Isomorphism classes and automorphisms of finite Hopf algebras of type $A_n$

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1. Introduction

In [2] we classified a large class of finite-dimensional pointed Hopf algebras up to isomorphism. However the following problem was left open for Hopf algebras of type $A, D$ or $E_6$, that is whose Cartan matrix is connected and allows a non-trivial automorphism of the corresponding Dynkin diagram. In this case we described the isomorphisms between two such Hopf algebras with the same Cartan matrix only implicitly. The problem is whether it is possible to compute the isomorphisms in terms of the defining families of parameters.

In the present paper we solve this problem for type $A$. To our surprise there are closed formulas for these isomorphisms. They are based on an action of the non-trivial automorphism $\sigma$ of the Dynkin diagram on the parameter spaces of the Hopf algebras of type $A$.

The Hopf algebras $u(D, \mu)$ of type $A_n$ can be defined as follows. For more details and references to the literature we refer to our survey paper [1]. Let $n \geq 2$ and $(a_{ij})_{1 \leq i,j \leq n}$ the Cartan matrix of type $A_n$ in the form

\begin{equation}
    a_{ij} = \begin{cases} 
    2, & \text{if } i = j, \\
    -1, & \text{if } |i - j| = 1, \\
    0, & \text{if } |i - j| > 1.
    \end{cases}
\end{equation}

Let $\Gamma$ be a finite abelian group, $g_i \in \Gamma$ and $\chi_i$ characters of $\Gamma$ for all $1 \leq i \leq n$. Define $g_{ij} = \chi_j(g_i)$, $1 \leq i, j \leq n$. Then

$$
D = D(\Gamma, (g_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n})
$$

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is a datum of Cartan type if there is a root of unity \( q \) of order \( N > 1 \) in \( k \) such that

\[
(1.2) \quad q_{ii} = q \quad \text{for all} \quad 1 \leq i \leq n, \quad \text{and}
\]

\[
(1.3) \quad q_{ij}q_{ji} = \begin{cases} q^{-1}, & \text{if } |i - j| = 1, \\ 1, & \text{if } |i - j| > 1. \end{cases}
\]

For simplicity we assume that \( N \) is odd. The case when \( N \) is even could be treated in the same way.

Let \( \Phi^+ \) be the positive roots of the root system of type \( A_n \), and let \( k\Phi^+ \) be the set of all families \( \mu = (\mu_{ij})_{1 \leq i < j \leq n+1} \) of scalars in \( k \). A family of root vector parameters for \( D \) is a family \( \mu \in k\Phi^+ \) satisfying the following two conditions.

\[
(\text{R1}) \quad \mu_{ij} = 0 \quad \text{for all} \quad 1 \leq i < j \leq n + 1 \quad \text{with} \quad (g_i g_{i+1} \cdots g_{j-1})^N = 1.
\]

\[
(\text{R2}) \quad \mu_{ij} = 0 \quad \text{for all} \quad 1 \leq i < j \leq n + 1 \quad \text{with} \quad (\chi_i \chi_{i+1} \cdots \chi_{j-1})^N \neq 1.
\]

In (2.9) we associate to any family \( \mu \in k\Phi^+ \) satisfying (R2) a family \((u_{ij}(\mu))_{1 \leq i < j \leq n+1}\) of elements in the group algebra \( k[\Gamma] \). If \( \mu \) satisfies (R2) we can always normalize it such that \( \mu \) becomes a family of root vector parameters without changing the elements \( u_{ij}(\mu) \). This normalization process is discussed in Lemma 2.2. The Hopf algebra \( u(D, \mu) \) is generated as an algebra by the group \( \Gamma \), that is, by generators of \( \Gamma \) satisfying the relations of the group, and \( x_1, \ldots, x_n \), with the relations:

\[
(\text{Action of the group}) \quad gx_i g^{-1} = \chi_i(g)x_i, \quad \text{for all} \quad i, \quad \text{and all} \quad g \in \Gamma,
\]

\[
(\text{Serre relations}) \quad \text{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \quad \text{for all} \quad i \neq j,
\]

\[
(\text{Root vector relations}) \quad x_{ij}^N = u_{ij}(\mu), \quad \text{for all} \quad 1 \leq i < j \leq n + 1.
\]

The coalgebra structure is given by

\[
\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad \Delta(g) = g \otimes g, \quad \text{for all} \quad 1 \leq i \leq \theta, g \in \Gamma.
\]

The Serre relations are the deformed Serre relations where

\[
(\text{ad}_c(x_i))(x_{j_1} \cdots x_{j_s}) = x_i x_{j_1} \cdots x_{j_s} - q_{ij_1} \cdots q_{ij_s} x_{j_1} \cdots x_{j_s} x_i, s \geq 1,
\]

is the braided adjoint action. The root vectors \( x_{ij} \) are iterated braided commutators. They are defined in (2.3).

The non-trivial automorphism of the Dynkin diagram of \( A_n \) is the permutation \( \sigma \in S_n \) defined by \( \sigma(i) = n - i + 1 \) for all \( 1 \leq i \leq n \). For each
\(\mathcal{D}\) we have an action of \(\sigma\) on the parameter spaces by an explicitly defined morphism of affine algebraic varieties

\[\sigma^D : k^{\Phi^+} \to k^{\Phi^+}, \mu \mapsto (\sigma^D_{ij}(\mu))_{1 \leq i < j \leq n+1}.\]

The polynomials \(\sigma^D_{ij}(\mu)\) of degree \(j - i\) are defined in (4.4). By Theorem 2 they define an isomorphism of affine algebraic varieties between the subspaces of all elements of \(k^{\Phi^+}\) satisfying (R2) for \(\mathcal{D}\) resp. for \(\mathcal{D}^\sigma\). Here

\[\mathcal{D}^\sigma = \mathcal{D}(\Gamma, (g_{\sigma(i)})_{1 \leq i \leq n}, (\chi_{\sigma(i)})_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n}).\]

In Corollary 4.4 we show

\[u(\mathcal{D}^\sigma, \sigma^D(\mu)) \cong u(\mathcal{D}, \mu)\]

for all \(\mu \in k^{\Phi^+}\) satisfying (R2).

Our main result is Theorem 5.1, where we compute all Hopf algebra isomorphisms between two Hopf algebras \(u(\mathcal{D}', \mu')\) and \(u(\mathcal{D}, \mu)\) of type \(A_n\). The polynomials \(\sigma^D_{ij}\) play an important role in this theorem. The first essential steps in the proof of Theorem 5.1 is Theorem 3.1, where we compute the basis representation of the \(N\)-th powers of the “reverse root vectors” in the usual PBW-basis formed by the root vectors. The second essential step is Theorem 4.3, where we prove that the images of the \(N\)-th powers of the reverse root vectors in \(u(\mathcal{D}, \mu)\) are the elements \(u^D_{ij}(\sigma^D(\mu))\).

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2. Finite Hopf algebras of type \(A_n\)

2.1. Diagrams of type \(A_n\), root vectors, and reverse root vectors

Let \(n \geq 2\) and \((a_{ij})_{1 \leq i, j \leq n}\) the Cartan matrix of type \(A_n\) in the form (1.1). Let \(\mathbb{Z}[I]\) be the free abelian group with basis \(\alpha_1, \ldots, \alpha_n\). The Weyl group \(W\) of \((a_{ij})\) is the subgroup of \(\text{Aut}(\mathbb{Z}[I])\) generated by the simple reflections \(s_1, \ldots, s_n\) defined by \(s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i\) for all \(1 \leq i, j \leq n\). The root system \(\Phi\) of \((a_{ij})\) is defined by \(\Phi = \cup_{i=1}^n W(\alpha_i)\). It has the basis \(\alpha_1, \ldots, \alpha_n\), and the positive roots with respect to this basis are the elements

\[\alpha_{ij} = \sum_{l=i}^{j-1} \alpha_l, 1 \leq i < j \leq n + 1.\]

Let \(w_0\) be the longest element in \(W\). We choose the reduced representation

\[w_0 = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} \cdots s_1\]
of \( w_0 \) of length \( p = \frac{n(n+1)}{2} \). The corresponding convex ordering of the positive roots

\[
\beta_l = s_{i_l} \cdots s_{i_{l-1}}(\alpha_{i_l}), 1 \leq l \leq p,
\]
is the lexicographic ordering, that is,

\[
\alpha_{12} < \alpha_{13} < \cdots < \alpha_{1,n+1} < \alpha_{23} < \cdots \alpha_{2,n+1} < \cdots < \alpha_{n,n+1}.
\]

Let \( \Gamma \) be a finite abelian group,

\[
\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n})
\]
a Cartan datum and define \( q_{ij} = \chi_j(g_i), 1 \leq i, j \leq n \). Then the Cartan condition \( q_{ij}q_{ji} = q_{a_{ij}}(x_i) = 0 \), \( \forall 1 \leq i, j \leq n, i \neq j \) is equivalent to the following: There is a root of unity \( q \) of order \( N > 1 \) in \( k \) such that (1.2) and (1.3) hold.

For simplicity we assume that \( N \) is odd; as we said, this is not essential and the case when \( N \) is even could be treated in the same way.

