} \overset{\omega}{\bullet}$	2	$\Bbbk\langle x_1, x_2, x_3 \mid x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,$ $x_1x_3 - q_{12}x_3x_1, z_2, x_3^3, z_1^3, z_{1,0}^3\rangle$
$\mathfrak{C}_1$	$\boxminus \overset{(-1,1)}{\rule{2em}{0.4pt}} \overset{-1}{\bullet}$	2	$\Bbbk\langle x_1, x_2, x_3 \mid x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21},$ $x_3^2, f_0^2, f_1^2, z_1^2, x_{21}x_3 - q_{12}^2x_3x_{21},$ $x_2z_1 + q_{12}z_1x_2 - q_{12}f_0x_2 - \frac{1}{2}f_1\rangle$

We next deal with the following situation: $V = V_1 \oplus V_2$, where V_1 has dimension 2 and is of diagonal type, V_2 has dimension 1 but the braiding between them is not diagonal. Concretely, V is a braided vector space of dimension 3 with braiding given in the basis $(x_i)_{i\in\mathbb{I}_3}$, for some $\varepsilon, q_{ij} \in \Bbbk^\times$, $i,j\in\mathbb{I}_2$, by

$$(c(x_i\otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} \varepsilon x_1\otimes x_1 & \varepsilon x_2\otimes x_1 & q_{12}x_3\otimes x_1 \\ \varepsilon x_1\otimes x_2 & \varepsilon x_2\otimes x_2 & q_{12}x_3\otimes x_2 \\ q_{21}x_1\otimes x_3 & q_{21}(x_2+x_1)\otimes x_3 & q_{22}x_3\otimes x_3 \end{pmatrix}. \tag{84}$$

Theorem 9. *[4] Let V be as above. Then GK-dim $\mathscr{B}(V) = \infty$ if and only if $\varepsilon = -1$ and either of the following holds:*

1. $q_{12}q_{21} = 1$ *and* $q_{22} = \pm 1$*; in this case GK-dim* $\mathscr{B}(V) = 1$.
2. $q_{22} = -1 = q_{12}q_{21}$*; in this case GK-dim* $\mathscr{B}(V) = 2$.

In conclusion, let G be an abelian group and $V \in {}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$ of dimension 3 but not of diagonal type. Then GK-dim $\mathscr{B}(V) < \infty$ if and only if as a braided vector space, it has the shape (83) or (84), and is determined by Theorems 8 and 9.

4.4 Rack type, infinite dimension

From now on, any rack is assumed to be isomorphic to a conjugacy class of a finite group. The problem we deal with is:

For every finite rack X, every finite-dimensional vector space W and a every 2-cocycle $\mathfrak{q}: X\times X \to GL(W)$, see (27), compute the Nichols algebra $\mathscr{B}(X,\mathfrak{q}) := \mathscr{B}(V)$, where $V = \Bbbk X\otimes W$ and the braiding is given by (28).

Specifically, decide when $\dim \mathscr{B}(X,\mathfrak{q}) < \infty$ or GK-dim $\mathscr{B}(X,\mathfrak{q}) < \infty$.

This is an enormous task and we are far away from a complete answer.[2] Fortunately there are methods to reduce the problem. Before stating them, we make some comments.

Remark 6. Let X, W and $\mathfrak{q}$ as above. Suppose that Y is an abelian subrack. Then $U = \Bbbk Y\otimes W$ is a braided vector subspace of diagonal type of $(V, c^{\mathfrak{q}})$. Thus, if $\dim \mathscr{B}(Y, \mathfrak{q}_{|Y\times Y}) = \infty$, what can be verified from [45], then $\dim \mathscr{B}(X,\mathfrak{q}) = \infty$.

Remark 7. For every finite rack X and every finite-dimensional vector space W, we would need first to compute all 2-cocycles $\mathfrak{q}: X\times X\to GL(W)$, up to some natural equivalence. When $\dim W = 1$ and X is indecomposable, an explicit description of these 2-cocycles was given in [40].

[2] Technically, it is enough to assume that $\mathfrak{q}$ is faithful, what means that the map $X \to GL(V)$, $x \mapsto (e_y w \mapsto e_{x\triangleright y}\mathfrak{q}_{xy}(w))$ is injective, but we omit this requirement for an easier exposition. As well, for the classification of finite-dimensional pointed Hopf algebras, it is enough to assume that $\mathfrak{q}$ is finite, i.e. that its image is contained in a finite subgroup of $GL(W)$.

4.4.1 Criteria of types C, D, F

The optimist sees the opportunity in every difficulty, and we proposed:

Definition 11. [11, 2.2] A finite rack X *collapses* if $\dim \mathscr{B}(X,\mathfrak{q}) = \infty$ for any $\mathfrak{q}$.

Actually, this Definition accompanied the discovery of the criterion of type D [11, 3.5]; later we found the criteria of type F [6, 2.4] and C [8, 2.3]. Let us first state concretely these criteria and then discuss their implications

Definition 12. We say that a rack X is of type

- **C** when there are a decomposable subrack such that $Y = R \coprod S$, with

$$R = \mathscr{O}_r^{\operatorname{Inn} Y}, \qquad S = \mathscr{O}_s^{\operatorname{Inn} Y}, \qquad \min\{|R|,|S|\} > 2 \text{ or } \max\{|R|,|S|\} > 4;$$

  (see Exercise 8 for $\operatorname{Inn} Y$); and elements $r \in R$, $s \in S$ satisfying

$$r \triangleright \neq s; \tag{85}$$

- **D** if there are a decomposable subrack $Y = R \coprod S$, $r \in R$, $s \in S$ such that

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s; \tag{86}$$

- **F** if there are subracks $(R_a)_{a\in\mathbb{I}_4}$ and elements $r_a \in R_a$, $a \in \mathbb{I}_4$, such that
  - $R_a \triangleright R_b = R_b$, $a,b \in \mathbb{I}_4$;
  - $R_a \cap R_b = \emptyset$, $a \neq b \in \mathbb{I}_4$;
  - $r_a \triangleright r_b \neq r_b$ for $a \neq b \in \mathbb{I}_4$.

First of all, these definitions are well adapted to our goal.

Theorem 10. *[11, 6, 8]. A rack X of type D, F or C collapses.*

The proof of this Theorem follows is based on results from [26, 48, 53].

