On observable module categories

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Introduction

Module categories should play the same fundamental role in the theory of tensor categories, as representations do in the theory of groups. They were introduced in [Be] and studied in several papers in category theory; they appeared also in mathematical physics, see for example [FK]. Recently a systematic study of module categories over fusion, or more generally finite, tensor categories was undertaken in [O1, O2, ENO, EO]. In particular, indecomposable module categories over the category of representations of a finite group were classified in [O1] (characteristic 0) and [EO] (arbitrary characteristic).

In the papers [ENO, EO], the authors consider rigid tensor categories with appropriate finiteness conditions. One of the motivations for the discussions in the present paper is the study of module categories over the tensor category ${}_G\mathcal{M}$ of rational modules over an algebraic group G. Furthermore, we are interested in the induction functor from the category ${}_H\mathcal{M}$ of representations of a closed subgroup H of G to ${}_G\mathcal{M}$. For this, one needs to consider any rational module, not only the rigid (= finite-dimensional) ones. We are naturally led to the notions of *ind-rigid* and *geometric* tensor categories, see Definition 1.2.

The purpose of the present paper is to begin the study of a class of module categories over a tensor category $\mathcal{C}$ that we call *observable module categories*. These module categories are simple in a suitable sense (that we introduce in this paper). If $\mathcal{C} = {}_G\mathcal{M}$ then the archetypical example is the module category ${}_H\mathcal{M}$ where H is an observable subgroup of G. We extend some well-known results on observable subgroups to the setting of quotients of Hopf algebras.

The notion of observable subgroup has the following geometric characterization: a closed subgroup H of an algebraic group G is observable iff the homogeneous space G/H is quasi-affine. This suggests that the study of observable module categories could have a "non-commutative algebra" flavor: they should correspond to "non-commutative quasi-affine varieties".

Throughout, $\Bbbk$ denotes an arbitrary algebraically closed field, and all vector spaces, algebras, varieties, etc. are considered over $\Bbbk$.

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1. **Preliminaries**

In this section we recall some basic definitions of the theory of tensor categories and of the theory of affine algebraic groups. We also introduce the main definitions we shall work with.

1.1. **Serre subcategories.**

Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}_0$ be a full abelian subcategory of $\mathcal{A}$. Recall that

- $\mathcal{A}_0$ is a *Serre subcategory* of $\mathcal{A}$ if subobjects, quotients and finite direct sums of objects in $\mathcal{A}_0$, are again in $\mathcal{A}_0$.
- $\mathcal{A}_0$ is a *closed* subcategory of $\mathcal{A}$ if any object in $\mathcal{A}$ that is a colimit of objects in $\mathcal{A}_0$ is itself in $\mathcal{A}_0$.
- $\mathcal{A}_0$ *generates* $\mathcal{A}$ if any object in $\mathcal{A}$ is colimit of objects in $\mathcal{A}_0$.

An example of the use of these abstract notions in a very general version of duality theory is the following result. See also references in [NT].

Lemma 1.1. [NT, Lemma 3.2]. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Grothendieck categories. Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be two generating Serre subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. If $F : \mathcal{A}_0 \to \mathcal{B}_0$, $G : \mathcal{B}_0 \to \mathcal{A}_0$ is an equivalence (resp. duality) then there exists a unique extension $\overline{F} : \mathcal{A} \to \mathcal{B}$, $\overline{G} : \mathcal{B} \to \mathcal{A}$ which is an equivalence (resp. a pair of contravariant right adjoint functors).* □

1.2. **Tensor categories.**

In this paper, by *tensor category* we understand an abelian $\Bbbk$-linear tensor category such that the unit object $\mathbf{1}$ is simple as in [BK], except that we do *not* assume rigidity. This is because we are primarily interested in the category of rational modules over an algebraic group, which is not rigid.

Given an object X in a tensor category $\mathcal{C}$, a *right dual* of X is an object X^* provided with maps

$$\mathrm{ev} : X^* \otimes X \to \mathbf{1}, \quad \mathrm{coev} : \mathbf{1} \to X \otimes X^*,$$

called "evaluation" and "coevaluation", such that the compositions

$$X \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} X \otimes X^* \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} X \tag{1.1}$$

$$X^* \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} X^* \tag{1.2}$$

are the identities of X and of X^*, respectively.

Note that if $X \neq 0$ has a right dual then $\mathrm{coev} : \mathbf{1} \to X \otimes X^*$ is monic. Indeed, if coev is not monic then it is 0, as $\mathbf{1}$ is simple. But $\mathrm{coev} \otimes \mathrm{id}$ is monic, thus $X = 0$. Analogously, $\mathrm{ev} : X^* \otimes X \to \mathbf{1}$ is epi.