Let \( V \in \mathcal{YD} \) with basis \( x_i \in V g_i, 1 \leq i \leq n \); that is \( g \cdot x_i = \chi_i(g)x_i \) for all \( g \in \Gamma \), and \( \delta(x_i) = g_i \otimes x_i \). Then

\[
R = R(\mathcal{D}) = k\langle x_1, \ldots, x_n | \text{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \forall 1 \leq i, j \leq n, i \neq j \rangle
\]
is a Hopf algebra in the braided category \( \mathcal{YD} \). For \( x, y \in R \) we define the braided commutator

\[
[x, y]_c = xy - \mu c(x \otimes y),
\]
where \( c \) denotes the braiding and \( \mu \) the multiplication map of \( R \). As in [1, (6-7) and (6-8)] we define root vectors \( x_{ij} \), \( 1 \leq i < j \leq n+1 \), in \( R \) inductively by

\[
\begin{align*}
(2.1) \quad x_{i,i+1} &= x_i \text{ for all } 1 \leq i \leq n, \\
(2.2) \quad x_{ij} &= [x_{i,i+1}, x_{i+1,j}]_c \text{ for all } 1 \leq i < j \leq n+1, j - i > 1.
\end{align*}
\]

Then

\[
(2.3) \quad x_{ij} = [x_{il}, x_{lj}]_c \text{ for all } 1 \leq i < l < j \leq n+1,
\]
and the root vectors \( x_{ij} \) in the lexicographic order define a PBW-basis of \( R \) [1, Theorem (6.13)].

In addition we define inductively reverse root vectors \( x_{ji} \), \( 1 \leq i < j \leq n+1 \), in \( R \) by

\[
\begin{align*}
(2.4) \quad x_{i+1,i} &= x_i \text{ for all } 1 \leq i \leq n, \\
(2.5) \quad x_{ji} &= [x_{j,j-1}, x_{j-1,j}]_c \text{ for all } 1 \leq i < j \leq n+1, j - i > 1.
\end{align*}
\]
Again it follows that

\[(2.6) \quad x_{ji} = [x_{jl}, x_{li}]_c \text{ for all } 1 \leq i < l < j \leq n + 1.\]

Thus for all \(1 \leq i < j \leq n + 1\), \(x_{ij}\) is any bracketing of the elements \(x_i, x_{i+1}, \ldots, x_{j-1}\) in this order, and \(x_{ji}\) is any bracketing of the reverse sequence \(x_{j-1}, x_{j-2}, \ldots, x_i\).

### 2.2. Root vector parameters and normalization

For any positive root we define elements in the group and characters of the group by

\[(2.7) \quad g_{ij} = \prod_{i \leq l < j} g_i, \chi_{ij} = \prod_{i \leq l < j} \chi_l, 1 \leq i < j \leq n + 1.\]

A family of root vector parameters for \(\mathcal{D}\) is a family \(\mu = (\mu_{ij})_{1 \leq i < j \leq n+1}\) of scalars \(\mu_{ij} \in k\) satisfying the following two conditions.

(R1) \(\mu_{ij} = 0\) for all \(1 \leq i < j \leq n + 1\) with \(g_{Nij}^N = 1\).

(R2) \(\mu_{ij} = 0\) for all \(1 \leq i < j \leq n + 1\) with \(\chi_{Nij}^N \neq 1\).

For all \(1 \leq i < j \leq n + 1\) let

\[I_{ij} = \{(i_1, \ldots, i_r) \mid r \geq 2, i = i_1 < i_2 < \cdots < i_r = j\} \text{.}\]

We denote the set of all families \(\mu = (\mu_{ij})_{1 \leq i < j \leq n+1}\) of elements in \(k\) by \(k^{\Phi^+}\).

For any \(\mu \in k^{\Phi^+}\) we define for all \(1 \leq i < j \leq n + 1\) scalars

\[(2.8) \quad \mu(i_1, \ldots, i_r) = \mu_{i_1i_2} \cdots \mu_{i_{r-1}i_r} \text{ for all } (i_1, \ldots, i_r) \in I_{ij},\]

and elements in the group algebra

\[(2.9) \quad u_{ij}^{\mathcal{D}}(\mu) = \sum_{(i_1, \ldots, i_r) \in I_{ij}} (q - 1)^{N(r-1)} \mu(i_1, \ldots, i_r)(1 - g_{N_{i_{r-1}i_r}}^N).\]

Thus

\[u_{ij}^{\mathcal{D}}(\mu) = \mu_{ij}(1 - g_{ij}^N) + \sum_{i < p < j} \left( \sum_{(i_1, \ldots, i_r) \in I_{ip}} (q - 1)^{N(r-1)} \mu(i_1, \ldots, i_r) \right) \mu_{pj}(1 - g_{pj}^N).\]

We will write \(u_{ij}(\mu) = u_{ij}^{\mathcal{D}}(\mu)\) when \(\mathcal{D}\) is fixed. Recall that \(q = \chi_i(g_i)\) for all \(1 \leq i \leq n\) also depends on \(\mathcal{D}\). It is easy to see that the family
of elements in the group algebra can inductively be defined by

\[(u_{ij}(\mu))_{1 \leq i < j \leq n+1}\]

\[u_{ij}(\mu) = \mu_{ij}(1 - g_{ij}^N) + \sum_{i<p<j} (q-1)^N \mu_{ip}u_{pj}(\mu),\]

with \(1 \leq i < j \leq n + 1\).

This definition agrees with the inductive definition in [1, Theorem 6.18] when \(\mu\) is a family of root vector parameters for \(D\). There we defined

\[C_{kl}^j = (1 - q^{-1})^N \chi_{kl}(g_{ij}) \frac{N(N-1)}{2}, i \leq k < l \leq j,\]

\[u_{ij}(\mu) = \mu_{ij}(1 - g_{ij}^N) + \sum_{i<p<j} C_{ip}^j \mu_{ip}u_{pj}(\mu),\]

\[1 \leq i < j \leq n + 1.\]

Since \(N\) is odd it follows from (R2) that

\[C_{kl}^j \mu_{kl} = (q-1)^N \mu_{kl}\]

for all \(i \leq k < l \leq j\). Thus both definitions do agree.

**Lemma 2.1.** Let \(\mu \in k^\Phi^+\).

1. Suppose \(\mu\) satisfies (R2). Then

\[u_{ij}(\mu) = 0 \text{ for all } 1 \leq i < j \leq n + 1 \text{ with } \chi_{ij}^N \neq 1.\]

2. Suppose \(\mu\) satisfies (R1) and (R3). Then \(\mu\) satisfies (R2), that is, \(\mu\) is a family of root vector parameters for \(D\).

**Proof:** This follows by induction on \(j - i\) from (2.10) since for all \(i < p < j\) the inequality \(\chi_{ij}^N \neq 1\) implies that \(\chi_{ip}^N \neq 1\) or \(\chi_{pj}^N \neq 1\). □

By [1, Theorem 6.18] the families \((u_{ij}(\mu))_{1 \leq i < j \leq n+1}\) are exactly the solutions of the equations

\[\Delta(u_{ij}) = u_{ij} \otimes 1 + g_{ij}^N \otimes u_{ij} + \sum_{i<p<j} (q-1)^N u_{ip}g_{pj}^N \otimes u_{pj}\]

in \(k[\Gamma] \otimes k[\Gamma]\) for all \(1 \leq i < j \leq n + 1\). This characterization of the \(u_{ij}(\mu)\) is used to prove the next lemma. It shows how to “normalize” an arbitrary sequence \(\mu\) so that (R1) is satisfied.

**Lemma 2.2.** Let \(\mu \in k^\Phi^+\). Then there is exactly one family \(\mu' \in k^\Phi^+\) satisfying (R1) such that

\[u_{ij}(\mu) = u_{ij}(\mu') \text{ for all } 1 \leq i < j \leq n + 1.\]

If \(\mu\) satisfies (R2) then \(\mu'\) is a family of root vector parameters for \(D\).
Proof: Let \( u_{ij} = u_{ij}(\mu) \) for all \( 1 \leq i < j \leq n + 1 \). We define the elements \( \mu'_{ij} \) by induction on \( j - i \). Let \( j = i + 1 \). Let

\[
\mu'_{i,i+1} = \begin{cases} 
\mu_{i,i+1}, & \text{if } g_{i,i+1}^N \neq 1, \\
0, & \text{if } g_{i,i+1}^N = 1.
\end{cases}
\]

Then \( (2.12) \) for \( (i, i + 1) \) holds since \( u_{i,i+1}(\mu) = \mu_{i,i+1}(1 - g_{i,i+1}^N) \).

Let \( k > 1 \). Suppose we have already defined \( \mu'_{ij} \) whenever \( j - i \leq k - 1 \) such that \( (2.12) \) holds if \( j - i \leq k - 1 \). Let \( 1 \leq i < j \leq n + 1 \) and assume that \( j - i = k \). If \( g_{ij}^N = 1 \), then we define \( \mu'_{ij} = 0 \). If \( g_{ij}^N \neq 1 \), we define \( \mu'_{ij} \in k \) to be the unique scalar satisfying

\[
(2.13) \quad u_{ij} = \mu'_{ij}(1 - g_{ij}^N) + \sum_{i<p<j} (q-1)^N \mu'_{ip} u_{pj}.
\]

The existence of the scalar \( \mu'_{ij} \) follows from the argument in the induction step of the proof of [1, Theorem 6.18].

Thus we have shown the existence of the family \( \mu' \). Uniqueness follows easily by induction on \( j - i \) from \( (2.10) \).

Suppose that \( \mu \) satisfies (R2). Then \( \mu \) satisfies (R3) by Lemma 2.1 (1), and \( \mu' \) satisfies (R2) by Lemma 2.1 (2). \( \square \)

For any \( \mu \in k^{\Phi^+} \) we define

\[
(2.14) \quad \nu^D(\mu) = \mu', \quad \nu_{ij}^D(\mu) = \mu'_{ij} \text{ for all } 1 \leq i < j \leq n+1,
\]

where \( \mu' \) is the family constructed from \( \mu \) in Lemma 2.2. We call \( \nu^D(\mu) \) the normalization of \( \mu \).