Second, these criteria can be phrased in group terms; that is, if we realize X as a conjugacy class in a finite group G, then

- (85) means that $rs \neq sr$;
- (86) means that $(rs)^2 \neq (sr)^2$;
- the other requirements can be stated in terms of suitable subgroups of G.

In other words, the criteria are really problems in finite group theory. Third, there is another advantage, but to state it succinctly, we introduce more terminology.

Definition 13. A rack is *austere* if every subrack generated by two elements is either abelian or indecomposable; *sober* if every subrack is either abelian or indecomposable; *kthulhu* if it is neither of type C, D nor F.

It is easy to see that sober $\implies$ austere $\implies$ kthulhu. Although the proof of the following result is straightforward, it shows that the criteria are meaningful.

Proposition 8. *[11, 6, 8] Let $X \to Y$ be a surjective morphism of racks. If Y is not kthulhu, then X is not kthulhu.*

In fact, every finite rack projects onto a simple rack, by an evident recursive argument.

Corollary 1. *Let X be a rack that admits a surjective morphism of racks $X \to Y$ with Y simple and not kthulhu. Then X collapses.*

In other words, we do not need to compute cocycles, even less Nichols algebras, for racks as in the Corollary (if we are interested in finite-dimensional Nichols algebras).

Question 1. Are the criteria of types C, D, F valid, or adjustable, to finite Gelfand-Kirillov dimension?

In conclusion, we arrive at the next problem.

Determine all simple racks that are not kthulhu.

Remembering now Theorem 2, we overview the present status of this problem. The most substantial results are on simple racks associated to triples (L,t,θ). We consider only the case $t=1$, so that the racks in question are either conjugacy classes in the non-abelian simple group L or in the semidirect product $L \rtimes \langle\theta\rangle$. Indeed, the racks associated to triples (L,t,θ) with $t>1$ represent an even more serious challenge. Some partial results appear in [24].

Finally, the affine simple racks seem to be insensible to these arguments. For instance, the dihedral rack $\mathscr{D}_n$, see Example 12, where $n \geq 3$ is odd, is sober.

4.4.2 Alternating and symmetric groups

We start by the alternating groups $\mathbb{A}_m$, $m \in \mathbb{N}_{\geq 5}$. Recall that $\operatorname{Aut}\mathbb{A}_m = \mathbb{S}_m$, except for $m=6$. Thus we need to deal with conjugacy classes in $\mathbb{A}_m$ and $\mathbb{S}_m$. The conjugacy class $\mathscr{O}_\sigma^{\mathbb{S}_m}$ of σ in $\mathbb{S}_m$ is determined by its type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$, saying that the action of σ on $\mathbb{I}_m$ has n_1 fixed points, n_2 orbits of 2 elements, etc. If $\sigma \in \mathbb{A}_m$, then $\mathscr{O}_\sigma^{\mathbb{S}_m} \cap \mathbb{A}_m$ is either the conjugacy class $\mathscr{O}_\sigma^{\mathbb{A}_m}$ in $\mathbb{A}_m$, or else the union of two conjugacy classes that are isomorphic as racks. Thus, the type is also an appropriate label for them. We need a name for the set

$$\mathscr{F} = \left\{ p \in \mathbb{N} : p \text{ prime},\ p = \frac{r^k - 1}{r-1},\ \text{where } r \text{ is a prime power and } k \in \mathbb{N} \right\}.$$

Theorem 11. *[11, 34] Let $\mathscr{O}$ be either $\mathscr{O}_\sigma^{\mathbb{S}_m}$, if $\sigma \notin \mathbb{A}_m$, or else $\mathscr{O}_\sigma^{\mathbb{A}_m}$ if $\sigma \in \mathbb{A}_m$. If $\mathscr{O}$ is not listed in Table 2, then it collapses.*

Table 2 Kthulhu classes in a symmetric or alternating group

G	type	Reference
$\mathbb{S}_m$	$(1^{m-2},2)$	kthulhu, [11, Remark 4.2]
$\mathbb{A}_m$	$(1^{m-3},3)$	austere, idem
$\mathbb{A}_p$, $p=5,7$ or $\notin \mathscr{F}$	(p)	sober, [34, Remark 3.2 (b)]
$\mathbb{A}_{p+1}$, $p=5$ or $\notin \mathscr{F}$	$(1,p)$	sober, [34, Remark 3.2 (c)]
$\mathbb{A}_8$	(2^4)	austere, [11, Remark 4.2]
$\mathbb{A}_7$	$(2^2,3)$	austere, idem
$\mathbb{S}_6$	2^3	kthulhu, isomorphic to the class of type $(1^4,2)$
$\mathbb{A}_6$	(3^2), $(1^2,2^2)$	austere, [11, Remark 4.2]
$\mathbb{S}_5$	$(2,3)$	sober, idem
$\mathbb{A}_5$	$(1,2^2)$	idem

4.4.3 Finite simple groups of Lie type

The first examples of these appeared in the seminal paper of Evariste Galois! We start by observing:

- The finite simple groups of Lie type are (related to) the kernels of the so-called Steinberg endomorphisms of simple algebraic groups in positive characteristic. An exposition of their construction and description, even assuming the classification of the simple algebraic groups, is beyond the limits of this monograph. The interested reader may consult the beautiful account [84] of the classification of the finite simple groups, or the book [69] for a detailed presentation. Steinberg endomorphisms of simple algebraic groups fall into three possible classes [69, 21 & 22.5], hence there are three families of finite simple groups of Lie type: Chevalley, Steinberg and Suzuki-Ree groups. The complete list of the simple groups in each family also appears in [6, p. 38].
- For each finite simple group of Lie type, the classification of the conjugacy classes is a classical problem whose answer, again, is long and difficult. However there are two special classes, namely unipotent and semisimple, a terminology that correctly suggests a relation with the theory of the Jordan form of a linear transformation.

Here is the main result on these conjugacy classes, summarizing [6, 7, 8, 9].

Theorem 12. *Let* **G** *be a Chevalley or Steinberg group and let* $\mathscr{O}$ *be a non-trivial unipotent conjugacy class in* **G**. *If* $\mathscr{O}$ *is not listed in Table 3, then it collapses.*

Remark 8. Let $\mathscr{O}$ be a *non semisimple* class in a finite simple group of Lie type **G**. Then $\mathscr{O}$ has a subrack that is a unipotent conjugacy class in a smaller group and we may argue inductively, as was effectively performed for $PSL_n(q)$ in [6]. Semisimple classes appear to be more difficult to tackle, see partial results in [8].