Similarly, a left dual of $Y \in \mathcal{C}$ is an object ${}^*Y$ provided with maps

$$\mathrm{ev} : Y \otimes {}^*Y \to \mathbf{1}, \quad \mathrm{coev} : \mathbf{1} \to {}^*Y \otimes Y,$$

still called "evaluation and coevaluation", such that the compositions

$$ {}^*Y \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} {}^*Y\otimes Y\otimes {}^*Y \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} {}^*Y \tag{1.3}$$

$$ Y \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} Y\otimes {}^*Y\otimes Y \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} Y \tag{1.4}$$

are the identities of ${}^*Y$ and of Y, respectively.

An object is *rigid* if it has a right and left dual.

If $X\in\mathcal{C}$ is rigid and $Y, Z\in\mathcal{C}$ then

$$\mathcal{C}(Y, X\otimes Z)\simeq\mathcal{C}(X^*\otimes Y, Z), \tag{1.5}$$

$$\mathcal{C}(Y, Z\otimes X^*)\simeq\mathcal{C}(Y\otimes X, Z). \tag{1.6}$$

See [BK, Lemma 2.1.6].

A tensor category is rigid if every object is rigid. Our main interest here is in a weaker notion of "ind-rigid tensor categories".

Let $\mathcal{C}$ be a tensor category and let $\mathcal{C}_{\mathrm{rig}}$ be the full subcategory of rigid objects. Then $\mathcal{C}_{\mathrm{rig}}$ is a rigid monoidal subcategory of $\mathcal{C}$, but not necessarily a Serre subcategory.

A subobject or a quotient object of a rigid object need not be rigid. For example, let R be a commutative $\Bbbk$-algebra (resp. any associative $\Bbbk$-algebra with unit) and $\mathcal{C} = {}_R\mathcal{M}$ the category of R-modules (resp. $\mathcal{C} = {}_R\mathcal{M}_R$ the category of R-bimodules) with $\otimes = \otimes_R$. Then an object M is rigid if and only if it is finitely generated and projective as R-module (resp. both as a left and right module); but quotients and submodules of finitely generated projective modules do not inherit in general these properties.

On the other hand, if X and Y are rigid objects then $X\oplus Y$ is rigid.

Definition 1.2. An object in a tensor category is *ind-rigid* if it is the colimit of a family of rigid objects.

A tensor category is *ind-rigid* if

(a) any object is ind-rigid,
(b) any subobject or quotient object of a rigid object is rigid.

An ind-rigid tensor category is *geometric* if in addition

(c) the tensor product functor is exact in both variables,
(d) and preserves colimits (on both sides).

Remark 1.3. If $\mathcal{C}$ is an ind-rigid tensor category then any object of $\mathcal{C}$ is the colimit of a family of rigid sub-objects– take the epi-monic decomposition of the arrows of an arbitrary colimit of rigid objects.

Remark 1.4. If $\mathcal{C}$ is an ind-rigid tensor category then $\mathcal{C}_{\mathrm{rig}}$ is a Serre subcategory generating $\mathcal{C}$. Thus, if $\mathcal{C}$ is a Grothendieck category then the duality (contravariant) functor $* : \mathcal{C}_{\mathrm{rig}} \to \mathcal{C}_{\mathrm{rig}}$ extends uniquely to $* : \mathcal{C}\to\mathcal{C}$.

Remark 1.5. In an ind-rigid tensor category, the simple objects are necessarily rigid. Indeed, let S be a simple object and express it as a colimit of rigid objects $\{d_j : j\in J\}$. Then at least one of the arrows $d_j\to S$ of the associated cone is non-zero– otherwise $S=0$– hence $S=\operatorname{Im} d_j$ is rigid.

Thus, not every tensor category is ind-rigid. For instance, the category of representations of a Hopf algebra H is not ind-rigid if H has a simple infinite-dimensional module, *e.g.* $H = U(\mathfrak{sl}(2,\Bbbk))$.

Remark 1.6. If $\mathcal{C}$ is an ind-rigid tensor category and

(d') the tensor product functor preserves colimits on one side, then (d) holds. Indeed, assume that it preserves colimits on the left side. Given $x \in \mathcal{C}$, write it as a colimit $x = \operatorname{colim}_j x_j$ of rigid objects x_j. If $y = \operatorname{colim}_i y_i$ in $\mathcal{C}$ then

$$\begin{aligned} x \otimes y &= (\operatorname{colim}_j x_j) \otimes (\operatorname{colim}_i y_i) \simeq \operatorname{colim}_j \big(x_j \otimes (\operatorname{colim}_i y_i)\big) \\ &\simeq \operatorname{colim}_j \operatorname{colim}_i \big(x_j \otimes y_i\big) \simeq \operatorname{colim}_i \operatorname{colim}_j \big(x_j \otimes y_i\big) \\ &\simeq \operatorname{colim}_i \big((\operatorname{colim}_j x_j) \otimes y_i\big) \simeq \operatorname{colim}_i \big(x \otimes y_i\big), \end{aligned}$$

the first isomorphism since $x_j \otimes \_\_$ is exact because x_j is rigid.

