On the structure of Hopf algebras

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I. Introduction. $\Bbbk$ algebraically closed field.

A algebra: product $\mu : A \otimes A \to A$, unit $u : \Bbbk \to A$

Associative:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

Unitary:

$$\begin{tikzcd} \Bbbk \otimes A \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\sim"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \Bbbk \arrow[l, "\mathrm{id} \otimes u"'] \arrow[dl, "\sim"] \\ & A & \end{tikzcd}$$

C coalgebra: coproduct $\Delta : C \to C \otimes C$, counit $\varepsilon : C \to \mathbb{k}$

Co-associative:

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "\mathrm{id} \otimes \Delta"'] \\ C \otimes C \arrow[u, "\Delta \otimes \mathrm{id}"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

Co-unitary:

$$\begin{tikzcd} \mathbb{k} \otimes C \arrow[dr, "\sim"'] & C \otimes C \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \varepsilon"] & C \otimes \mathbb{k} \arrow[dl, "\sim"] \\ & C \arrow[u, "\Delta"] & \end{tikzcd}$$

Hopf algebra: $(H, \mu, u, \Delta, \varepsilon)$

- (H, μ, u) algebra
- (H, Δ, ε) coalgebra
- Δ, ε algebra maps
- There exists $\mathcal{S} : H \to H$ (the antipode) such that

$$
\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[drr, "\varepsilon"] & H \otimes H \arrow[rr, "\mathcal{S}\otimes\mathrm{id}", "\mathrm{id}\otimes\mathcal{S}"'] & & H \otimes H \arrow[r, "\mu"] & H \\
 & & \Bbbk \arrow[urr, "u"] & &
\end{tikzcd}
$$

Example:

- Γ finite group
- $H = \mathcal{O}(\Gamma) =$ algebra of functions $\Gamma \to \mathbb{k}$
- $\Delta : H \to H \otimes H \simeq \mathcal{O}(\Gamma \times \Gamma)$, $\Delta(f)(x, y) = f(x.y)$.
- $\varepsilon : H \to \mathbb{k}$, $\varepsilon(f) = f(e)$.
- $\mathcal{S} : H \to H$, $\mathcal{S}(f)(x) = f(x^{-1})$.

Remark: $(H, \mu, u, \Delta, \varepsilon)$ finite-dimensional Hopf algebra
$\Longrightarrow$ $(H^*, \Delta^t, \varepsilon^t, \mu^t, u^t)$ Hopf algebra

Example: $H = \mathcal{O}(\mathit{\Gamma})$; for $x \in \mathit{\Gamma}$, $E_x \in H^*$, $E_x(f) = f(x)$. Then

$$E_x E_y = E_{xy}, \qquad \mathcal{S}(E_x) = E_{x^{-1}}.$$

Hence $H^* = \Bbbk\mathit{\Gamma}$, group algebra of $\mathit{\Gamma}$.

Remark: $(H, \mu, u, \Delta, \varepsilon)$ Hopf algebra with $\dim H = \infty$,
H^* NOT a Hopf algebra,
but contains a largest Hopf algebra with operations transpose to those of H.

Example:

- Γ affine algebraic group

- $H = \mathcal{O}(\Gamma) =$ algebra of regular (polynomial) functions $\Gamma \to \Bbbk$ is a Hopf algebra with analogous operations.

- $H^* \supset \Bbbk\Gamma$
- $H^* \supset \mathcal{U} :=$ algebra of distributions with support at e; this is a Hopf algebra

- If $\operatorname{char} \Bbbk = 0$, then $\mathcal{U} \simeq U(\mathfrak{g})$, $\mathfrak{g} =$ Lie algebra of Γ

- If $\mathfrak{g}$ is any Lie algebra, then the enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$.

Short history:

• Since the dictionary *Lie groups* $\leftrightsquigarrow$ *Lie algebras* fails when char > 0, Dieudonné studied in the early 50's the hyperalgebra $\mathcal{U}$. Pierre Cartier introduced the abstract notion of hyperalgebra (cocommutative Hopf algebra) in 1955.

• Armand Borel considered algebras with a coproduct in 1952, extending previous work of Hopf. He coined the expression *Hopf algebra*.

• George I. Kac introduced an analogous notion in the context of von Neumann algebras.

• The first appearance of the definition (that I am aware of) as it is known today is in a paper by Kostant (1965).

First invariants of a Hopf algebra H:
$G(H) = \{x \in H - 0 : \Delta(x) = x \otimes x\}$, group of grouplikes.

$\mathrm{Prim}(H) = \{x \in H : \Delta(x) = x \otimes 1 + 1 \otimes x\}$, Lie algebra of primitive elements.

$\tau : V \otimes W \to W \otimes W$, $\tau(v \otimes w) = w \otimes v$ the *flip*.
H is commutative if $\mu\tau = \mu$. H is cocommutative if $\tau\Delta = \Delta$.