Table 3 Kthulhu unipotent classes in a finite simple Chevalley or Steinberg group

G	q	type	Reference
$PSL_2(q)$	even or not a square	(2)	sober, [6, Lemma 3.5]
$PSL_3(2)$		(3)	sober, [6, Lemma 3.7 (b)]
$PSp_{2n}(q)$, $n \geq 2$	even	$W(1)^a \oplus V(2)$	austere, [8, Lemma 2.14]
	odd, 9 or not a square	$(1^{r_1}, 2)$	idem
$PSp_4(q)$	even	$W(2)$	idem
$PSU_n(q)$	even	$(2, 1, \dots, 1)$	austere, [9, Lemma 5.16]

Remark 9. Since $PSL_3(2) \simeq PSL_2(7)$, the unipotent class of type (3) is really a semisimple class in the former group.

Also, $PSL_2(q) \simeq PSp_2(q)$, so we really have two families of kthulhu unipotent classes in Table 3:

- the class $\mathsf{Sp}_{2n,q}$ inside $PSp_{2n}(q)$, $n \geq 1$, and
- the class $\mathsf{SU}_{m,q}$ inside $PSU_m(q)$.

They both correspond to the partition $(2, 1, ..., 1)$, and, up to rack isomorphism, are represented by $x_\beta(1)$, where β is the highest root, see [7].

Indeed, the class $W(2)$ in $PSp_4(q)$ for q even is due to the existence of a non-standard graph automorphism in C_2, in even characteristic, that interchanges short with long roots. Hence, this class is isomorphic, as a rack, to $\mathsf{Sp}_{4,q}$.

These families are related: first, if $q|q'$, then

$$\mathsf{Sp}_{2n,q} \leq \mathsf{Sp}_{2n,q'}, \qquad \mathsf{SU}_{2n,q} \leq \mathsf{SU}_{2n,q'}.$$

Next the morphism of groups $Sp_{2n}(q) \hookrightarrow Sp_{2n+2}(q)$ implies that

$$\mathsf{Sp}_{2n,q} \leq \mathsf{Sp}_{2n+2,q}.$$

When $q = 2^t$ and $m = 2n$ are even, $Sp_{2n}(q) \leq SU_{2n}(q)$, hence

$$\mathsf{Sp}_{2n,2^t} \leq \mathsf{SU}_{2n,2^t}.$$

Finally, there are inclusions between the unitary groups that induce

$$\mathsf{SU}_{n,q} \leq \mathsf{SU}_{n+2,q}, \qquad \mathsf{SU}_{2n,q} \leq \mathsf{SU}_{2n+1,q}.$$

Naturally, we are eager to know:

Are there cocycles for $\mathsf{Sp}_{2n,q}$ or $\mathsf{SU}_{m,q}$ such that the corresponding Nichols algebras are finite-dimensional?

4.4.4 Sporadic groups

The classification of the finite simple groups contains, besides the alternating groups and those of Lie type, 26 more examples that are called the *sporadic* groups; here we discuss also the so-called Tits group. We refer to [84, Chapter 5] for an introduction to these groups.

Theorem 13. *[12, 35] Let* $\mathbf{G}$ *be a sporadic simple group different from the Moster* M *and let* $\mathscr{O}$ *be a non-trivial conjugacy class in* $\mathbf{G}$ *or* $\operatorname{Aut}\mathbf{G}$. *If* $\mathscr{O}$ *is not listed in Table 4, then it collapses.*

The proof of this last result was done using the information in the online version of the Atlas, with the computer program GAP.

Remark 10. As for the Monster group M, these conjugacy classes are not known to be of type D: 32A, 32B, 41A, 46A, 46B, 47A, 47B, 59A, 59B, 69A, 69B, 71A, 71B, 87A, 87B, 92A, 92B, 94A, 94B. All the rest are of type D.

The criteria of type C and F were not applied neither to these classes nor to those in Table 4.

Table 4 Classes in sporadic, or automorphism of sporadic, groups not of type D

Group	Classes	Group	Classes	Group	Classes
T	2A	M_{11}	8A, 8B, 11A, 11B	$\operatorname{Aut}(M_{22})$	2B
M_{12}	11A, 11B	M_{22}	11A, 11B	$\operatorname{Aut}(HS)$	2C
M_{23}	23A, 23B	M_{24}	23A, 23B	$\operatorname{Aut}(Fi_{22})$	2D
Ru	29A, 29B	Suz	3A	$\operatorname{Aut}(J_3)$	34A, 34B
HS	11A, 11B	McL	11A, 11B	$\operatorname{Aut}(ON)$	38A, 38B, 38C
Co_1	3A	Co_2	2A, 23A, 23B	$\operatorname{Aut}(McL)$	22A, 22B
Co_3	23A, 23B	J_1	15A, 15B, 19A, 19B, 19C	$\operatorname{Aut}(Fi'_{24})$	2C
J_2	2A, 3A	J_3	5A, 5B, 19A, 19B	J_4	29A, 43A, 43B, 43C
Ly	37A, 37B, 67A, 67B, 67C	$O'N$	31A, 31B	Fi_{23}	2A
Fi_{22}	2A, 22A, 22B	Fi'_{24}	29A, 29B	B	2A, 46A, 46B, 47A, 47B

4.5 Rack type, finite dimension

Here we discuss finite-dimensional Nichols algebras of rack type. We first present some examples that were computed by ad-hoc techniques. Then we summarize the main results on Nichols algebras of decomposable Yetter-Drinfeld modules from [53, 54].

The quadratic approximations of Nichols algebras associated to racks an abelian cocycles are not difficult to describe explicitly by generators and relations, see [37] for a general formulation. Thus the problem is either to see whether the Nichols algebra is quadratic or else to find higher degree relations.