The elements \( \mu'_{ij} = \nu^D_{ij}(\mu) \) can be computed inductively. Let \( 1 \leq i < j \leq n + 1 \) and assume that \( g_{ij}^N \neq 1 \). Then we have \( u_{ij}(\mu) = u_{ij}(\mu') \). We replace \( u_{ij}(\mu) \) and \( u_{ij}(\mu') \) by the right hand sides of \( (2.9) \) and collect all terms with coefficient \( (1 - g_{ij}^N) \). This gives the equality

\[
(2.15) \quad \mu_{ij} + \sum_{i<p<j} \sum_{(i_1,\ldots,i_r)\in I_p} (q-1)^{N(r-1)} \mu(i_1,\ldots,i_r) \mu'_{pj} = \mu'_{ij} + \sum_{i<p<j} \sum_{(i_1,\ldots,i_r)\in I_p} (q-1)^{N(r-1)} \mu'(i_1,\ldots,i_r) \mu'_{pj},
\]

where \( \mu'(i_1,\ldots,i_r) \) is defined in \( (2.8) \) for \( \mu' \). Thus \( \mu'_{ij} \) is a function of \( \mu \) and \( \mu'_{ab}, b - a < j - i \). In particular, we see that \( \nu_{ij}^D \) is a polynomial in the
variables \((\mu_{ij})\) with coefficients in \(\mathbb{Z}[q]\). The polynomial \(\nu^D_{ij}\) depends on \(\mathcal{D}\), more precisely on \(q = \chi_i(g_i)\) and on the numbers

\[
d_{ab} = \begin{cases} 1, & \text{if } g_{ab}^N \neq 1, \\ 0, & \text{if } g_{ab}^N = 1. \end{cases}
\]

For example assume that \(g_{i,i+2}^N \neq 1\). Then

\[
\mu'_{i,i+2} = \begin{cases} \mu_{i,i+2}, & \text{if } g_{i,i+2}^N \neq 1, \\ \mu_{i,i+2} + \mu(i, i+1, i+2), & \text{if } g_{i,i+2}^N = 1. \end{cases}
\]

### 2.3. The Hopf algebras \(u(\mathcal{D}, \mu)\) and isomorphisms

As in [1, 2] we define for any family of root vector parameters \(\mu\) for \(\mathcal{D}\) a finite-dimensional Hopf algebra by

\[(2.16) \quad u(\mathcal{D}, \mu) = R(\mathcal{D}) \# k[\Gamma]/(x_{ij}^N - u_{ij}(\mu) | 1 \leq i < j \leq n + 1).\]

We can extend this definition to all \(\mu \in k^{\Phi^+}\) satisfying (R2) since by Lemma 2.2 then \(\nu^D(\mu)\) is a family of root vector parameters, and \(u_{ij}(\mu) = u_{ij}(\nu^D(\mu))\) for all \(1 \leq i < j \leq n + 1\). By [1, Theorem 6.25] a finite-dimensional pointed Hopf algebra \(A\) is of the form \(A \cong u(\mathcal{D}, \mu)\) if and only if \(\text{gr}(A) \cong u(\mathcal{D}, 0) \# k[\Gamma]\), where \(\text{gr}(A)\) is the graded Hopf algebra associated to the coradical filtration of \(A\).

Let \(\rho \in S_n\) be a diagram automorphism of \((a_{ij})\), that is,

\[a_{ij} = a_{\rho(i)\rho(j)} \text{ for all } 1 \leq i, j \leq n.\]

Then \(\rho = \text{id}\) or \(\rho = \sigma\), where

\[(2.17) \quad \sigma(i) = n - i + 1 \text{ for all } 1 \leq i \leq n.\]

As in [2, Theorem 7.5] let

\[\mathcal{D}^\rho = \mathcal{D}(\Gamma, (g_i^\rho)_{1 \leq i \leq n}, (\chi_i^\rho)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n})\]

be the Cartan datum with \(g_i^\rho = g_{\rho(i)}, \chi_i^\rho = \chi_{\rho(i)}\) for all \(1 \leq i \leq n\). Let \(V^\rho \in \hat{\mathcal{Y}}\mathcal{D}\) with basis \(x_i^\rho \in (V^\rho)_{\chi_i^\rho}\) for all \(1 \leq i \leq n\). Then

\[F^\rho : R(\mathcal{D}^\rho) \rightarrow R(\mathcal{D}), \quad x_i^\rho \mapsto x_{\rho(i)} \text{ for all } 1 \leq i \leq n,\]

defines an isomorphism of braided Hopf algebras in \(\hat{\mathcal{Y}}\mathcal{D}\). For all \(1 \leq i < j \leq n + 1\), we denote the root vector of \(\alpha_{ij}\) in \(R(\mathcal{D}^\rho)\) by \(x_{ij}^\rho\).
Let $\mathcal{D}' = \mathcal{D}(\Gamma', (g'_i)_{1 \leq i \leq n}, (\lambda'_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n})$ be another Cartan datum with finite abelian group $\Gamma'$ and the same Cartan matrix of type $A_n$ as $\mathcal{D}$. Let $\varphi : \Gamma' \to \Gamma$ be a group isomorphism, $\rho \in \mathbb{S}_n$ a diagram automorphism of $(a_{ij})$ and $s = (s_i)_{1 \leq i \leq n}$ a family of non-zero elements in $k$. Let
\begin{equation}
(2.18) \quad s_{ij} = \prod_{i \leq l < j} s_l \text{ for all } 1 \leq i < j \leq n + 1.
\end{equation}

Then $\pi : R(\mathcal{D}) \# k[\Gamma] \to u(\mathcal{D}, \mu)$ be the canonical projection.

The triple $(\varphi, \rho, (s_i))$ is called an isomorphism from $(\mathcal{D}', \mu')$ to $(\mathcal{D}, \mu)$ if the following conditions are satisfied:
\begin{align}
(2.19) & \quad \varphi(g'_i) = g_{\rho(i)}, \chi'_i = \chi_{\rho(i)} \varphi \text{ for all } 1 \leq i \leq n. \\
(2.20) & \quad \varphi(u^\mathcal{D}_i(x'_ij)) = s_{ij}^\mathcal{D} \varphi((x'_ij)^N) \text{ for all } 1 \leq i < j \leq n + 1.
\end{align}

Let $\text{Isom}(\mathcal{D}', \mu'), (\mathcal{D}, \mu)$ be the set of all isomorphisms from $(\mathcal{D}', \mu')$ to $(\mathcal{D}, \mu)$. For Hopf algebras $A', A$ we denote by $\text{Isom}(A', A)$ the set of all Hopf algebra isomorphisms from $A'$ to $A$. Then

**Theorem 2.3.** [2, Theorem 7.2] The map
$$
\text{Isom}(\mathcal{D}', \mu'), (\mathcal{D}, \mu)) \to \text{Isom}(u(\mathcal{D}', \mu'), u(\mathcal{D}, \mu))
$$
given by $(\varphi, \rho, (s_i)) \mapsto F$, where $F(x'_i) = s_i x_{\rho(i)}$ and $F(g' = \varphi(g')$ for all $1 \leq i \leq \theta$ and $g' \in \Gamma'$, is bijective.

The main result in this paper is the explicit computation of the set $\text{Isom}(\mathcal{D}', \mu'), (\mathcal{D}, \mu)$. In Section 3 we first compute the elements $F^\sigma(x^\sigma_{ij})^N$ in terms suitable for our purpose. The next lemma shows that these elements are reverse root vectors. This lemma also allows to derive (2.6) from (2.3).

**Lemma 2.4.** For all $1 \leq i < j \leq n + 1$,
$$
F^\sigma(x^\sigma_{ij}) = x_{n-i+2,n-j+2}.
$$

**Proof:** This follows by induction on $j - i$. Suppose that $j = i + 1$. Then $x_{ij} = x_i$ and $F^\sigma(x^\sigma_{ij}) = x_{\sigma(i)} = x_{n-i+1} = x_{n-i+2,n-j+2}$.

If $j - i \geq 2$, then
$$
F^\sigma(x^\sigma_{ij}) = F^\sigma([x^\sigma_{i}, x^\sigma_{i+1,j}] \sigma) = [x_{\sigma(i)}, F^\sigma(x^\sigma_{i+1,j})] \sigma = [x_{n-i+1}, x_{n-i+1,n-j+2}]_c = x_{n-i+2,n-j+2}
$$
(by induction)

\[\square\]
3. The reverse root vectors

In the next theorem we compute the basis representation of the $N$-th powers of the reverse root vectors in the standard PBW-basis.

As in the last section we fix a diagram $\mathcal{D}$ of Cartan type $A_n$ and let $R = R(\mathcal{D})$. For all $1 \leq i < j \leq n + 1$ we define

$$\tau_{ij} = \prod_{i \leq k < l < j} q_{lk}^N. \quad (3.1)$$

$$\tau(i_1, \ldots, i_r) = \tau_{i_1 i_2} \tau_{i_2 i_3} \cdots \tau_{i_{r-1} i_r}, \text{ for all } (i_1, \ldots, i_r) \in I_{ij}. \quad (3.2)$$

Note that $\tau_{ij} = \prod_{i < l < j} \chi_i^N(g_l)$. We write $\tau_{ij}^D$ instead of $\tau_{ij}$ if we want to emphasize the datum $\mathcal{D}$.

**Theorem 3.1.** Assume that $1 \leq i < j \leq n + 1$. For all $(i_1, \ldots, i_r) \in I_{ij}$ define

$$t(i_1, \ldots, i_r) = (-1)^{j-i-r+1}(q-1)^{N(r-2)} \tau(i_1, \ldots, i_r) - \frac{N-1}{2} \frac{N+1}{2} \tau_{ij}. \quad (3.3)$$

Then

$$x_{ji}^N = \sum_{(i_1, \ldots, i_r) \in I_{ij}} t(i_1, \ldots, i_r) x_{i_1 i_2}^N x_{i_2 i_3}^N \cdots x_{i_{r-1} i_r}^N. \quad (3.3)$$

The proof of Theorem 3.1 will be done after Lemma 3.8.

To compute the coefficients $t(i_1, \ldots, i_r)$ we first change the notation using characteristic functions. We can assume that $j - 2 \geq i$ since $x_{i+1} = x_i$. For natural numbers $k < l$ let $[k, l] = \{k, k+1, \ldots, l\}$.