Group algebras, enveloping algebras, hyperalgebras are cocommutative.

Theorem. (Cartier-Kostant, early 60's). $\mathrm{char}\,\Bbbk = 0$.
Any cocommutative Hopf algebra is of the form $U(\mathfrak{g}) \# \Bbbk\Gamma$.

$H = \Bbbk[X]$, $\Delta(X) = X \otimes 1 + 1 \otimes X$. Then

$$\Delta(X^n) = \sum_{0 \le j \le n} \binom{n}{j} X^j \otimes X^{n-j}.$$

If $\operatorname{char} \Bbbk = p > 0$, then $\Delta(X^p) = X^p \otimes 1 + 1 \otimes X^p$.

Thus $\Bbbk[X]/\langle X^p \rangle$, $\Delta(X) = X \otimes 1 + 1 \otimes X$ is a Hopf algebra, commutative and cocommutative, dim p.

(Kulish, Reshetikhin and Sklyanin, 1981). Quantum $SL(2)$: if $q \in \Bbbk$, $q \neq 0, \pm 1$, set

$$\begin{aligned} U_q(\mathfrak{sl}(2)) = \Bbbk\langle E, F, K, K^{-1} | KK^{-1} &= 1 = K^{-1}K \\ KE &= q^2 EK, \\ KF &= q^{-2} FK, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}} \rangle \end{aligned}$$

$$\begin{aligned} \Delta(K) &= K \otimes K, \\ \Delta(E) &= E \otimes 1 + K \otimes E, \\ \Delta(F) &= F \otimes K^{-1} + 1 \otimes F. \end{aligned}$$

These Hopf algebras, neither commutative nor cocommutative, are analogues of the enveloping algebra of $\mathfrak{sl}(2)$.

(Lusztig, 1989). If q is a root of 1 of order N odd, then

$$\begin{aligned}\mathfrak{u}_q(\mathfrak{sl}(2)) = \Bbbk\langle E, F, K, K^{-1} | &\text{same relations plus} \\ K^N = 1, \quad E^N = F^N = 0\rangle.\end{aligned}$$

These Hopf algebras, neither commutative nor cocommutative, are analogues of the Frobenius kernel of $\mathfrak{sl}(2)$.

There are dual Hopf algebras, analogues of the algebra of regular functions of $SL(2)$.

$$\mathcal{O}_q(SL(2)) = \mathbb{k}\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} | ab = qba, \quad ac = qca, \quad bc = cb,$$
$$bd = qdb, \quad cd = qdc, \quad ad - da = (q - q^{-1})bc,$$
$$ad - qbc = 1\rangle.$$
$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(Manin). If q is a root of 1 of order N odd, then

$$\mathfrak{o}_q(\mathfrak{sl}(2)) = \mathbb{k}\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} |\text{same relations plus}$$
$$a^N = 1 = d^N, \quad b^N = c^N = 0\rangle.$$

In 1983, Drinfeld and Jimbo introduced quantized enveloping algebras $U_q(\mathfrak{g})$, for q as above and $\mathfrak{g}$ any simple Lie algebra.

- Quantum function algebras $\mathcal{O}_q(G)$: Faddeev-Reshetikhin and Takhtajan (for $SL(N)$) and Lusztig (any simple G).

- Finite-dimensional versions when q is a root of 1.

Motivation: A braided vector space is a pair (V, c), where V is a vector space and $c : V \otimes V \to V \otimes V$ is a linear isomorphism that satisfies

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

This is called the braid equation (closely related to the quantum Yang-Baxter equation).

- Any Hopf algebra (with bijective antipode) gives a machine of solutions of the braid equation.

- The solutions associated to $U_q(\mathfrak{g})$ are very important in low dimensional topology and some areas of theoretical physics.

Braided Hopf algebra: $(R, c, \mu, u, \Delta, \varepsilon)$

- (R, c) braided vector space
- (R, μ, u) algebra, (R, Δ, ε) coalgebra
- Δ, ε algebra maps, with the multiplication μ_2 in $R \otimes R$

$$\begin{tikzcd} R \otimes R \otimes R \otimes R \arrow[rr, "\mathrm{id} \otimes c \otimes \mathrm{id}"] \arrow[dr, "\mu_2"'] & & R \otimes R \otimes R \otimes R \arrow[dl, "\mu \otimes \mu"] \\ & R \otimes R & \end{tikzcd}$$

- There exists $\mathcal{S} : R \to R$, the antipode.

Braided Hopf algebras appear in nature:

Let $\pi : H \to K$ be a surjective morphism of Hopf algebras that admits a section $\iota : K \to H$, also a morphism of Hopf algebras. Then

$$R = \{x \in H : (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$

is a braided Hopf algebra; it bears an action and a coaction of K. Also

$$H \simeq R \# K.$$

We say that H is the bosonization of R by K.

II. On the structure of Hopf algebras.

Goal: classify finite-dimensional Hopf algebras.