Remark 1.7. Condition (c) fails in general, *e.g.* in ${}_R\mathcal{M}$ or ${}_R\mathcal{M}_R$. It is true in rigid tensor categories, see [BK, Prop. 2.1.8]. Actually, the proof of [BK, Prop. 2.1.8] shows that $X \otimes \_\_$ and $\_\_ \otimes X$ are exact in an arbitrary tensor category if X is rigid.

We do not know if (c) and (d) follows from (a) and (b), *i. e.* if any ind-rigid tensor category is geometric.

A natural example of geometric tensor category is the category of comodules over a Hopf algebra with bijective antipode.

1.3. **Tensor functors and module categories.** Recall that a tensor functor between tensor categories is an additive functor that preserves the tensor and the unit "up to a natural isomorphism". Technically, a tensor functor from a tensor category $\mathcal{C}$ to a tensor category $\mathcal{D}$ is a triple (F, ζ, ϕ) where $F : \mathcal{C} \to \mathcal{D}$ is an additive functor, $\phi : \mathbf{1} \to F(\mathbf{1})$ is an isomorphism in $\mathcal{D}$, and $\zeta_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ is a natural transformation such that

$$F(a_{X,Y,Z}) \zeta_{X \otimes Y, Y} (\zeta_{X,Y} \otimes \mathrm{id}_{F(Z)}) = \zeta_{X, Y \otimes Z} (\mathrm{id}_{F(X)} \otimes \zeta_{Y,Z}) a_{F(X),F(Y),F(Z)}, \tag{1.7}$$

$$F(l_X) \zeta_{\mathbf{1},X} (\phi \otimes \mathrm{id}_{F(X)}) = l_{F(X)}, \tag{1.8}$$

$$F(r_X) \zeta_{X,\mathbf{1}} (\mathrm{id}_{F(X)} \otimes \phi) = r_{F(X)}, \tag{1.9}$$

for all $X, Y, Z \in \mathcal{C}$.

If $F : \mathcal{C} \to \mathcal{C}'$ is a tensor functor and X is a rigid object of $\mathcal{C}$ then $F(X)$ is a rigid object of $\mathcal{C}'$ and F preserves duals. Thus F induces a monoidal functor $F_{\text{rig}} : \mathcal{C}_{\text{rig}} \to \mathcal{C}'_{\text{rig}}$.

A module category over a tensor category $\mathcal{C}$ is an abelian $\Bbbk$-linear category $\mathcal{M}$ provided with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (the *action*) and natural isomorphisms

$$m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M), \qquad l_M : \mathbf{1} \otimes M \to M,$$

$X, Y \in \mathcal{C}$, $M \in \mathcal{M}$, such that the following diagrams commute:

$$\begin{tikzcd} & ((X \otimes Y) \otimes Z) \otimes M \arrow[dl, "a_{X,Y,Z} \otimes id"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\ (X \otimes (Y \otimes Z)) \otimes M \arrow[d, "m_{X, Y \otimes Z, M}"] & & (X \otimes Y) \otimes (Z \otimes M) \arrow[d, "m_{X,Y,Z \otimes M}"'] \\ X \otimes ((Y \otimes Z) \otimes M) \arrow[rr, "id \otimes m_{Y,Z,M}"] & & X \otimes (Y \otimes (Z \otimes M)) \end{tikzcd} \tag{1.10}$$

and

$$\begin{tikzcd} (X \otimes \mathbf{1}) \otimes M \arrow[rr, "m_{X,\mathbf{1},Y}"] \arrow[dr, "r_X \otimes id"'] & & X \otimes (\mathbf{1} \otimes M) \arrow[dl, "id \otimes l_M"] \\ & X \otimes M & \end{tikzcd} \tag{1.11}$$

for all $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$.

In the definition, "exact bifunctor" means "it is exact in both variables". By analogy with the axioms (c) and (d) in the definition of geometric tensor category, we shall also assume that the action satisfies the following axiom:

- the action functor $\otimes$ preserves colimits (on both sides).

Let $\mathcal{M}, \mathcal{M}'$ be two module categories over a tensor category $\mathcal{C}$. A *module functor* from $\mathcal{M}$ to $\mathcal{M}'$ is a triple (F, b, u) where $F : \mathcal{M} \to \mathcal{M}'$ is an additive functor, $b_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$ is a natural isomorphism and $u : F(\mathbf{1}) \to \mathbf{1}$ is an isomorphism, satisfying some natural compatibilities, see [O2].

By a *submodule category* of a module $\mathcal{M}$ we understand a Serre subcategory stable under the action.