Let $E_{ij}$ be the set of all functions $e : [i, j-2] \to \mathbb{N}$ with values in $\{0, 1\}$. We consider the bijection

$$\Omega : I_{ij} \to E_{ij}$$

given for all $(i_1, \ldots, i_r) \in I_{ij}$ and $l \in [i, j-2]$ by

$$\Omega(i_1, \ldots, i_r)(l) = \begin{cases} 1, & \text{if } l \in \{i_2 - 1, \ldots, i_r - 1\}, \\ 0, & \text{otherwise.} \end{cases}$$

For any $e \in E_{ij}$ define

$$|e| = |\{l \mid i \leq l \leq j - 2, e(l) = 0\}|.$$

If $e = \Omega(i_1, \ldots, i_r)$ then

$$|e| = j - i - r + 1. \quad (3.4)$$

The constant function in $E_{ij}$ with value 1 resp. 0 will be denoted by (1) resp. (0). Thus $(1) = \Omega(i, i+1, \ldots, j)$ and $(0) = \Omega(i, j)$. 


For \(e, f \in E_{ij}\) we write \(e \leq f\) if for all \(i \leq l \leq j - 2\), \(e(l) = 0\) implies \(f(l) = 0\).

**Lemma 3.2.** Let \(1 \leq i < j \leq n + 1\), \(j - i \geq 2\), and \(f \in E_{ij}\), \((1) \neq f\). Then

\[
\sum_{e \in E_{ij}, e \leq f} (-1)^{|e|} = 0,
\]

\[
\sum_{(i_1, \ldots, i_r) \in I_{ij}} (-1)^r = 0.
\]

**Proof:** Since \(f \neq (1)\) we can choose an index \(l\) with \(f(l) = 0\). Then \(\{e \in E_{ij} | e \leq f\}\) is the disjoint union of elements \(e\) with \(e(l) = 0\) and with \(e(l) = 1\), and (3.5) is obvious. To prove (3.6) we consider the case of (3.5) with \(f = (0)\). Then \(\sum_{e \in E_{ij}} (-1)^{|e|} = 0\), and (3.6) follows from the bijection \(\Omega\) and (3.4). \(\square\)

For all \(e = \Omega(i_1, \ldots, i_r) \in E_{ij}\) let

\[
\tau_e = (q - 1)^{|e|}(\tau_{i_1i_2} \tau_{i_2i_3} \cdots \tau_{i_r-1i_r})^{N-1},
\]

\[
t_e = (-1)^{|e|} \tau_e^{-1}(q - 1)^{N(j - i - 1)} \tau_{ij}^{N+1},
\]

\[
x_e^N = x_{i_1}^N x_{i_2}^N \cdots x_{i_r-1}^N.
\]

Note that \(t_e = t(i_1, \ldots, i_r)\) if \(e = \Omega(i_1, \ldots, i_r)\). This follows from the definitions using (3.4). Hence (3.3) in Theorem 3.1 can be restated as

\[
x_{ji}^N = \sum_{e \in E_{ij}} t_e x_e^N.
\]

The idea of the proof of Theorem 3.1 is to project \(R\) onto skew-polynomial rings \(R_e\), one for each \(e \in E_{ij}\). Before we begin the proof we establish some technical results on these projections.

**Definition 3.3.** For any \(e \in E_{ij}\), let \(R_e\) be the algebra generated by \(x_i, x_{i+1}, \ldots, x_{j-1}\) with relations

\[
x_i x_{i+1} - q_{i,i+1} x_{i+1} x_i = 0, \quad \text{if } e(l) = 1, i \leq l \leq j - 2,
\]

\[
x_{i+1} x_l - q_{i+1,i} x_l x_{i+1} = 0, \quad \text{if } e(l) = 0, i \leq l \leq j - 2,
\]

\[
x_k x_l - q_{k,l} x_l x_k = 0, \quad \text{if } i \leq k, l \leq j - 1, |k - l| \geq 2.
\]

**Lemma 3.4.** For any \(f \in E_{ij}\), the natural projection

\[
\pi_f : R \rightarrow R_f, \pi_f(x_l) = \begin{cases} 
    x_l, & \text{if } i \leq l \leq j - 1, \\
    0, & \text{otherwise},
\end{cases}
\]

where \(1 \leq l \leq n\).
is a well-defined algebra map, and for all \( i \leq u < v \leq j, v - u \geq 2 \),

\[
\pi_f(x_{uv}) = 0, \quad \text{if } f(l) = 1 \text{ for some } u \leq l < l + 2 \leq v,
\]

\[
\pi_f(x_{vu}) = 0, \quad \text{if } f(l) = 0 \text{ for some } u \leq l < l + 2 \leq v.
\]

**Proof:** The Serre relations can be reformulated according to the following identities

\[
-q_{l+1,l} \text{ad}_c(x_l)^2(x_{l+1}) = x_l[x_{l+1}, x_l]_c - q_{l+1,l} [x_{l+1}, x_l]_c x_l,
\]

\[
-q_{l,l+1} \text{ad}_c(x_{l+1})^2(x_l) = x_{l+1}[x_l, x_{l+1}]_c - q_{l,l+1} [x_l, x_{l+1}]_c x_{l+1},
\]

in the free algebra \( k\langle x_1, \ldots, x_n \rangle \) for all \( 1 \leq l \leq n - 1 \). Hence both Serre relations \( \text{ad}_c(x_l)^2(x_{l+1}) = 0 \) and \( \text{ad}_c(x_{l+1})^2(x_l) = 0 \) hold in \( R_f \) for all \( i \leq l < j \), since \( [x_l, x_{l+1}]_c = 0 \) by (3.8) if \( f(l) = 1 \), and \( [x_{l+1}, x_l]_c = 0 \) by (3.9) if \( f(l) = 0 \). Thus \( \pi_f \) is well-defined.

To prove (3.11) note that by (2.3)

\[
x_{uv} = \begin{cases} [x_{ul}, x_{l+2}]_c, & \text{if } u < l < l + 2 = v, \\ [x_{ul}, [x_{l+2}, x_{l+2,v}]]_c, & \text{if } u < l < l + 2 < v, \end{cases}
\]

and that \( \pi_f(x_{l+2}) = 0 \) by definition of \( \pi_f \). In the same way (3.12) follows from (2.6). \( \square \)

We note the following obvious rule in skew polynomial rings.

**Lemma 3.5.** Let \( x_1, \ldots, x_m \) be elements in an algebra such that

\[ x_l x_k = p_{kl} x_k x_l \text{ for all } k < l, \]

where \( p_{kl} \in k \) for all \( k < l \). Then for any natural number \( N \),

\[ (x_1 \cdots x_m)^N = \prod_{k<l} p_{lk}^{N(N-1)/2} x_1^N \cdots x_m^N. \]

\( \square \)

**Lemma 3.6.** Let \( f \in E_{ij} \) and \( i \leq k < l \leq j \) with \( k \leq l - 2 \). Suppose that \( f(k) = f(k+1) = \cdots = f(l-2) = 0 \). Then

\[ \pi_f(x_{kl}^N) = (q-1)^{(N(k-1)/2)} \frac{N!}{t_{kl}} x_k^N x_{k+1}^N \cdots x_{l-1}^N. \]
PROOF: We first prove by induction on \( l - k \) that
\[
\pi_f(x_{kl}) = (1 - q^{-1})^{l-1}x_kx_{k+1} \cdots x_{l-1}.
\] (3.15)

Suppose that \( l = k + 2 \). Then
\[
\pi_f(x_{k,k+2}) = x_kx_{k+1} - q_{k,k+1}x_{k+1}x_k
\]
\[
= x_kx_{k+1} - q_{k,k+1}q_{k+1,k}x_kx_{k+1}
\]
\[
= (1 - q^{-1})x_kx_{k+1}.
\]
(by (3.11) and (3.9) and \( f(k) = 0 \))

This proves (3.15) for \( l = k + 2 \).

For the induction step let \( l - k > 2 \). We obtain by induction
\[
\pi_f(x_{kl}) = \left[ x_k, x_{k+1}, \ldots, x_l \right]_e
\]
\[
= \left[ x_k, (1 - q^{-1})^{l-k-2}x_{k+1}x_{k+2} \cdots x_{l-1} \right]_e
\]
\[
= (1 - q^{-1})^{l-k-2}(x_kx_{k+1} \cdots x_{l-1} - q_{k,k+1}q_{k+1,k} \cdots q_{k,l-1}x_kx_{k+1} \cdots x_{l-1}x_k).
\]

Since \( x_{k+1}x_{k+2} \cdots x_{l-1}x_k = q_{l-1,k}q_{l-2,k} \cdots q_{k+1,k}x_kx_{k+1} \cdots x_{l-1} \) in \( R_f \) by \( f(k) = 0 \), (3.9) and (3.10), and
\[
q_{k,k+1}q_{k+1,k} \cdots q_{k,l-1}q_{l-1,k}q_{l-2,k} \cdots q_{k+1,k} = q^{-1}
\]
by (1.3), equation (3.15) for \( k, l \) follows.