We describe a method, joint with J. Cuadra, generalizing previous work joint with H.-J. Schneider.

Let C be a coalgebra, $D, E \subset C$. Then

$D \wedge E = \{x \in C : \Delta(x) \in D \otimes C + C \otimes E\}$.

$\wedge^0 D = D$, $\wedge^{n+1} D = (\wedge^n D) \wedge D$.

Invariants of a Hopf algebra H:

- The coradical $H_0 =$ sum of all simple subcoalgebras of H.
- The *Hopf coradical* $H_{[0]}$ is the subalgebra generated by H_0.
- The *standard filtration* is $H_{[n]} = \wedge^{n+1} H_{[0]}$.
- The *associated graded Hopf algebra* $\operatorname{gr} H = \oplus_{n \in \mathbb{N}} H_{[n]}/H_{[n-1]}$.

It turns out that $\operatorname{gr} H \simeq R \# H_{[0]}$, where

- $R = \oplus_{n \in \mathbb{N}} R^n$ is a graded connected algebra and it is a braided Hopf algebra.

Example:

$H = U_q(\mathfrak{b}) = \Bbbk\langle E, K, K^{-1} | KK^{-1} = 1 = K^{-1}K,\ KE = q^2 EK\rangle$,

$\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes 1 + K \otimes E$.

- $H_0 = \Bbbk\langle K, K^{-1}\rangle = H_{[0]} \simeq \Bbbk\mathbb{Z}$.
- $H_n = H_{[n]} =$ subspace spanned by $K^j E^n$, $j \in \mathbb{Z}$.
- $H \simeq \operatorname{gr} H \simeq R\#\Bbbk\langle K, K^{-1}\rangle$, where
- $R = \Bbbk\langle E\rangle$, $c(E^i \otimes E^j) = q^{2ij} E^j \otimes E^i$; $\Delta(E) = E \otimes 1 + 1 \otimes E$.

Example: $H = U_q(\mathfrak{sl}(2))$

- $H_0 = \Bbbk\langle K, K^{-1}\rangle = H_{[0]} \simeq \Bbbk\mathbb{Z}$.

- $H_n = H_{[n]} =$ subspace spanned by $K^j E^i F^{n-i}$, $j \in \mathbb{Z}$, $i \in \mathbb{N}$.

- $\operatorname{gr} H = \Bbbk\langle X, Y, K, K^{-1} | KK^{-1} = 1 = K^{-1}K,$
$KX = q^2 XK$, $KY = q^{-2} YK$, $XY - qYX = 0\rangle$.

$$\Delta(X) = X \otimes 1 + K \otimes X,\ \Delta(Y) = Y \otimes 1 + K^{-1} \otimes Y.$$

- $R = \Bbbk\langle E\rangle$, $c(X \otimes Y) = q^2 Y \otimes X$, $c(Y \otimes X) = q^{-2} X \otimes Y$.

$\Delta(X) = X \otimes 1 + 1 \otimes X$, $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$.

Theorem. (A.– Cuadra).

Any Hopf algebra with injective antipode is a deformation of the bosonization of connected graded braided Hopf algebra by a Hopf algebra generated by a cosemisimple coalgebra.

To provide significance to this result, we should address some fundamental questions.

Question I. Let C be a finite-dimensional cosemisimple coalgebra and $T : C \to C$ a bijective morphism of coalgebras. Classify all finite-dimensional Hopf algebras L generated by C such that $\mathcal{S}_{|C} = T$.

Question II. Given L as in the previous item, classify all finite-dimensional connected graded Hopf algebra in ${}^{L}_{L}\mathcal{YD}$.

Question III. Given L and R as in the previous items, classify all deformations, or liftings, H, that is, such that $\operatorname{gr} H \simeq R\#L$.

About Question I:

Theorem. (Stefan).

Let H *be a Hopf algebra and* C *an* $\mathcal{S}$*-invariant 4-dimensional simple subcoalgebra. If* $1 < \operatorname{ord} \mathcal{S}^2_{|C} = n < \infty$*, then there are a root of unity* ω *and a Hopf algebra morphism* $\mathcal{O}_{\sqrt{-\omega}}(SL_2(\Bbbk)) \to H$*.*

- Classification of finite-dimensional quotients of $\mathcal{O}_q(SL_N(\Bbbk))$: E. Müller.

- Classification of quotients of $\mathcal{O}_q(G)$: A.–G. A. García.

About Question II:

The most important examples of connected graded braided Hopf algebras are Nichols algebras:

given a braided vector space (V, c), its Nichols algebra is a braided Hopf algebra

$$\mathfrak{B}(V) = \oplus_{n \in \mathbb{N}_0} \mathfrak{B}^n(V),$$

- $\mathfrak{B}^0(V) = \Bbbk$, $\mathfrak{B}^1(V) = V$,
- $\mathrm{Prim}(\mathfrak{B}(V)) = V$,
- V generates $\mathfrak{B}(V)$ as an algebra.