Remark 1.8. As explained in [EO], it is worth to restrict the attention to the class of *exact* module categories. Recall that $\mathcal{M}$ is an exact tensor category over a finite tensor category $\mathcal{C}$ if $P \otimes X \in \mathcal{M}$ is projective for any $P \in \mathcal{C}$ projective an any $X \in \mathcal{M}$. Module functors from an exact module category are always exact. Now, the inclusion functor from a full abelian subcategory of an abelian category is exact precisely when the domain is a Serre subcategory. This justifies our definition of submodule category.

The trivial module category is $\mathcal{M} = 0$.

A module category is *indecomposable* if it is not isomorphic to a direct sum of two nontrivial submodule categories [O1].

Definition 1.9. A non-trivial module category over a tensor category $\mathcal{C}$ is *simple* if any proper submodule category is trivial.

1.4. **Algebraic groups.** In this paper, we consider affine algebraic groups defined over $\Bbbk$; by "subgroup" we mean a closed subgroup. If G is an affine algebraic group, we denote as $\Bbbk[G]$ the Hopf algebra of polynomial functions on G.

Also we denote as ${}_G\mathcal{M}$, the category of all rational left G-modules (for the basic properties of this category see [FR]). In particular it is well-known that ${}_G\mathcal{M}$ is an abelian ind-rigid tensor category, where ${}_G\mathcal{M}_{\rm rig}$ is the full subcategory of finite-dimensional rational G-modules. Clearly $\Bbbk[G] \in {}_G\mathcal{M}$ when it is equipped with the left translation: $(x, f) \mapsto x \cdot f$, for $x \in G$ and $f \in \Bbbk[G]$.

If G is an affine algebraic group and H a closed subgroup, then the coset space G/H is naturally an algebraic G–variety. In the above situation the restriction functor $\mathrm{Res}_H^G : {}_G\mathcal{M} \to {}_H\mathcal{M}$ is a tensor functor. Then we can consider ${}_H\mathcal{M}$ as module category over ${}_G\mathcal{M}$, that is $X \otimes N = X|_H \otimes N$, for a rational H-module N and a rational G-module X. The restriction functor has an adjoint, that we denote as $\mathrm{Ind}_H^G : {}_H\mathcal{M} \to {}_G\mathcal{M}$, called the induction functor. Explicitly,

$$\mathrm{Ind}_H^G(M) = {}^H(\Bbbk[G] \otimes M), \qquad M \in {}_H\mathcal{M},$$

where the superscript H means that we are taking H-invariants. Note that Ind_H^G has usually infinite dimension, this is why we can not work directly with rigid tensor categories. The counit of this adjunction $E_M : \mathrm{Ind}_H^G(M) \to M$ for $M \in {}_H\mathcal{M}$, is called the evaluation map.

1.5. **Observable subgroups.** Let G be an algebraic group and H a closed subgroup. Recall that a rational character $\eta : H \to \Bbbk$ is extendible, if there is a non–zero polynomial $f \in k[G]$ such that for all $x \in H$, $x \cdot f = \eta(x) f$.

The following properties are equivalent:

(a) G/H is a quasi–affine variety.
(b) All rational characters $\chi : H \to \Bbbk$ are extendible.
(c) For all H–modules $M \in {}_H\mathcal{M}$, the evaluation map E_M is surjective.

See [FR, Ch. 10] for the relevant definitions and proofs. If any of (a), (b), (c) holds then one says that H is *observable* in G.

Here are some examples of subgroups that are observable (or not):

- If the only rational character $\chi : H \to \Bbbk$ is the trivial one, then H is observable in any algebraic group containing it. Thus, semisimple and unipotent algebraic subgroups are always observable.
- Finite subgroups are observable.
- Normal subgroups are observable.
- Finite-index subgroups are observable.
- A parabolic subgroup of a semisimple algebraic group is not observable.

2. **Observable tensor categories**

2.1. **Definition and basic properties.**

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between arbitrary tensor categories $\mathcal{C}$ and $\mathcal{D}$. Consider $\mathcal{D}$ as module category over $\mathcal{C}$ via F; that is,

$$c \otimes d := F(c) \otimes d, \qquad c \in \mathcal{C},\ d \in \mathcal{D}.$$

To illustrate the concepts in action, we prove the following result.

Lemma 2.1. *Assume that F has a right adjoint $I : \mathcal{D} \to \mathcal{C}$. Then the full subcategory $\mathcal{D}_0$ of $\mathcal{D}$ with objects*

$$\mathcal{D}_0 = \{d \in \mathcal{D} : I(d) = 0\}$$

is stable under action by objects of $\mathcal{C}_{\mathrm{rig}}$. If I preserves colimits then $\mathcal{D}_0$ is a full abelian subcategory with the same kernels and cokernels as in $\mathcal{D}$.