Since \( f(k) = f(k + 1) = \cdots = f(l - 2) = 0 \), (3.9) and (3.10) imply by Lemma 3.5 that
\[
(x_kx_{k+1} \cdots x_{l-1})^N = \prod_{k \leq \mu < \nu < l} q_{\nu, \mu}^N x_k^N x_{k+1}^N \cdots x_{l-1}^N.
\]

Hence Lemma 3.6 follows from (3.15) by taking \( N \)-th powers. Note that
\[
(1 - q^{-1})^N = (q - 1)^N
\]
since \( q^N = 1 \). \( \square \)

**Lemma 3.7.** Let \( e, f \in E_{ij} \). Then
\[
\pi_f(x_e^N) = \begin{cases} 
\tau_e x_i^N x_{i+1}^N \cdots x_{j-1}^N, & \text{if } e \leq f, \\
0, & \text{otherwise}.
\end{cases}
\]

PROOF: Let \( e = \Omega(i_1, \ldots, i_r) \). Suppose that \( e \not\leq f \), that is \( f(l) = 1 \) and \( e(l) = 0 \) for some \( i \leq l \leq j - 2 \). Then \( l + 1 \not\in \{i_1, \ldots, i_r\} \), since \( e(l) = 0 \). Hence there is an index \( s \) with \( i_s \leq l < l + 2 \leq i_{s+1} \). Since \( f(l) = 1 \), it follows by (3.11) that \( \pi_f(x_{i_s,i_{s+1}}) = 0 \), and thus \( \pi_f(x_e^N) = 0 \).
Now assume $e \leq f$. Since $\pi_f$ is an algebra map, it is enough to show for all $1 \leq s < r$ that
\begin{equation}
(3.16) \quad \pi_f(x_{i_s,i_{s+1}}^N) = (q - 1)^{N(i_{s+1} - i_s - 1)} \frac{N-1}{r_{i_s,i_{s+1}}} x_{i_{s}}^{N} x_{i_{s+1}}^{N} \cdots x_{i_{s+1}-1}^{N}.
\end{equation}

Note that by (3.4) \( \sum_{s=1}^{r-1} (i_{s+1} - i_s - 1) = j - i - r + 1 = |e| \).

If $i_s + 1 = i_{s+1}$, then $x_{i_{s},i_{s+1}} = x_{i_{s}}$ and (3.16) is obvious.

If $i_s \leq i_{s+1} - 2$, then $e(l) = 0$ for all $i_s \leq l \leq i_{s+1} - 2$ by definition of the function $\Omega$. Hence $f(l) = 0$ for all $i_s \leq l \leq i_{s+1} - 2$ since $e \leq f$, and (3.16) follows from Lemma 3.6. □

**Lemma 3.8.**
\[ \pi(1)(x_{ji}^N) = t(1)x_{i+1}x_{i+2} \cdots x_{j-1}. \]

**Proof:** We first prove by induction on $j - i$ that
\begin{equation}
(3.17) \quad \pi(1)(x_{ji}) = (q - 1)^{j-i-1} \left( \prod_{i \leq k < l < j} q_{lk} \right) x_{i}x_{i+1} \cdots x_{j-1}.
\end{equation}

Suppose that $j = i + 2$. Then
\begin{equation}
(3.18) \quad x_{i+1}x_{i} = q_{i+1}^{-1}x_{i}x_{i+1} \text{ in } R(1).
\end{equation}

Hence
\begin{align*}
\pi(1)(x_{i+2,i}) &= x_{i+1}x_{i} - q_{i+1,i}x_{i}x_{i+1} \\
&= (q_{i+1}^{-1} - q_{i+1,i})x_{i}x_{i+1} \\
&= (q - 1)q_{i+1,i}x_{i}x_{i+1} \quad \text{(since } q_{i,i+1}q_{i+1,i} = q^{-1} \text{ by (1.3)).}
\end{align*}

For the induction step let $j - i > 2$. Then by induction
\begin{align*}
\pi(1)(x_{ji}) &= [x_{j-1}, x_{j-1}, i]c \\
&= [x_{j-1}, (q - 1)^{j-i-2} \left( \prod_{i \leq k < l < j-1} q_{lk} \right) x_{i}x_{i+1} \cdots x_{j-2}]c \\
&= (q - 1)^{j-i-2} \left( \prod_{i \leq k < l < j-1} q_{lk} \right) (x_{j-1}x_{i}x_{i+1} \cdots x_{j-2} \\
&\quad - q_{j-1,i}q_{j-1,i+1} \cdots q_{j-1,j-2}x_{i}x_{i+1} \cdots x_{j-2}x_{j-1}).
\end{align*}

Since in $R(1)$
\begin{equation}
(3.19) \quad x_{j-1}x_{j-2} = q_{j-2,j-1}^{-1}x_{j-2}x_{j-1} \\
= q_{j-1,j-2}q_{j-2}x_{j-2}x_{j-1},
\end{equation}
and hence
\[ x_{j-1}x_{i+1} \cdots x_{j-2} = q_{j-1,j}q_{j-1,i+1} \cdots q_{j-1,j-2}q x_{i+1} \cdots x_{j-1}, \]
it follows that
\[ \pi(1)(x_{ji}) = (q - 1)^{j-i-2} \prod_{i \leq k < l < j} q_{lk} \prod_{k=i}^{j-2} q_{j-1,k}(q - 1)x_{i+1} \cdots x_{j-1} \]
\[ = (q - 1)^{j-i-1} \prod_{i \leq k < l < j} q_{lk}x_{i+1} \cdots x_{j-1}. \]
This finishes the proof of (3.17).

By (3.8), (3.10) and Lemma 3.5
\[ (x_{i+1} \cdots x_{j-1})^N = q^{\frac{N(N-1)}{2}(j-i-1)} \prod_{i \leq k < l < j} q_{lk}^N x_{i+1}^N \cdots x_{j-1}^N. \]
Hence Lemma 3.8 follows from (3.17) by taking $N$-th powers. Note that $q^{\frac{N(N-1)}{2}} = 1$ since $N$ is odd by assumption. □

We now prove Theorem 3.1.

PROOF: Since the root vectors $x_{ij}$ in the lexicographic order define a PBW-basis of $R$, there are uniquely determined coefficients $\tilde{t}_e \in k$, $e \in E_{ij}$, with
\[ x_{ji}^N = \sum_{e \in E_{ij}} \tilde{t}_e x_e^N, \]
 cf. [2, Th. 2.6 (2)] – and compare with [1, Lemma 6.9]. By (3.7) we have to show that $t_e = t_e$ for all $e \in E_{ij}$.

To prove that $\tilde{t}_e = t_e$, we apply $\pi(1)$ to both sides of (3.20). For all $(1) \neq e \in E_{ij}$ we see from Lemma 3.7 that $\pi(1)(x_e^N) = 0$, since $e \not\leq (1)$. Hence $\pi(1)(\sum_{e \in E_{ij}} \tilde{t}_e x_e^N) = \tilde{t}_e x_e^N x_{i+1}^N \cdots x_{j-1}^N$, and $\tilde{t}_e = t_e$ by Lemma 3.8.

Let $(1) \neq f \in E_{ij}$. Then $f(l) = 0$ for some $i \leq l \leq j - 2$ and $\pi_f(x_{ji}) = 0$ by (3.12). Hence applying $\pi_f$ to both sides of (3.20) and using Lemma 3.7 we obtain $0 = \sum_{e \leq f} \tilde{t}_e \tau_e x_e^N x_{i+1}^N \cdots x_{j-1}^N$, hence
\[ \sum_{e \leq f} \tilde{t}_e \tau_e = 0. \]
Note that by definition
\[ t_f = (-1)^{|f|} \tau_f^{-1} t_{(1)} \text{ for all } f \in E_{ij}. \]
To finish the proof of the theorem we therefore show by induction on $|f|$ that
\begin{equation}
\tilde{t}_f \tau_f = (-1)^{|f|} t_{(1)} \text{ for all } f \in E_{ij}.
\end{equation}

Suppose that $|f| = 0$. Then $f = (1)$. Since $\tilde{t}_{(1)} = t_{(1)}$ and $\tau_{(1)} = 1$, (3.22) follows for $f = (1)$.

For the induction step we note that $|e| < |f|$ for all $e \leq f, e \neq f$. Hence we get by induction from (3.21) for all $f \neq (1)$
\begin{equation}
\tilde{t}_f \tau_f = -\sum_{e \leq f, e \neq f} \tilde{t}_e \tau_e = -\sum_{e \leq f, e \neq f} (-1)^{|e|} t_{(1)}.
\end{equation}

By (3.5) $\sum_{e \leq f} (-1)^{|e|} = 0$ for $f \neq (1)$, and (3.22) follows from (3.23).

4. The action of the diagram automorphism on root vector parameters

As in Section 2 let
\[ D = D(\Gamma, (g_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n}) \]
be a datum with finite abelian group $\Gamma$ and Cartan matrix (1.1) of Type $A_n$. Recall that $\sigma$ denotes the non-trivial diagram automorphism of $(a_{ij})$ given by (2.17). In this section we will construct for each $\mu \in k^\Phi^+$ satisfying (R2) a family $\sigma^D(\mu) \in k^\Phi^+$ satisfying (R2) such that the isomorphism
\[ F^\sigma \# \text{id} : R(D^\sigma) \# k[\Gamma] \to R(D) \# k[\Gamma] \]
induces an isomorphism $u(D^\sigma, \sigma^D(\mu)) \to u(D, \mu)$. We begin with a technical lemma to simplify the constants $\tau(i_1, \ldots, i_r)$ in Theorem 3.1 when they appear as factors of certain root vector parameters.

Lemma 4.1. Let $1 \leq i < j \leq n + 1, (l_1, \ldots, l_m) \in I_{ij}, (k_1, \ldots, k_r) \in I_{1m}$. Let $\mu$ be a family of root vector parameters for $D$. Then
\[ \mu(l_1, \ldots, l_m) \tau_{ij} = \mu(l_1, \ldots, l_m) \tau(l_{k_1}, l_{k_2}, \ldots, l_{k_r}). \]
Let this holds for $S(4.1)$.