If $\mathscr{B}(V)$ is finite-dimensional, then there exists $N \in \mathbb{N}$ such that $\mathscr{B}^N(V) \neq 0$, $\mathscr{B}^{N+1}(V) = 0$; we call N the *top degree* of $\mathscr{B}(V)$. Notice that $\mathscr{B}^N(V)$ is the space of integral of $\mathscr{B}(V)$, hence $\dim \mathscr{B}^N(V) = 1$ and $\mathscr{B}(V)$ satisfies Poincaré duality $\dim \mathscr{B}^j(V) = \dim \mathscr{B}^{N-j}(V)$ for all $j \in \mathbb{I}_{0,N}$.

4.5.1 Fomin-Kirillov algebras

Let $m \geq 3$. We consider two Nichols algebras associated to the conjugacy class $\mathscr{O}_2^m$ of transpositions in $\mathbb{S}_m$, with respect to the following cocycles:

$$\varepsilon \equiv -1; \qquad \chi(\nu, \zeta) = \begin{cases} 1 & \nu(i) < \nu(j), \\ -1 & \nu(i) > \nu(j), \end{cases} \qquad \text{where } \zeta = (i\ j), i < j.$$

The braided vector spaces $(\Bbbk\mathscr{O}_2^m, c^{\varepsilon})$ and $(\Bbbk\mathscr{O}_2^m, c^{\chi})$ can be realized as Yetter-Drinfeld modules M_1 and M_2 over $\Bbbk\mathbb{S}_m$. Furthermore, if $M \in {}^{\Bbbk\mathbb{S}_m}_{\Bbbk\mathbb{S}_m}\mathscr{YD}$, $M \not\simeq M_1, M_2$, and $m > 6$, then $\dim \mathscr{B}(M) = \infty$ [11, Theorem 1.1].

We start by the quadratic approximations of $\mathscr{B}(\mathscr{O}_2^m, c^{\varepsilon})$ and $\mathscr{B}(\mathscr{O}_2^m, c^{\chi})$.

Definition 14. [36, 70] Let $\mathtt{FK_m}$ be the algebra presented by generators $(x_{(ij)})_{i<j\in\mathbb{I}_m}$ and relations

$$\begin{aligned} x_{(ij)}^2 &= 0, && i < j \in \mathbb{I}_m, \\ x_{(ij)}x_{(kl)} - x_{(kl)}x_{(ij)} &= 0, && i,j,k,l \in \mathbb{I}_m, \text{ all different}, \\ x_{(jk)}x_{(ik)} - x_{(ij)}x_{(jk)} + x_{(ik)}x_{(ij)} &= 0, && i < j < k \in \mathbb{I}_m, \\ x_{(ik)}x_{(jk)} - x_{(jk)}x_{(ij)} + x_{(ij)}x_{(ik)} &= 0, && i < j < k \in \mathbb{I}_m. \end{aligned}$$

This is the quadratic approximation of $\mathscr{B}(\mathscr{O}_2^m, c^{\chi})$; it is called the m-th Fomin-Kirillov algebra since it appeared first in [36], albeit rediscovered in [70].

Proposition 9. *If $m = 3$, 4 or 5, then the dimension, the Hilbert-Poincaré series $\mathscr{H}_{\mathtt{FK_m}}(t)$ and its top degree of the Fomin-Kirillov algebra $\mathtt{FK_m}$ are given in Table 5.*

Table 5 Fomin-Kirillov algebras $\mathtt{FK_m}$ and their relatives $\mathtt{B_m}$, $m = 3,4,5$

m	dimension	top degree	Hilbert-Poincaré series
3	12	4	$\mathscr{H}_{\mathtt{FK}_3}(t) = (2)_t^2(3)_t$
4	576	12	$\mathscr{H}_{\mathtt{FK}_5}(t) = (2)_t^2(3)_t^2(4)_t^2$
5	8294400	40	$\mathscr{H}_{\mathtt{FK}_5}(t) = (4)_t^4(5)_t^2(6)_t^4$

We turn to the quadratic approximation of $\mathscr{B}(\mathscr{O}_2^m, c^{\varepsilon})$.

Definition 15. [70] Let $\mathtt{B_m}$ be the algebra generated $(x_{(ij)})_{i<j\in\mathbb{I}_m}$ with relations

$$\begin{aligned} x_{(ij)}^2 &= 0, & & i<j\in\mathbb{I}_m,\\ x_{(ij)}x_{(kl)}+x_{(kl)}x_{(ij)} &= 0, & & i,j,k,l\in\mathbb{I}_m, \text{ all different},\\ x_{(jk)}x_{(ik)}+x_{(ik)}x_{(ij)}+x_{(ij)}x_{(jk)} &= 0, & & i<j<k\in\mathbb{I}_m,\\ x_{(ik)}x_{(jk)}+x_{(jk)}x_{(ij)}+x_{(ij)}x_{(ik)}, &= 0, & & i<j<k\in\mathbb{I}_m. \end{aligned}$$

These algebras are closely related; cf. Example 30.

Proposition 10. *[83] $\mathscr{B}(\Bbbk\mathscr{O}_2^m, c^{\varepsilon})$ and $\mathscr{B}(\Bbbk\mathscr{O}_2^m, c^{\chi})$ are twist-equivalent.*

Corollary 2. *If $m=3$, 4 or 5, then $\mathtt{B_m}$ has the same dimension and the same Hilbert-Poincaré series as those of $\mathtt{FK_m}$ and thus they are also given in Table 5.*

Indeed, it is enough to prove Proposition 9 or Corollary 2, as they are equivalent. If $m=3$ or 4, then Proposition 9 was proved in [36]; if $m=5$, is due to Jan-Erik Roos, with a computer program. If $m=3$ or 4, then Corollary 2 was proved in [70] using Grobner basis.

Theorem 14. *If $m=3$, 4 or 5, then $\mathtt{FK_m}\simeq\mathscr{B}(\mathscr{O}_2^{\mathtt{m}},\mathtt{c}^{\chi})$.*

The proof appears in [70] for $\mathtt{B_m}$ and $m\leq 4$, and verified by Graña for $m=5$ using Deriva–see details in [37]. By Proposition 10, it translates at once to $\mathtt{FK_m}$.

Let $m\geq 6$. The following three assertions are open Questions:

- $\mathtt{FK_m}\simeq\mathscr{B}(\mathscr{O}_2^{\mathtt{m}},\mathtt{c}^{\chi})$ is a Nichols algebra (i.e. $\mathscr{B}(\mathscr{O}_2^m,c^{\chi})$ is quadratic).
- The dimension of $\mathtt{FK_m}$ is finite.
- The dimension of $\mathscr{B}(\mathscr{O}_2^m,c^{\chi})$ is finite.