Proof. Let $d \in \mathcal{D}_0$. If $c \in \mathcal{C}_{\mathrm{rig}}$, $x \in \mathcal{C}$, then

$$\begin{aligned}\mathcal{C}(x, I(c\otimes d)) &\simeq \mathcal{C}(x, I(F(c)\otimes d)) \simeq \mathcal{D}(F(x), F(c)\otimes d)\\ &\simeq \mathcal{D}(F(c)^*\otimes F(x), d) \simeq \mathcal{D}(F(c^*\otimes x), d)\\ &\simeq \mathcal{C}(c^*\otimes x, I(d)) = 0,\end{aligned}$$

where we have used (1.5). Thus $c\otimes d \in \mathcal{D}_0$ for any $c \in \mathcal{C}_{\mathrm{rig}}$ as claimed. Since I is a right adjoint, it preserves limits [M, Ch. V, §5, Th. 1].

If I preserves colimits then $\mathcal{D}_0$ is closed under colimits. Thus $\mathcal{D}_0$ is closed under kernels and cokernels and is a full abelian subcategory of $\mathcal{D}$. □

Here is one of the main definitions in the present paper.

Definition 2.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between arbitrary tensor categories $\mathcal{C}$ and $\mathcal{D}$ and assume that F has a right adjoint $I : \mathcal{D} \to \mathcal{C}$. We say that the category $\mathcal{D}$ is *observable* over $\mathcal{C}$ if the counit for the adjunction $\varepsilon_d : F(I(d)) \to d$ is an epimorphism for all $d \in \mathcal{D}$.

Here recall that ε_d corresponds to $\mathrm{id}_{I(d)}$ via $\mathcal{D}(F(I(d)), d) \simeq \mathcal{C}(I(d), I(d))$. The motivation for the above definition comes from the theory of algebraic groups, see Section 1.4. Indeed, a subgroup H is observable in an algebraic group G if and only if ${}_H\mathcal{M}$ is observable in ${}_G\mathcal{M}$.

Theorem 2.3. *In the notations above, if $\mathcal{D}$ is ind-rigid and observable over $\mathcal{C}$, then it is simple as a module category over $\mathcal{C}$.*

Proof. Let $\mathcal{N} \subset \mathcal{D}$ be a non-zero submodule category. Take $0 \neq m \in \mathcal{N}$ and write it as a colimit in $\mathcal{D}$ of rigid objects $\{d_j : j \in J\}$, $d_j \in \mathcal{D}_{\mathrm{rig}}$. One of the arrows of the cone $f_j : d_j \to m$ is non-zero, since otherwise $m = 0$. Using the standard epi, monic factorization of f_j we obtain a pair of morphisms in $\mathcal{D}$ and an object $0 \neq n \in \mathcal{D}$, $h : d_j \to n$, $g : n \to m$

with $f_j = gh$, h epi and g monic. As $\mathcal{N}$ is closed under subobjects, it follows that $n \in \mathcal{N}$ and as n is a quotient of d_j, it is rigid. In other words, $\mathcal{N}$ contains at least one rigid object n.

In this situation the morphism $\varepsilon_{n^*} \otimes \mathrm{id} : I(n^*) \otimes n = F(I(n^*)) \otimes n \to n^* \otimes n$ is an epimorphism by the observability hypothesis. Since $\mathcal{N}$ is a submodule category, $I(n^*) \otimes n \in \mathcal{N}$ and we conclude that $n^* \otimes n \in \mathcal{N}$, because $\mathcal{N}$ is closed under quotients. Using the monomorphism given by the coevaluation map $\mathrm{coev} : \mathbf{1} \to n^* \otimes n$, we conclude that $\mathbf{1} \in \mathcal{N}$.

Consider now an arbitrary object $d \in \mathcal{D}$; in this situation the morphism $\varepsilon_d \otimes \mathrm{id} : I(d) \otimes \mathbf{1} = F(I(d)) \otimes \mathbf{1} \to d \otimes \mathbf{1} \simeq d$ is epi and we conclude that $d \in \mathcal{N}$. Thus $\mathcal{N} = \mathcal{D}$. □

2.2. **Observable quotients of Hopf algebras.**

Let now A be a Hopf algebra with bijective antipode and let $\mathcal{M}^A$ be the tensor category of right A-comodules. It is well-known that $\mathcal{M}^A$ satisfies the properties required to be geometric.

Let $\pi : A \to B$ be a Hopf algebra quotient and let $\mathrm{Res} = \mathrm{Res}_A^B : \mathcal{M}^A \to \mathcal{M}^B$ be the corresponding "restriction" functor; that is, if M is a right comodule over A via $\rho : M \to M \otimes A$ then $\mathrm{Res}\, M = M_{|B}$ is the vector space M considered as a right comodule over B via $(\mathrm{id} \otimes \pi)\rho$. Clearly, Res is a tensor functor.