The Lemma is true for $\text{Lemma 4.2.}$ since ($\text{Proof:}$

Assume that $k < m$. Then $(\chi_{l,k_{k+1}})^N = 1$ for all $1 \leq k < m$. In particular $(\chi_{l_1,l_{k_1}})^N = 1$ for all $2 \leq s \leq r$. Therefore

$$
\tau_{ij} = \prod_{i<l<j} (\chi_{il})^N(g_i)
= \prod_{l_{k_1} < l < l_{k_2}} (\chi_{l_{k_1}l})^N(g_l) \prod_{l_{k_2} \leq l < l_{k_3}} (\chi_{l_{k_2}l})^N(g_l) \cdots \prod_{l_{k_r-1} \leq l < l_{k_r}} (\chi_{l_{k_r-1}l})^N(g_l)
= \tau_{l_{k_1},l_{k_2}} \cdots \tau_{l_{k_r-1},l_{k_r}} = \tau(l_{k_1}, l_{k_2}, \ldots, l_{k_r}),
$$

since $(\chi_{l_1,l})^N = (\chi_{l_1,l_2}^N) (\chi_{l_1,l}^N)$ for all $2 \leq s < r$ and $l_{k_s} \leq l$. □

We also need the following identity in group algebras.

**Lemma 4.2.** Let $G$ be a group, $m \geq 2$ and $h_{st}$ elements in $G$ for all $1 \leq s < t \leq m$. Assume

$$
h_{rs} h_{s,s+1} = h_{r,s+1} \text{ for all } 1 \leq r \leq 2, 1 \leq s < m.
$$

Then

$$
\sum_{(k_1, \ldots, k_r) \in I_{1m}} (-1)^r (1 - h_{k_{r-1}k_r})
= \sum_{(k_1, \ldots, k_r) \in I_{1m}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1}).
$$

**Proof:** Let

$$
S_m = \sum_{(k_1, \ldots, k_r) \in I_{1m}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1}).
$$

The Lemma is true for $m = 2$. We show by induction on $m > 2$ that

(4.1) $S_m = h_{2m} - h_{1m}$ if $m > 2$.

This holds for $m = 3$ since

$$
S_3 = -(1 - h_{12})(1 - h_{23}) + 1 - h_{12} = (1 - h_{12})h_{23} = h_{23} - h_{13}.
$$

The induction step follows from

$$
S_{m+1} = \sum_{(k_1, \ldots, k_r) \in I_{1,m+1}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1})
+ \sum_{(k_1, \ldots, k_r) \in I_{1,m+1}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1})
= -S_m (1 - h_{m,m+1}) + S_m
= S_m h_{m,m+1}.
$$
On the other hand if \( m > 2 \) then
\[
\sum_{(k_1, \ldots, k_r) \in I_{1m}} (-1)^r (1 - h_{k_{r-1}k_r})
\]
\[
= 1 - h_{1m} + \sum_{1 < p < m} \sum_{(k_1, \ldots, k_r) \in I_{1p}} (-1)^r (1 - h_{k_{r-1}k_r})
\]
\[
= 1 - h_{1m} + \sum_{1 < p < m} \sum_{(k_1, \ldots, k_r) \in I_{1p}} (-1)^{-1} (1 - h_{pm})
\]
\[
= 1 - h_{1m} - (1 - h_{2m}) \text{ by (3.6)}
\]
\[
= h_{2m} - h_{1m}.
\]
\( \Box \)

We introduce the notation
\[ \tilde{i} = n - i + 2 \text{ for all } 1 \leq i \leq n + 1 \]
for the non-trivial diagram automorphism of \( A_{n+1} \). Note that the map
\[
(4.2) \quad I_{\tilde{j} \tilde{i}} \to I_{ij}, (i_1, \ldots, i_r) \mapsto (\tilde{i}_r, \ldots, \tilde{i}_1),
\]
is bijective for all \( 1 \leq i < j \leq n + 1 \). Recall that
\[
\mathcal{D}^\sigma = \mathcal{D}(\Gamma, (g^\sigma_i)_{1 \leq i \leq n}, (\chi^\sigma_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n}),
\]
where \( g^\sigma_i = g_{\sigma(i)} \), \( \chi^\sigma_i = \chi_{\sigma(i)} \) for all \( 1 \leq i \leq n \). Then
\[
(4.3) \quad g^\sigma_{ij} = \prod_{i \leq l < j} g^\sigma_l = g^\sigma_{\tilde{j} \tilde{i}}, \text{ for all } 1 \leq i < j \leq n + 1,
\]
since \( g^\sigma_{ij} = g^\sigma_{i+1} \cdots g^\sigma_{j-1} = g_{n-i+1} g_{n-i} \cdots g_{n-j+2} = g^\sigma_{\tilde{j} \tilde{i}} \). For all \( \mu \in k^{\Phi^+} \) and all \( 1 \leq i < j \leq n + 1 \) we define
\[
(4.4) \quad \sigma^\mathcal{D}_D(\mu) = \tau_{\tilde{j} \tilde{i}} (-1)^{-i+1} \sum_{(i_1, \ldots, i_r) \in I_{ij}} (q - 1)^{N(r-2)} \mu(\tilde{i}_r, \ldots, \tilde{i}_1),
\]
\[
\sigma^\mathcal{D}(\mu) = (\sigma^\mathcal{D}_D(\mu))_{1 \leq i < j \leq n+1}.
\]
Here \( \tau_{\tilde{j} \tilde{i}} = \tau_{\tilde{j} \tilde{i}}^\mathcal{D} \) and \( q \) depend on \( \mathcal{D} \), or more precisely on the braiding matrix \( (g_{ij}) \) of \( \mathcal{D} \). Note that \( q = \chi_i(g_i) = \chi^\sigma_i(g^\sigma_i) \) for all \( 1 \leq i \leq n \).

We will see in the next theorem that \( \sigma^\mathcal{D} \) defines an isomorphism of affine algebraic varieties between the subspaces of all elements of \( k^{\Phi^+} \) satisfying (R2) for \( \mathcal{D} \) resp. for \( \mathcal{D}^\sigma \).

By abuse of notation we will denote the images of the reverse root vectors in the quotient Hopf algebras \( u(\mathcal{D}, \mu) \) again by \( x_{ji} \).
Theorem 4.3. Let $\mu \in k^{d^+}$ satisfying (R2) for $D$. Then

1. $u_{ij}^{D^\sigma}(\sigma^D(\mu)) = (x_{ij})^N$ for all $1 \leq i < j \leq n + 1$.

2. The family $\sigma^D(\mu)$ satisfies (R2) for $D^\sigma$ and $\sigma^{D^\sigma}(\sigma^D(\mu)) = \mu$.

Proof: (1) Let $1 \leq i < j \leq n + 1$. We write

$$\mu' = \sigma^D(\mu).$$

We first compute $u_{ij}^{D^\sigma}(\mu')$. By (2.9) and (4.3),

$$u_{ij}^{D^\sigma}(\mu') = \sum_{(i_1, \ldots, i_r) \in I_{ij}} (q - 1)^{N(r-2)} \mu'(i_1, \ldots, i_r)(1 - (g_{i_r i_{r-1}})^N).$$

By (4.4) we can write for all $1 \leq i \leq i_{t-1} < i_t \leq j$

$$\mu'_{i_{t-1} i_t} = (-1)^{i_t - i_{t-1} + 1} \tau_{i_t i_{t-1}} \sum_{(l_i^1, \ldots, l_i^r) \in I_{i_{t-1} i_t}} (q - 1)^{N(s_t - 2)} \mu(l_{i_1}^1, \ldots, l_{i_r}^1).$$

Hence we obtain

$$u_{ij}^{D^\sigma}(\mu') = \sum_{(i_1, \ldots, i_r) \in I_{ij}} (q - 1)^{N(r-2)}(-1)^{j-i+r-1} \tau_{i_r i_1} \prod_{t=1}^{r-1} \sum_{(l_t^{s_t}, \ldots, l_t^1) \in I_{i_{t-1} i_t}} (q - 1)^{N(s_t - 2)} \mu(l_{i_1}^{s_t}, \ldots, l_{i_1}^1).$$

(4.5)

On the other hand by Theorem 3.1

$$(x_{ij})^N = \sum_{(i_1, \ldots, i_r) \in I_{ij}} t(i_r, \ldots, i_1) \mu_{i_r i_{r-1}} \cdots u_{ij}^{D^\sigma}(\mu').$$
By (2.9) and (4.2) we have for all \(1 \leq i \leq i_{t-1} < i_t \leq j\)
\[
    u_{i_{t-1}i_t} = \sum_{(\tilde{l}^i_1, \ldots, \tilde{l}^i_n) \in I_{i_{t-1}i_t}} (q - 1)^{N(s_t-2)} \mu(\tilde{l}^i_1, \ldots, \tilde{l}^i_n)(1 - (g_{i_{t-1}i_t})^N).
\]

Again we get a large sum of products as before:
\[
    (x_{ij})^N = \sum_{(i_1, \ldots, i_r) \in I_{ij}} t(i_{r+1}, \ldots, i_1) \times \left( \sum_{(l^2_1, \ldots, l^2_n, l'_1, \ldots, l'_m) \in I_{i_1i_2}} (q - 1)^{N(s_2-2)} \mu(l^2_1, \ldots, l^2_n) \right) \times \cdots \times \left( \sum_{(l^r_1, \ldots, l^r_n) \in I_{i_{r-1}i_r}} (q - 1)^{N(s_r-2)} \mu(l^r_1, \ldots, l^r_n) \right) \times \left( 1 - (g_{i_{r-1}i_r})^N \right) \cdots \left( 1 - (g_{i_{r-1}i_r})^N \right).
\]