Some authors suggest that the last two assertions are false, see e.g. [67].

Needless to say, the analogous Question is stated for $\mathtt{B_m}$ and $\mathscr{B}(\mathscr{O}_2^m,c^{\varepsilon})$; but both Questions are equivalent by Proposition 10.

The following Example is close to $\mathtt{B_4}$.

Example 32. [15] The Nichols algebra of the conjugacy class $\mathscr{O}_4^4$ of 4-cycles in $\mathbb{S}_4$ with the constant cocycle -1 is quadratic and has the same dimension and Hilbert-Poincaré series as as those of $\mathtt{FK_4}$ and thus are given in Table 5.

4.5.2 Finite-dimensional Nichols algebras of some affine racks

In the Examples below, we consider simple affine racks $\mathrm{Aff}(\mathbb{F}_q,T)$ and the constant cocycle $\mathfrak{q}\equiv -1$. We set $\mathscr{B}(\mathbb{F}_q,T):=\mathscr{B}(\mathrm{Aff}(\mathbb{F}_q,T),\mathfrak{q})$. Notice that $\mathrm{Aff}(\mathbb{F}_3,2)\simeq\mathscr{O}_2^3$; as we have seen, $\dim\mathscr{B}(\mathbb{F}_3,2)=12=3.2^2$ and the top degree is $4=2^2$.

Example 33. [41] Let $\omega \in \mathbb{F}_4$ such that $\omega^2+\omega+1=0$. The tetrahedron rack is $\mathscr{T} := \mathrm{Aff}(\mathbb{F}_4,\omega)$. Then $\mathscr{B}(\mathbb{F}_4,\omega)$ is generated by $(x_i)_{i\in\mathbb{F}_4}$ with relations

$$\begin{aligned}
&x_i^2 = 0, && i\in\mathbb{F}_4;\\
&x_ix_j + x_{(\omega+1)i+\omega j}x_i + x_jx_{(\omega+1)i+\omega j} = 0, && i\neq j\in\mathbb{F}_4;\\
&x_\omega x_1x_0x_\omega x_1x_0 + x_1x_0x_\omega x_1x_0x_\omega + x_0x_\omega x_1x_0x_\omega x_1 = 0. &&
\end{aligned}$$

Also, $\dim\mathscr{B}(\mathbb{F}_4,\omega) = 72 = 4.2.3^2$ (observe that $2=\varphi(4)$ where φ is the Euler function) and the top degree is $9=3^2$. The Hilbert-Poincaré series is the polynomial

$$\begin{aligned}
(1+t)^2(1+t+t^2)^2(1+t^3) = t^9+4t^8+8t^7+11t^6&\\
+12t^5+12t^4+11t^3+8t^2+4t+1.&
\end{aligned}$$

Example 34. [46] There is a cocycle $\mathfrak{q}$ on $\mathscr{T} = \mathrm{Aff}(\mathbb{F}_4,\omega)$ that takes values $\pm\xi$, where $\xi\in\mathbb{G}_3'$, such that $\dim\mathscr{B}(\mathscr{T},\mathfrak{q}) = 5184$. The Nichols algebra $\mathscr{B}(X_{4,\omega},\mathfrak{q})$ can be presented by generators $(x_i)_{i\in\mathbb{F}_4}$ with defining relations

$$\begin{gathered}
x_0^3 = x_1^3 = x_\omega^3 = x_{\omega^2}^3 = 0,\\
\xi^2x_0x_1 + \xi x_1x_\omega - x_\omega x_0 = 0, \quad \xi^2x_0x_\omega + \xi x_\omega x_{\omega^2} - x_{\omega^2}x_0 = 0,\\
\xi x_0x_{\omega^2} - \xi^2x_1x_0 + x_{\omega^2}x_1 = 0, \quad \xi x_1x_{\omega^2} + \xi^2x_\omega x_1 + x_{\omega^2}x_\omega = 0,\\
x_0^2x_1x_\omega x_1^2 + x_0x_1x_\omega x_1^2x_0 + x_1x_\omega x_1^2x_0^2 + x_\omega x_1^2x_0^2x_1 + x_1^2x_0^2x_1x_\omega + x_1x_0^2x_1x_\omega x_1\\
+x_1x_\omega x_1x_0^2x_\omega + x_\omega x_1x_0x_1x_0x_\omega + x_\omega x_1^2x_0x_\omega x_0 = 0.
\end{gathered}$$

Example 35. (Graña, see [15]). We consider the affine racks $\mathrm{Aff}(\mathbb{F}_5,2)$, $\mathrm{Aff}(\mathbb{F}_5,3)$. First, $\mathscr{B}(\mathbb{F}_5,2)$ is generated by $(x_i)_{i\in\mathbb{F}_5}$ with relations

$$\begin{aligned}
&x_i^2 = 0, && i\in\mathbb{F}_5;\\
&x_ix_j + x_{-i+2j}x_i + x_{3i-2j}x_{-i+2j} + x_jx_{3i-2j} && i\neq j\in\mathbb{F}_5;\\
&x_1x_0x_1x_0 + x_0x_1x_0x_1. &&
\end{aligned}$$

Also, $\dim\mathscr{B}(\mathbb{F}_5,2) = 1280 = 5.4^4$ and the top degree is $16 = 4^2$. The Hilbert-Poincaré series is the polynomial

$$\begin{aligned}
(1+t)^2(1+t+t^2+t^3)(1+t+2t^2+2t^3+2t^4+t^5+t6)(1+t+2t^2+2t^3+t^4+t^5)&\\
= t^{16}+5t^{15}+15t^{14}+35t^{13}+66t^{12}+105t^{11}+145t^{10}+175t^9&\\
+186t^8+175t^7+145t^6+105t^5+66t^4+35t^3+15t^2+5t+1.&
\end{aligned}$$

Next the braided vector space associated to $\mathrm{Aff}(\mathbb{F}_5,3)$ with $\mathfrak{q}\equiv -1$ is dual to the preceding; hence $\dim\mathscr{B}(\mathbb{F}_5,3) = 1280$ and the Hilbert-Poincaré series is the same.