If $g \in G(B)$ is a group-like and M is a right B-comodule then M^g denotes the isotypic component of type g, that is

$$M^g = \{m \in M : \rho(m) = m \otimes g\}.$$

If $g = 1$, we denote $M^B = M^1$ the subcomodule of invariant elements.

Let $M \in \mathcal{M}^B$. If we consider $A \in \mathcal{M}^A$ via the comultiplication, then $A_{|B} \otimes M \in \mathcal{M}^B$ with coaction ρ_1; explicitly,

$$\rho_1(a \otimes m) = a_{(1)} \otimes m_{(0)} \otimes \pi(a_{(2)}) m_{(1)}, \qquad a \in A, m \in M.$$

There is another right coaction of A on $A \otimes M$ by

$$\rho_2(a \otimes m) = a_{(2)} \otimes m \otimes \mathcal{S}(a_{(1)}), \qquad a \in A, m \in M.$$

It is easy to see that these two coactions commute. Then the space of invariants $(A_{|B} \otimes M)^B$ with respect to ρ_1 is a right comodule over A via ρ_2. In this situation we can define a map $E_M : (A_{|B} \otimes M)^B \to M$– the evaluation map– by

$$E_M\big(\sum_i f_i \otimes m_i\big) = \sum_i \varepsilon(f_i) m_i, \qquad \sum_i f_i \otimes m_i \in A \otimes M.$$

Then E_M is a map of B-comodules, natural on M.

The following result is well-known; see [F]. We sketch a proof for completeness. The proof does not differ from the analogous proof for algebraic groups, see [FR, Th. 6.6.11].

Proposition 2.4. *The "induction" functor* $\mathrm{Ind} = \mathrm{Ind}_B^A : \mathcal{M}^B \to \mathcal{M}^A$*, given by*

$$\mathrm{Ind}_B^A(M) = (A_{|B} \otimes M)^B, \qquad M \in \mathcal{M}^B,$$

is a right adjoint to Res_A^B*. The counit for the adjunction is the map* E_M*.*

Proof. (Sketch). Let $N \in \mathcal{M}^A$. Given $\varphi \in \mathrm{Hom}_B(N, M)$, define $\widetilde{\varphi} \in \mathrm{Hom}(N, A_{|B} \otimes M)$ by

$$\widetilde{\varphi}(n) = \sum \mathcal{S}^{-1}(n_{(1)}) \otimes n_{(0)}, \qquad n \in N.$$

Straightforward calculations show that:

(a) The image of $\widetilde{\varphi}$ is contained in $(A_{|B} \otimes M)^B$.

(b) $\widetilde{\varphi}$ is a morphism of A-comodules and $E_M\widetilde{\varphi} = \varphi$.

$$\begin{tikzcd} N \arrow[r, "\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & (A_{|B}\otimes M)^B \arrow[d, "E_M"] \\ & M \end{tikzcd}$$

(c) A morphism of A-comodules $N \to (A_{|B}\otimes M)^B$ such that $E_M\widetilde{\varphi} = \varphi$ coincides necessarily with $\widetilde{\varphi}$.

This proves the Proposition. $\square$

We are then in the situation of Definition 2.2; it is natural to introduce the following notion.

Definition 2.5. A Hopf algebra quotient $\pi : A \to B$ is *observable* over A (or B is observable over A) if the evaluation map $E_M : (A_{|B}\otimes M)^B \to M$ is an epimorphism for all $M \in \mathcal{M}^B$.

As for algebraic groups, there are alternative characterizations of observability. The proof of the following result mimics the proof of the analogous result for algebraic groups, see [FR, Ch. 10].

Theorem 2.6. *Let $\pi : A \to B$ be a Hopf algebra quotient. The following are equivalent:*

(a) *B is observable over A.*
(b) *For every finite-dimensional $M \in \mathcal{M}^B$ there exists a finite-dimensional $N \in \mathcal{M}^A$ and an epimorphism of B-comodules $\varphi : N_{|B} \to M$.*
(c) *For every finite-dimensional $M \in \mathcal{M}^B$ there exists a finite-dimensional $N \in \mathcal{M}^A$ and a monomorphism of B-comodules $M \to N_{|B}$.*

Furthermore, if any of these conditions hold then

(d) *For every $g \in G(B)$, there exists $f \in (A_{|B})^g$ such that $\pi(f) = g$.*

Proof. (a) $\implies$ (b): Let $s : M \to (A_{|B}\otimes M)^B \to M$ be a linear section of the evaluation map E_M. The A-comodule N generated by the image $s(M)$ is finite-dimensional and $\varphi = \pi_{|N} : N_{|B} \to M$ is an epimorphism of B-comodules.