The point of the proof is to change the order of the summation indices. We have to sum over all sequences \((l^2_1, \ldots, l^2_n, l'_1, \ldots, l'_m) \in I_{ij}\)
with \(l^2_1 = i_1, l^2_2 = i_2 = l^3_1, \ldots, l^r_m = i_r\), where each sequence has \(m = s_2 + \cdots + s_r - r + 2\) elements. Equivalently we can start with an arbitrary sequence \((l_1, \ldots, l_m) \in I_{ij}\), then take all subsequences \((l_{k_1}, \ldots, l_{k_r})\), \((k_1, \ldots, k_r) \in I_{i_1m}\), of \((l_1, \ldots, l_m)\) and define \((i_1, \ldots, i_r) \in I_{ij}\) by \(i_p = l_{k_p}\) for all \(p\), \(1 \leq p \leq r\). Thus the right hand side of (4.5) becomes
\[
    \sum_{(l_1, \ldots, l_m) \in I_{ij}} \sum_{(k_1, \ldots, k_r) \in I_{i_1m}} (-1)^{j-i-r+1}(q - 1)^{N(m-2)} \mu(l_m, \ldots, l_1) \tau(l_{k_r}, l_{k_{r-1}}, \ldots, l_{k_1})(1 - (g_{l_{k_{r-1}}l_{k_r}})^N),
\]

The point of the proof is to change the order of the summation indices. We have to sum over all sequences \((l^2_1, \ldots, l^2_n, l'_1, \ldots, l'_m) \in I_{ij}\)
with \(l^2_1 = i_1, l^2_2 = i_2 = l^3_1, \ldots, l^r_m = i_r\), where each sequence has \(m = s_2 + \cdots + s_r - r + 2\) elements. Equivalently we can start with an arbitrary sequence \((l_1, \ldots, l_m) \in I_{ij}\), then take all subsequences \((l_{k_1}, \ldots, l_{k_r})\), \((k_1, \ldots, k_r) \in I_{i_1m}\), of \((l_1, \ldots, l_m)\) and define \((i_1, \ldots, i_r) \in I_{ij}\) by \(i_p = l_{k_p}\) for all \(p\), \(1 \leq p \leq r\). Thus the right hand side of (4.5) becomes
\[
    \sum_{(l_1, \ldots, l_m) \in I_{ij}} \sum_{(k_1, \ldots, k_r) \in I_{i_1m}} (-1)^{j-i-r+1}(q - 1)^{N(m-2)} \mu(l_m, \ldots, l_1) \tau(l_{k_r}, l_{k_{r-1}}, \ldots, l_{k_1})(1 - (g_{l_{k_{r-1}}l_{k_r}})^N),
\]
and the right hand side of (4.6) becomes

\[(4.8) \quad \sum_{(l_1, \ldots, l_m) \in I_{ij}} \sum_{(k_1, \ldots, k_r) \in I_{1m}} (q-1)^{N(m-r)} \mu(\widetilde{l}_m, \ldots, \widetilde{l}_1)t(l_{kr}, \widetilde{l}_{kr-1}, \ldots, \widetilde{l}_{k_1}) \cdot (1 - (g_{l_{kr-1}l_{kr-1}})^N) \cdot (1 - (g_{l_{k_1+1}l_{k_1}})^N).\]

Both expressions (4.7) and (4.8) can be simplified. By Lemma 4.1 we can write in (4.7)

\[\mu(\widetilde{l}_m, \ldots, \widetilde{l}_1) \tau(\widetilde{l}_{kr}, \widetilde{l}_{kr-1}, \ldots, \widetilde{l}_{k_1}) = \mu(\widetilde{l}_m, \ldots, \widetilde{l}_1) \tau_{ij}.\]

Similarly by Lemma 4.1 and since \(N\) is odd we have in (4.8)

\[\mu(\widetilde{l}_m, \ldots, \widetilde{l}_1)t(l_{kr}, \widetilde{l}_{kr-1}, \ldots, \widetilde{l}_{k_1}) = \mu(\widetilde{l}_m, \ldots, \widetilde{l}_1)(-1)^{j-i-r+1}(q-1)^{N(r-2)} \cdot \tau(\widetilde{l}_{kr}, \widetilde{l}_{kr-1}, \ldots, \widetilde{l}_{k_1}) \cdot \frac{N-1}{2} \cdot \frac{N+1}{2} \cdot \tau_{ij}.\]

After this simplification we finally obtain

\[(4.9) \quad u_{ij}^{D^\sigma}(\mu') = (-1)^{j-i+1} \cdot \sum_{(l_1, \ldots, l_m) \in I_{ij}} (q-1)^{N(m-2)} \mu(\widetilde{l}_m, \ldots, \widetilde{l}_1) \cdot \sum_{(k_1, \ldots, k_r) \in I_{1m}} (1 - (g_{l_{kr}l_{kr-1}})^N),\]

and

\[(4.10) \quad (x_{ij})^N = (-1)^{j-i+1} \cdot \sum_{(l_1, \ldots, l_m) \in I_{ij}} (q-1)^{N(m-2)} \mu(\widetilde{l}_m, \ldots, \widetilde{l}_1) \cdot \sum_{(k_1, \ldots, k_r) \in I_{1m}} (1 - (g_{l_{kr}l_{kr-1}})^N) \cdot (1 - (g_{l_{k_1+1}l_{k_1}})^N),\]

and (1) follows from Lemma 4.2 with

\[h_{st} = (g_{l_{l_1}l_s})^N \text{ for all } 1 \leq s < t \leq m\]

and for each sequence \((l_1, \ldots, l_m)\).

(2) To prove that \(\mu'\) satisfies (R2) for \(D^\sigma\) let \(\chi'_{ij} \neq 1\). Then \(\chi'_{ji} \neq 1\). Hence for all \((i_1, \ldots, i_r) \in I_{ij}\) we have \(\chi'_{i_1i_{s-1}} \neq 1\) for some \(1 \leq s \leq r\). Thus \(\mu(\widetilde{i}_r, \ldots, \widetilde{i}_1) = 0\), and \(\mu'_{ij} = 0\) by (4.4).
The proof of the equality $\sigma^{D^\sigma}(\sigma^D(\mu)) = \mu$ is similar to the proof of (1). Let $1 \leq i < j \leq n+1$. By definition
\[
\sigma^{D^\sigma}_{ij}(\sigma^D(\mu)) = \tau^{D^\sigma}_{ij}(-1)^{j-i+1} \sum_{(i_1, \ldots, i_r)} (q - 1)^{N_2} \mu'(\tilde{i}_r, \ldots, \tilde{i}_1).
\]
For all $1 \leq i \leq i_{t-1} < i_t \leq j$ we have
\[
\mu'_{i_t i_{t-1}} = (-1)^{i_{t-1} - i_t + 1} \tau_{i_{t-1} i_t} \sum_{(l_1', \ldots, l_m')} (q - 1)^{N_3} \mu(l_1', \ldots, l_m').
\]
As before for $u^{D^\sigma}_{ij}(\mu')$ we now obtain
\[
\sigma^{D^\sigma}_{ij}(\sigma^D(\mu)) = \tau_{ij} \sum_{(l_1, \ldots, l_m) \in I_{ij}} (q - 1)^{N_4} \mu(l_1, \ldots, l_m) \sum_{(k_1, \ldots, k_r) \in I_{1m}} (-1)^r = \tau_{ij} \mu_{ij} \mu_{i-t} \mu_{i+1} (by \ (3.6)).
\]
This proves the claim since
\[
\tau_{ij} = \prod_{j \leq i < k < i} (\chi_{g_k}^N) = \prod_{i < k < j} (\chi_{g_k}^N) = \prod_{i \leq k < l < j} (\chi_{g_k}^N) = 1.
\]
\[
\tau_{ij} = \prod_{i \leq k < l < j} (\chi_{g_k}^N) = \prod_{i \leq k < l < j} (\chi_{g_k}^N) = 1.
\]
\[
\square
\]
\textbf{Corollary 4.4.} Let $\mu \in k^{\Phi^+}$ satisfying (R2). Then the map
\[
u(D^\sigma, \sigma^D(\mu)) \to \nu(D, \mu)
\]
given by $x_i^\sigma \mapsto x_{\sigma(i)}, g \mapsto g, 1 \leq i \leq n, g \in \Gamma$, is an isomorphism of Hopf algebras.

\textbf{Proof:} Let $s_i = 1$ for all $1 \leq i \leq n$. Then the triple $(id_{\Gamma}, \sigma, (s_i))$ is an isomorphism from $(D^\sigma, \sigma^D(\mu))$ to $(D, \mu)$ by Theorem 4.3 and Lemma 2.4. Hence the claim follows from Theorem 2.3. $\square$
5. Hopf algebra isomorphisms

In this section let

\[ D = D(\Gamma, (g_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n}), \]

\[ D' = D(\Gamma', (g'_i)_{1 \leq i \leq n}, (\chi'_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n}) \]

be data with finite abelian groups \( \Gamma \) and \( \Gamma' \) and the same Cartan matrix (1.1) of Type \( A_n \). As before \( \sigma \) denotes the non-trivial diagram automorphism of \( (a_{ij}) \) given by (2.17).

For \( s = (s_i)_{1 \leq i \leq n} \in k^n \) and \( \mu \in k^\Phi^+ \) we define

\[
s^N = (s_i^N)_{1 \leq i \leq n}, \\
(\text{noting that}) \\
s \cdot \mu = (s_i \mu_{ij})_{1 \leq i < j \leq n+1}.
\]

Recall that

\[
s_{ij} = \prod_{i \leq i < j} s_i \text{ for all } 1 \leq i < j \leq n + 1.
\]

Then

\[
(5.1) \quad u_{ij}(s \cdot \mu) = s_{ij}u_{ij}(\mu) \text{ for all } 1 \leq i < j \leq n + 1
\]

since \( s_{ij} = s_{i_1i_2} \cdots s_{i_r-1i_r} \) for all \((i_1, \ldots, i_r) \in I_{ij}\).

An isomorphism of data of Cartan type from \( D' \) to \( D \) is a group isomorphism \( \varphi : \Gamma' \to \Gamma \) satisfying

\[
(5.2) \quad \varphi(g_i) = g_i, \chi'_i = \chi_i \varphi \text{ for all } 1 \leq i \leq n.
\]

We write \( \varphi : D' \cong D \) if \( \varphi \) is an isomorphism from \( D' \) to \( D \).