Example 36. (Graña). We consider the affine racks $\mathrm{Aff}(\mathbb{F}_7,3)$, $\mathrm{Aff}(\mathbb{F}_7,5)$. First, $\mathscr{B}(\mathbb{F}_7,3)$ is generated by $(x_i)_{i\in\mathbb{F}_7}$ with relations

$$\begin{aligned}
&x_i^2 = 0, && i \in \mathbb{F}_7;\\
&x_i x_j + x_{-2i+3j} x_i + x_j x_{-2i+3j} && i \neq j \in \mathbb{F}_7;\\
&x_2 x_1 x_0 x_2 x_1 x_0 + x_1 x_0 x_2 x_1 x_0 x_2 + x_0 x_2 x_1 x_0 x_2 x_1.
\end{aligned}$$

Also, $\dim \mathscr{B}(\mathbb{F}_7, 3) = 326592 = 7.6^6$ and the top degree is $36 = 6^2$. The Hilbert-Poincaré series is the polynomial

$$\begin{aligned}
&(1+t)^2(1+t+t^2)^2\\
&\quad \times (1+t+2t^2+3t^3+4t^4+5t^5+4t^6+5t^7+4t^8+3t^9+2t^{10}+t^{11}+t^{12})\\
&\quad \times (1+t+t^2+2t^3+2t^4+2t^5+2t^6+t^7+t^8+t^9)\\
&\quad \times (1+t+2t^2+2t^3+3t^4+3t^5+2t^6+2t^7+t^8+t^9)\\
&= t^{36}+7t^{35}+28t^{34}+84t^{33}+210t^{32}+462t^{31}+918t^{30}+1673t^{29}+2828t^{28}\\
&+4473t^{27}+6664t^{26}+9394t^{25}+12573t^{24}+16023t^{23}+19488t^{22}+22659t^{21}\\
&+25214t^{20}+26873t^{19}+27448t^{18}+26873t^{17}+25214t^{16}+22659t^{15}+19488t^{14}\\
&+16023t^{13}+12573t^{12}+9394t^{11}+6664t^{10}+4473t^9+2828t^8+1673t^7\\
&+918t^6+462t^5+210t^4+84t^3+28t^2+7t+1.
\end{aligned}$$

Next the braided vector space associated to $\mathrm{Aff}(\mathbb{F}_7, 5)$, $\mathfrak{q} \equiv -1$, is dual to the preceding; hence $\dim \mathscr{B}(\mathbb{F}_7, 5) = 326592$ and the Hilbert-Poincaré series is the same.

It was conjectured that the Examples in this §and the preceding exhaust all genuine finite-dimensional Nichols algebras over groups (besides those of diagonal type); see [47] for the precise formulation.

4.5.3 Decompositions with 2 summands

We start by the description of some decomposable braided vector spaces of rack type with finite dimensional Nichols algebra. Then we state the main result of [53]. For simplicity we assume that $\Bbbk$ is algebraically closed and $\operatorname{char} \Bbbk = 0$.

Example 37. Let $X = \mathscr{D}_4 = \mathbb{I}_2\,{}_\sigma\coprod_\sigma \mathbb{I}_2$, $\sigma \neq \mathrm{id}$, see Exercise 10. Concretely, $X = \{1,2\}_{(34)}\coprod_{(12)}\{3,4\}$. Then $\Bbbk X = V_1 \oplus V_2$, where V_1 is spanned by $(x_i)_{i \in \mathbb{I}_2}$, while V_2 is spanned by $(x_j)_{j \in \mathbb{I}_{3,4}}$. Let $p, q, r, t \in \Bbbk^\times$, $p \neq 1 \neq q$, and $\varepsilon, \varepsilon' \in \mathbb{G}_2$. Define a braiding on $\Bbbk X$ by

$$
\begin{aligned}
&c_{|V_1\otimes V_1} \text{ is of diagonal type with matrix } \begin{pmatrix} q & \varepsilon q \\ \varepsilon q & q \end{pmatrix}, \\
&c_{|V_2\otimes V_2} \text{ is of diagonal type with matrix } \begin{pmatrix} p & \varepsilon' p \\ \varepsilon' p & p \end{pmatrix}, \\
&\Big(c(x_i \otimes x_j)_{i\in\mathbb{I}_2, j\in\mathbb{I}_{3,4}} \Big) = \begin{pmatrix} x_4 \otimes x_1 & t^2 x_3 \otimes x_1 \\ \varepsilon' x_4 \otimes x_2 & \varepsilon' t^2 x_3 \otimes x_2 \end{pmatrix}, \\
&\Big(c(x_j \otimes x_i)_{j\in\mathbb{I}_{3,4}, i\in\mathbb{I}_2} \Big) = \begin{pmatrix} x_2 \otimes x_3 & r^2 x_1 \otimes x_3 \\ \varepsilon x_2 \otimes x_4 & \varepsilon r^2 x_1 \otimes x_4 \end{pmatrix}.
\end{aligned} \tag{87}
$$

Exercise 31. 1. Prove that this is indeed a braiding.

2. Assume that $\varepsilon = \varepsilon' = 1$. Consider the basis $(y_h)_{h\in\mathbb{I}_4}$ of $\Bbbk X$ where

$$y_1 = rx_1 + x_2, \qquad y_2 = -rx_1 + x_2, \qquad y_3 = tx_3 + x_4, \qquad y_4 = -tx_3 + x_4.$$

Then c on this basis is of diagonal type, with matrix

$$\begin{pmatrix} q & q & t & -t \\ q & q & t & -t \\ r & -r & p & p \\ r & -r & p & p \end{pmatrix}.$$

If $\dim \mathscr{B}(V) < \infty$, then $p = q = -1$. (Indeed, by Example 31, $p, q \in \mathbb{G}_2' \cup \mathbb{G}_3'$, then inspect the list in [45]). In this case the Dynkin diagram is

$$
\begin{array}{ccc} \overset{-1}{\underset{1}{\circ}} & \overset{rt}{\text{———}} & \overset{-1}{\underset{3}{\circ}} \\ {\scriptstyle -rt}\Big| & & \Big|{\scriptstyle -rt} \\ \overset{-1}{\underset{4}{\circ}} & \overset{rt}{\text{———}} & \overset{-1}{\underset{2}{\circ}}; \end{array}
\qquad \text{if } rt = 1: \quad
\begin{array}{ccc} \overset{-1}{\underset{1}{\circ}} & \qquad & \overset{-1}{\underset{3}{\circ}} \\ {\scriptstyle -1}\Big| & & \Big|{\scriptstyle -1} \\ \overset{-1}{\underset{4}{\circ}} & & \overset{-1}{\underset{2}{\circ}}; \end{array}
\qquad \text{if } rt = -1: \quad
\begin{array}{ccc} \overset{-1}{\underset{1}{\circ}} & \overset{-1}{\text{———}} & \overset{-1}{\underset{3}{\circ}} \\ & & \\ \overset{-1}{\underset{4}{\circ}} & \overset{-1}{\text{———}} & \overset{-1}{\underset{2}{\circ}}; \end{array}
$$