(b) $\implies$ (a): Assume first that M is finite-dimensional. Let $\varphi : N_{|B} \to M$ be an epimorphism of B-comodules and let $\widetilde{\varphi} \in \operatorname{Hom}(N, (A_{|B}\otimes M)^B)$ be the morphism of A-comodules such that $E_M\widetilde{\varphi} = \varphi$. Since φ is surjective, E_M is surjective.

If M has arbitrary dimension, then it is the union of its finite-dimensional subcomodules and the claim follows from the naturality of the evaluation map E_M.

(b) $\iff$ (c): by duality.

(b) $\implies$ (d): Let $M = \Bbbk g$ and let $N \in \mathcal{M}^A$ finite-dimensional provided with an epimorphism of B-comodules $\varphi : N_{|B} \to \Bbbk g$. If $n \in N$ satisfies $\varphi(n) = g$ and $\alpha \in N^*$ satisfies $\alpha(n) = 1$ then the matrix coefficient $f = \phi_{\alpha,n} \in A$ satisfies $\pi(f) = g$. $\square$

Remark 2.7. As we have seen, the implication (d) $\implies$ (b) is also valid for algebraic groups. It is likely that this implication is not always true. The proof in [FR] makes use of the modules of antisymmetric tensors of a given rational module, not a comodule in the case of a general Hopf algebra.

As for algebraic groups, cosemisimple Hopf algebra quotients are observable.

Theorem 2.8. *Let $\pi : A \to B$ be a Hopf algebra quotient. If B is cosemisimple then B is observable over A.*

Proof. Note that the following diagram commutes:

$$\begin{tikzcd} (A_{|B}\otimes M)^B \arrow[r, "\pi\otimes\,\mathrm{id}"] \arrow[dr, "E_M"'] & (B\otimes M)^B \arrow[d, "\widetilde{E}_M"] \\ & M \end{tikzcd}$$

where $\widetilde{E}_M : (B\otimes M)^B \to M$ is again given by $\widetilde{E}_M\big(\sum_i f_i\otimes m_i\big) = \sum_i \varepsilon(f_i)m_i$, $\sum_i f_i\otimes m_i \in B\otimes M$. Now it is clear that $\widetilde{E}_M$ is an isomorphism, since $(B\otimes M)^B = \mathrm{Ind}_B^B(M)$ is adjoint to the identity functor.

Thus, we have to prove that $\pi\otimes\mathrm{id} : (A_{|B}\otimes M)^B \to (B\otimes M)^B$ is surjective. But $A_{|B}\otimes M \simeq \ker(\pi\otimes\mathrm{id}) \oplus (B\otimes M)$ since B is cosemisimple, and the needed surjectivity follows taking the invariant submodules at both sides of this isomorphism. □

3. **Linearized Sheaves**

Let G be an affine algebraic group. Let X be a G–variety and let ${}_G\mathcal{M}_X$ be the abelian category of G–linearized sheaves on X, see [CPS, p. 453]. If $\mathcal{F}$ is a G–linearized and if $x \in X$, we denote by $\mathcal{F}_x$ the stalk of the sheaf at the point x. Then for every pair $x \in X$, $g \in G$ there exists a linear isomorphism $\varphi_{g,x} : \mathcal{F}_x \to \mathcal{F}_{g\cdot x}$, regular on the pair (g, x), in such a way that if $h \in G$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}_x \arrow[r, "\varphi_{g,x}"] \arrow[dr, "\varphi_{hg,x}"'] & \mathcal{F}_{g\cdot x} \arrow[d, "\varphi_{h,g\cdot x}"] \\ & \mathcal{F}_{(hg)\cdot x} \end{tikzcd}$$

In particular notice that the structure sheaf $\mathcal{O}_X \in {}_G\mathcal{M}_X$.

Then ${}_G\mathcal{M}_X$ is a module category over ${}_G\mathcal{M}$. Indeed, if $M \in {}_G\mathcal{M}$ and $\mathcal{F} \in {}_G\mathcal{M}_X$, we define $M \otimes \mathcal{F} \in {}_G\mathcal{M}_X$ as the presheaf associated to the sheaf $(M \otimes \mathcal{F})(U) = M \otimes \mathcal{F}(U)$, U open in X. Hence, for all $x \in X$, $(M \otimes \mathcal{F})_x = M \otimes \mathcal{F}_x$ and $\psi_{g,x} : M \otimes \mathcal{F}_x \to M \otimes \mathcal{F}_{g\cdot x}$ is $\psi_{g,x} = g\cdot \_\_ \otimes \varphi_{g,x}$.

If $T : Y \to X$ is a morphism of algebraic varieties and $\mathcal{G}$ is a sheaf on Y, then we denote as $T_*(\mathcal{G})$ the "direct image" sheaf on X defined as $T_*(\mathcal{G})(U) = \mathcal{G}(T^{-1}U)$, for $U \subset X$ open; see [H, p. 65]. Assume that X and Y are G–varieties and that T intertwines the G–actions. If the sheaf $\mathcal{G}$ is G–linearized, then $T_*(\mathcal{G})$ is also G–linearized. Moreover, in this situation, we have a module functor $T_* : {}_G\mathcal{M}_Y \to {}_G\mathcal{M}_X$.