Note that (5.2) implies for all \( 1 \leq i, j \leq n + 1 \) that

\[
\chi'_j(g'_i) = \chi_j(g_i), \text{ for all } 1 \leq i, j \leq n + 1,
\]

\[
\varphi(g'_{ij}) = g_{ij} \text{ for all } 1 \leq i < j \leq n + 1.
\]

Hence for all \( \mu' \in k^\Phi^+ \)

\[
(5.3) \quad \varphi(u_{ij}^{D'}(\mu')) = u_{ij}^{D}(\mu') \text{ for all } 1 \leq i < j \leq n + 1.
\]

Let \( k^\times = k \setminus \{0\} \) denote the multiplicative group of \( k \). In part (b) of (II) below, recall the definition of \( \nu^D \) in (2.14).
Theorem 5.1. Let $\mu$ and $\mu'$ be families of root vector parameters for $\mathcal{D}$ and $\mathcal{D}'$. Then the Hopf algebra isomorphisms $u(\mathcal{D}', \mu') \to u(\mathcal{D}, \mu)$ are given by

(I) $x_i \mapsto s_i x_i$, $g \mapsto \varphi(g)$, $1 \leq i \leq n$, $g \in \Gamma'$, where

(a) $\varphi : \mathcal{D}' \xrightarrow{\cong} \mathcal{D}$, and

(b) $s = (s_i) \in (k^\times)^n$ such that $\mu' = s^N \cdot \mu$,

and

(II) $x_i \mapsto s_i x_{\sigma(i)}$, $g \mapsto \varphi(g)$, $1 \leq i \leq n$, $g \in \Gamma'$, where

(a) $\varphi : \mathcal{D}' \xrightarrow{\cong} \mathcal{D}'$, and

(b) $s = (s_i) \in (k^\times)^n$ such that $\mu' = s^N \cdot \nu^D(\sigma^D(\mu))$.

Proof: By Theorem 2.3 the isomorphisms $u(\mathcal{D}', \mu') \to u(\mathcal{D}, \mu)$ are given by $x_i \mapsto s_i x_{\rho(i)}$, $g \mapsto \varphi(g)$, $1 \leq i \leq n$, $g \in \Gamma'$, where $\varphi : \Gamma' \to \Gamma$ is an isomorphism of groups, $\rho = \text{id}$ or $\rho = \sigma$, and $s = (s_i) \in (k^\times)^n$ such that

\[
\varphi(g_i') = g_{\rho(i)} \cdot \chi_i' = \chi_{\rho(i)} \varphi \quad \text{for all } 1 \leq i \leq n.
\]

\[
\varphi(u_{ij}^D(\mu')) = s_{ij}^N \pi(F^p(x_{ij}^D)^N) = s_{ij}^N u_{ij}^D(\mu') = u_{ij}^D(s^N \cdot \mu) \quad \text{(by (5.1)).}
\]

Hence (I)(b) and (5.5) are equivalent by the uniqueness in Lemma 2.2.

Let $\rho = \text{id}$ and assume (I)(a). Then the left hand side of (5.5) is $\varphi(u_{ij}^D(\mu')) = u_{ij}^D(\mu')$ by (5.3). For the right hand side of (5.5) we obtain

\[
s_{ij}^N \pi(F^p(x_{ij}^D)^N) = s_{ij}^N u_{ij}^D(\mu) = u_{ij}^D(s^N \cdot \mu) \quad \text{(by (5.1)).}
\]

Hence (I)(b) and (5.5) are equivalent by the uniqueness in Lemma 2.2.

Let $\rho = \sigma$ and assume (II)(a). Let $1 \leq i < j \leq n + 1$. Again it follows from (5.3) applied to $\mathcal{D}' \xrightarrow{\cong} \mathcal{D}'$ that $\varphi(u_{ij}^D(\mu')) = u_{ij}^D(\mu')$. We have shown in Lemma 2.4 and Theorem 4.3 that

\[
\pi(F^p(x_{ij}^D)^N) = u_{ij}^D(\sigma^D(\mu)).
\]

Hence

\[
s_{ij}^N \pi(F^p(x_{ij}^D)^N) = s_{ij}^N u_{ij}^D(\sigma^D(\mu)) = s_{ij}^N u_{ij}^D(\nu^D(\sigma^D(\mu))) = u_{ij}^D(s^N \cdot \nu^D(\sigma^D(\mu))) \quad \text{(by (5.1)).}
\]
Again it follows that (II)(b) and (5.5) are equivalent. □

We know from Theorem 4.3 that \( \nu^\sigma(\mu) \) satisfies (R2) for \( D^\sigma \). If we assume (II)(a), then \( \nu^D(\mu) \) also satisfies (R2) for \( D' \). Hence the normalization \( \nu^D(\sigma^D(\mu)) \) is a family of root vector parameters for \( D^\sigma \) and \( D' \). In general \( \sigma^D(\mu) \) is not a family of root vector parameters for \( D^\sigma \) since (R1) is not necessarily satisfied, and we have to pass to the normalization.

For example

\[
\sigma^D_{13}(\mu) = -\tau_{n-1,n+1}(\mu_{n-1,n+1} + (q-1)\mu_{n-1,n} \mu_{n,n+1}),
\]

and if \( (g^\sigma_{13})^N = g^N_{n-1,n+1} = 1 \) then \( \mu_{n-1,n+1} = 0 \), but \( \mu_{n-1,n} \mu_{n,n+1} \) can be non-zero if \( g^N_{n-1} \neq 1, g^N_n \neq 1 \) and \( \chi^N_{n-1} = 1, \chi^N_n = 1 \). As a realization of this situation take \( n = 2 \) and let \( \Gamma \cong \mathbb{Z}/(N^2) \times \mathbb{Z}/(N) \) with generators \( g \) of order \( N^2 \) and \( h \) of order \( N \). Let \( \zeta \in k \) be a root of 1 of order \( N \) and \( q = \zeta^2 \). Define \( g_1, g_2 \) and characters \( \chi_1, \chi_2 \) by

\[
g_1 = gh, g_2 = g^{-1}h, \chi_1(g) = \zeta, \chi_1(h) = \zeta, \chi_2(g) = \zeta^{-2}, \chi_2(h) = 1.
\]

Then \( \chi_1(g_1) = q = \chi_2(g_2) \), \( \chi_1(g_2) = q^{-1} \), and

\[
g_1^N = g^N_1 \neq 1, g_2^N = g^{-N} \neq 1, \chi_1^N = \chi_2^N = 1, \text{ and } g_{13}^N = g_1^N g_2^N = 1.
\]

Note that (R1) is trivially satisfied if \( g_{ij}^N \neq 1 \) for all \( 1 \leq i < j \leq n+1 \).

The next corollary follows immediately from Theorem 5.1.

**Corollary 5.2.** Let \( \mu, \mu' \) be families of root vector parameters for \( D \). Then the following are equivalent:

1. \( u(D, \mu') \cong u(D, \mu) \).

2. There is a family \( s \in (k^\times)^n \) such that

\[
\mu' = \begin{cases} 
  s^N \cdot \mu, & \text{if } D \ncong D^\sigma, \\
  s^N \cdot \mu \text{ or } s^N \cdot \nu^D(\sigma^D(\mu)), & \text{if } D \cong D^\sigma.
\end{cases}
\]

□

**Corollary 5.3.** Suppose there are \( 1 \leq i < j \leq n+1, j - i \geq 2 \), such that \( g_{ij}^N \neq 1 \) and \( g_{ij}^N \neq 1, \chi_{ij}^N = 1 \) for all \( i \leq l < j \). Then the number of isomorphism classes of Hopf algebras of the form \( u(D, \mu) \) is infinite.

**Proof:** By our assumption on \( D \) we can consider families of root vector parameters \( \mu, \mu' \) with \( \mu_{i,l+1} = \mu'_{i,l+1} = 1 \) for all \( i \leq l < j \), and with arbitrary
elements $\mu_{ij}, \mu'_{ij} \in k$. If $u(D', \mu') \cong u(D, \mu)$, then by Corollary 5.2 for all $i \leq l < j$ we have

$$
\mu'_{l,l+1} = \begin{cases} 
    s^N_l \mu_{l,l+1}, & \text{if } D \not\cong D^\sigma, \\
    s^N_l \mu_{l,l+1} \text{ or } s^N_l \mu_{\sigma(l),\sigma(l)+1}, & \text{if } D \cong D^\sigma,
\end{cases}
$$

hence $s^N_i = 1$. Thus $s^N_{ij} = 1$, and again by Corollary 5.2 it follows that

$$
\mu'_{ij} = \begin{cases} 
    \mu_{ij}, & \text{if } D \not\cong D^\sigma, \\
    \mu_{ij} \text{ or } \nu^D_{ij}(\sigma^D(\mu)), & \text{if } D \cong D^\sigma.
\end{cases}
$$

Therefore, we obtain infinitely many isomorphism classes of Hopf algebras $u(D, \mu)$. □

Theorem 5.1 gives the following description of the group of all Hopf algebra automorphisms of $u(D, \mu)$.

**Corollary 5.4.** Let $\mu$ be a family of root vector parameters for $D$. Then $\text{Hopfaut}(u(D, \mu))$ is isomorphic to the subgroup of $\text{Aut}(\Gamma) \times (k^\times)^n$ consisting of all pairs $(\varphi, s), \varphi \in \text{Aut}(\Gamma), s \in (k^\times)^n$, where

$$
\varphi : D \cong D, \mu = s^N \cdot \mu, \quad \text{or}
$$

$$
\varphi : D \cong D^\sigma, \mu = s^N \cdot \nu^D(\sigma^D(\mu)).
$$

□

We note the following special case.

**Corollary 5.5.** Let $\mu$ be a family of root vector parameters for $D$. Then the group of all Hopf algebra automorphisms of $u(D, \mu)$ is finite if $\mu_{i,i+1} \neq 0$ for all $1 \leq i \leq n$. □

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