Now $\overset{-1}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ}$, is a Dynkin diagram of Cartan type A_2 at -1. By elementary arguments, its Nichols algebra has dimension 8. Therefore, if $rt \in \mathbb{G}_2$, then $\dim \mathscr{B}(V) = 64$. If $rt \notin \mathbb{G}_2$, then $\dim \mathscr{B}(V) = \infty$ by inspection of the list in [45].

3. If $\varepsilon = \varepsilon' = -1$, then there is a twist ϕ as in Example 30 that reduces to the previous case.

When $\varepsilon, \varepsilon' \in \mathbb{G}_2$ are arbitrary, the same result holds but the proof requires the Weyl groupoid:

Theorem 15. *[48, Theorem 4.6] Let* $(V, c) = (\Bbbk\mathscr{D}, c)$ *where* c *is given by* (87). *Then* $\dim \mathscr{B}(V) = 64$.

Example 38. Let $X = \mathscr{D}_3 \coprod \{4\}$. Then $\Bbbk X = V_1 \oplus V_2$, where V_1 is spanned by $(x_i)_{i\in\mathbb{I}_3}$, while V_2 is spanned by x_4. Let $\omega \in \Bbbk^\times, \zeta \in \mathbb{G}_3, q_1, q_2 \in \Bbbk^\times$. Recall c^ε from §4.5.1 and define a braiding on $\Bbbk X$ by

$$\begin{aligned} c_{|V_1\otimes V_1} &= c^{\varepsilon}, & c(x_4\otimes x_4) &= -\omega x_4\otimes x_4, & \\ c(x_i\otimes x_4) &= q_1\zeta^{i-1}x_4\otimes x_i, & c(x_4\otimes x_i) &= q_2 x_i\otimes x_4, & i\in\mathbb{I}_3. \end{aligned} \tag{88}$$

Exercise 32. Check that (88) satisfies the braid equation.

Thus $\Bbbk X = V_1 \oplus V_2$ is a decomposition of braided vector spaces where V_1 is $(\Bbbk\mathscr{O}_2^3, c^{\varepsilon})$, V_2 is a point with label $-\omega \in \mathbb{G}'_6$ and the braiding between them is prescribed in the second line of (88).

Theorem 16. *[53, Theorem 8.2] Assume that $\omega \in \mathbb{G}'_3$ and that $q_1q_2 = -\omega^2$. Let $(V,c) = (\Bbbk(\mathscr{D}_3 \coprod\{4\}), c)$ where c is given by* (88). *Then* $\dim\mathscr{B}(V) = 10368 = 3^4 2^7$.

Example 39. Let $X = \mathscr{D}_3 \coprod\{4\}$ as in the previous Example. Let $V = V_1 \oplus V_2$, where $V_1 = \Bbbk\mathscr{D}_3$ is spanned by $(x_i)_{i\in\mathbb{I}_3}$, but now V_2 is $\Bbbk x_4\otimes\Bbbk^2$. Let $y_4 = x_4\otimes(1,0)$, $y_5 = x_4\otimes(0,1)$. Let $\zeta\in\mathbb{G}_3$, $q_1,q_2\in\Bbbk^{\times}$. Define a braiding on $\Bbbk X$ by

$$\begin{aligned} c_{|V_1\otimes V_1} &= c^{\varepsilon}, & c_{|V_2\otimes V_2} &= -\tau, & \\ c(x_i\otimes y_4) &= \zeta^{i-1}y_5\otimes x_i, & c(x_i\otimes y_5) &= q_1\zeta^{2(i-1)}y_4\otimes x_i, & \\ c(y_4\otimes x_i) &= q_2 y_4\otimes x_i, & c(y_5\otimes x_i) &= q_2 x_i\otimes y_5, & i\in\mathbb{I}_3. \end{aligned} \tag{89}$$

Thus $V = V_1\oplus V_2$ is a decomposition of braided vector spaces where V_1 is $(\Bbbk\mathscr{O}_2^3, c^{\varepsilon})$ as in §4.5.1, $V_2 = \Bbbk y_4 \oplus \Bbbk y_5$ has dimension 2 and the braiding between them is prescribed in the second and third lines of (89).

Exercise 33. Check that (89) satisfies the braid equation.

Theorem 17. *[53, Theorem 8.4] Let (V,c) be the braided vector space with c given by* (89). *Assume that $q_1q_2^2 = 1$. Then* $\dim\mathscr{B}(V) = 2304 = 3^2 2^8$.

Example 40. Let $X = \mathscr{D}_3\,{}_{(45)}\coprod_{(132),(123)}\mathbb{I}_{4,5}$; let $\sigma = (132)$. Let $V = \Bbbk X = V_1\oplus V_2$, where $V_1 = \Bbbk\mathscr{D}_3$ is spanned by $(x_i)_{i\in\mathbb{I}_3}$ and V_2 is spanned by x_4, x_5. Let $\zeta\in\mathbb{G}_3$, $a_1, q_1, q_2 \in \Bbbk^{\times}$. Define a braiding on V by

$$\begin{aligned} c_{|V_1\otimes V_1} &= c^{\varepsilon}, & c(x_i\otimes x_j) &= a_1\zeta^{2-\delta_{ij}}x_j\otimes x_i, & i,j\in\mathbb{I}_{4,5}; \\ c(x_i\otimes x_4) &= \zeta^{i-1}x_5\otimes x_i, & c(x_i\otimes x_5) &= q_1\zeta^{2(i-1)}x_4\otimes x_i, & \\ c(x_4\otimes x_i) &= q_2 x_{\sigma(i)}\otimes x_4, & c(x_5\otimes x_i) &= q_2 x_{\sigma^{-1}(i)}\otimes x_5, & i\in\mathbb{I}_3. \end{aligned} \tag{90}$$

Exercise 34. Check that (90) satisfies the braid equation and that $V = V_1\oplus V_2$ is a decomposition of braided vector spaces.