Lemma 3.1. *Let X be a G–variety and let Y be a closed G–stable subvariety of X. Then we can identify ${}_G\mathcal{M}_Y$ with a submodule category of ${}_G\mathcal{M}_X$.*

Proof. If Y is a closed subvariety of an arbitrary variety X and ι is the inclusion, then $\iota_*(\mathcal{G})_x = \mathcal{G}_x$ if $x \in Y$ and zero otherwise, for any sheaf $\mathcal{G}$ on Y [H, Ex. 1.19, p. 68]. This implies that we can identify ${}_G\mathcal{M}_Y$ with the full abelian subcategory of ${}_G\mathcal{M}_X$ consisting of those G–linearized sheaves $\mathcal{F}$ on X such that $\mathcal{F}_P$ for any $P \notin Y$. It is clear that subobjects and quotients of objects in ${}_G\mathcal{M}_Y$ are again in ${}_G\mathcal{M}_Y$. Thus, ${}_G\mathcal{M}_Y$ can be identified with a submodule category of ${}_G\mathcal{M}_X$. □

Remark 3.2. Actually, ${}_G\mathcal{M}_Y$ can be identified with a *closed* submodule category of ${}_G\mathcal{M}_X$, *cf.* subsection 1.1, because of [H, Ex. 1.12, p. 67]. We wonder whether it is important to distinguish closed submodule categories.

Lemma 3.3. *Let X be a G–variety. If the the category ${}_G\mathcal{M}_X$ is a simple module category, then the action of G on X is transitive.*

Assume further that the orbit map $G \to X$ is separable (this is automatic if $\Bbbk$ has characteristic 0). Then $X \simeq G/H$ for a closed subgroup H of G with the left regular action.

Proof. Let Y be a closed orbit for the action of G on X. The submodule category ${}_G\mathcal{M}_Y$ is not zero since $\mathcal{O}_Y \in {}_G\mathcal{M}_Y$. Thus, ${}_G\mathcal{M}_Y = {}_G\mathcal{M}_X$. Hence, the sheaf $\mathcal{O}_X$ is in ${}_G\mathcal{M}_Y$, and this means that $\mathcal{O}_{X,x} = 0$ for all $x \in X \setminus Y$, therefore $X = Y$. The second statement is a well-known fact in the theory of algebraic groups. □

Let H be a closed subgroup of G and let $\pi : G \to G/H$ be the canonical projection.

Theorem 3.4. *(a). The functor $\mathcal{L} : {}_H\mathcal{M} \to {}_G\mathcal{M}_{G/H}$ given by*

$$\mathcal{L}_N(U) = {}^H\big(\mathcal{O}_G(\pi^{-1}(U)) \otimes N\big)$$

for a rational H-module N and U open subset of G/H, is an equivalence of categories, whose inverse is given by taking the fiber at the identity:

$$\mathcal{F} \mapsto \mathcal{F}_{eH}, \qquad \mathcal{F} \in {}_G\mathcal{M}_{G/H}.$$

(b). $\mathcal{L}$ is a module functor. If K is a subgroup of H then the following diagram commutes:

$$\begin{array}{ccc} {}_H\mathcal{M} & \xrightarrow{\mathcal{L}} & {}_G\mathcal{M}_{G/H} \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ {}_K\mathcal{M} & \xrightarrow{\mathcal{L}} & {}_G\mathcal{M}_{G/K}, \end{array}$$

where $\pi : G/H \to G/K$ is the canonical projection.

Proof. (a) is [CPS, Th. 2.7]. The proof of (b) is straightforward. □

Theorem 3.5. *Assume that $\Bbbk$ has characteristic 0. If X is a G–variety then the following are equivalent:*

(a) *${}_G\mathcal{M}_X$ is isomorphic to an observable module category.*
(b) *$X \simeq G/H$ for a closed subgroup H observable in G.*

Proof. By Theorem 2.3, if such an isomorphism exists then ${}_G\mathcal{M}_X$ should be simple. Then Lemma 3.3 says that $X \simeq G/H$, and Theorem 3.4, that ${}_G\mathcal{M}_X \simeq {}_H\mathcal{M}$. The claim now follows from the characterization of observable subgroups, see subsection 1.4. □

The following problems arise naturally: Is the converse to Lemma 3.3 true? That is, what are the closed subgroups H such that ${}_H\mathcal{M}$ is simple? More generally, is any simple module category over ${}_G\mathcal{M}$ twist-equivalent (in some appropriate sense) to ${}_H\mathcal{M}$ with H closed subgroup of G?

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