Theorem 18. *[53, Theorem 8.1, 8.3] Let (V,c) be the braided vector space with c given by* (90).

1. *Assume that $\zeta\in\mathbb{G}'_3$, $a_1 = -\zeta^2$ and $q_1q_2^2 = \zeta^2$. Then* $\dim\mathscr{B}(V,c) = 10368$.
2. *Assume that $\zeta = 1$, $a_1 = -1$ and $q_1q_2^2 = 1$. Then* $\dim\mathscr{B}(V,c) = 2304$.

Example 41. Let $X = \mathscr{D}_4\,{}_{(56)}\coprod_{\sigma_1,\sigma_2}\mathbb{I}_{5,6}$. Here we number $\mathscr{D}_4$ as follows: $\mathscr{D}_4 = \{1,3\}\,{}_{\sigma}\coprod_{\sigma}\{2,4\}$, where $\sigma \neq \mathrm{id}$; that is, we change the numeration in Example 37 by $2 \leftrightarrow 3$. Also, $\sigma_1 = (1234)$, $\sigma_2 = (1432)$. Let $V = \Bbbk X = V_1 \oplus V_2$, where $V_1 = \Bbbk\mathscr{D}_4$ is spanned by $(x_i)_{i\in\mathbb{I}_4}$, and V_2 is spanned by x_5, x_6. Let $q_1, q_2 \in \Bbbk^\times$, $\zeta_1, \zeta_2 \in \mathbb{G}_4$. Define a cocycle on V by

$$
\begin{aligned}
&(\mathfrak{q}_{ij})_{i,j\in\mathbb{I}_4} = \begin{pmatrix} -1 & -\zeta_1^2 & -\zeta_1^2 & -\zeta_1^2 \\ -1 & -1 & -1 & -\zeta_1^2 \\ -\zeta_1^2 & -1 & -1 & -1 \\ -\zeta_1^2 & -\zeta_1^2 & -\zeta_1^2 & -1 \end{pmatrix}, \quad (\mathfrak{q}_{ij})_{i,j\in\mathbb{I}_{5,6}} = \begin{pmatrix} -1 & -\zeta_2^3 \\ -\zeta_2^3 & -1 \end{pmatrix}, \\
&(\mathfrak{q}_{ij})_{i\in\mathbb{I}_{5,6}, j\in\mathbb{I}_4} = \begin{pmatrix} 1 & q_2\zeta_1^3 & 1 & q_2\zeta_1 \\ \zeta_1^2 & q_2\zeta_1^3 & 1 & q_2\zeta_1^3 \end{pmatrix}, \qquad \mathfrak{q}_{ij} = \begin{cases} \zeta_2^{1-i}, & j = 5, \\ q_1\zeta_2^{i-1} & j = 6, \end{cases} \quad i \in \mathbb{I}_4.
\end{aligned} \tag{91}
$$

Exercise 35. Check that (91) satisfies the cocycle relation.

Theorem 19. *[52, Theorem 5.4] Let $(V, c^{\mathfrak{q}})$ be the braided vector space with $\mathfrak{q}$ given by* (91)*. Assume that $\zeta_1\zeta_2 = q_1q_2$ and $\zeta_2 \in \mathbb{G}_4'$. Then* $\dim\mathscr{B}(X,\mathfrak{q}) = 262144$.

Example 42. Let $X = \mathscr{T}\coprod\{5\}$. Let $V = \Bbbk X = V_1 \oplus V_2$, where $V_1 = \Bbbk\mathscr{T}$ is spanned by $(x_i)_{i\in\mathbb{I}_4}$, and V_2 is $\Bbbk x_5$. Let $a, q_1, q_2 \in \Bbbk^\times$. Define a braiding on V by

$$
\begin{aligned}
c_{|V_1\otimes V_1} &= c^{-1}, & c_{|V_2\otimes V_2} &= a\,\mathrm{id}, \\
c(x_i \otimes x_5) &= q_1 x_5 \otimes x_i, & c(x_5 \otimes x_i) &= q_2 x_i \otimes x_5, \quad i \in \mathbb{I}_4.
\end{aligned} \tag{92}
$$

Thus $V = V_1 \oplus V_2$ is a decomposition of braided vector spaces where V_1 is $(\Bbbk\mathscr{T}, c^{-1})$ as in Example 33 and $V_2 = \Bbbk x_5$ has dimension 1.

Exercise 36. Check that (92) satisfies the braid equation.

Theorem 20. *[52, Theorem 2.8] Let (V, c) be the braided vector space with c given by* (92)*. Assume that $-q_1q_2 \in \mathbb{G}_3'$ and $aq_1q_2 = 1$. Then* $\dim\mathscr{B}(X,\mathfrak{q}) = 80621568$.

The following remarkable result is the culmination of the series of papers [48, 51, 52, 53].

Theorem 21. *[53] Let G be a finite non-abelian group and $V = V_1 \oplus V_2 \in {}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$, where V_1 and V_2 are simple, the support of V generates G and $c^2_{|V_1\otimes V_2} \neq \mathrm{id}$. Assume that* $\dim\mathscr{B}(V) < \infty$*. Then as a braided vector space, V is isomorphic to one of the Examples 37, 38, 39, 40, 41 or 42.*

This formulation is simplified for the sake of the exposition; the actual result gives precise information of the possible groups G, it does not require $\Bbbk$ to be algebraically closed, and it extends to all characteristics, with new examples in characteristics 2 and 3.

4.5.4 Decompositions with $\theta > 2$ summands

The proof of Theorem 21 uses the Weyl groupoid and a detailed analysis of the subgroups of the enveloping group of the racks involved. With similar techniques, the same authors went on and obtained in [54] the analogous classification but for $\theta > 2$, again without restriction on the characteristic. The outcome is that essentially Dynkin diagrams of simple Lie algebras are, up to just a few exceptions, the main characters of the classification!

As the precise formulation of the main Theorem of [54] requires a careful preparation beyond the scope of these Notes, we refer the interested reader to the original source [54].

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