On pointed Hopf algebras associated to the symmetric groups

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Plan of the talk.

I. Lifting method.

II. Pointed Hopf algebras with non-abelian group.

III. Discarding infinite-dimensional pointed Hopf algebras.

I. Lifting method. N. A. and H.-J. Schneider, *Pointed Hopf Algebras*, MSRI Publications **43** (2002), 1-68, Cambridge Univ. Press.

H pointed Hopf alg. dim $H < \infty$, $\Gamma = G(H)$.

$0 = H_{-1} \subseteq H_0 \subseteq H_1 \subseteq \cdots \subseteq H_j \subseteq H_{j+1} \subseteq \ldots$

coradical filtration of H.

$$\operatorname{gr} H := \oplus_{n \geq 0} H_n / H_{n-1} \simeq \mathbb{C}\Gamma \# R,$$

(Radford-Majid)

$R = \oplus_{n\geq 0} R^n, \quad R(n) = R \cap H_n/H_{n-1}, \quad R' = \mathbb{C} < R(1) > \subseteq R.$

R and R' braided Hopf algebras $\equiv$ Hopf algebras in the braided category of Yetter-Drinfeld modules over $\mathbb{C}\Gamma$.

Method: • Determine all possible R' s. t. $\dim R' < \infty$.

• If $\dim R' < \infty$, then $R' = R$?

Conjecture (A.-Schneider) Any pointed Hopf alg., dim. $< \infty$ is generated by group-like and skew-primitive elements.

• Find all possible H s. t. $\operatorname{gr} H \simeq \mathbb{C}\Gamma \# R$ (Lifting).

Essential step: Determine all possible R' s. t. $\dim R' < \infty$. Why?

(V, c) braided vector space: $c \in GL(V \otimes V)$

$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$

$\rightsquigarrow \mathfrak{B}(V) = \oplus_{n \geq 0} \mathfrak{B}^n(V)$ (Nichols algebra)

In our case, $R' = \mathfrak{B}(V)$, where $V = R^1$!

Nichols algebra: $\mathfrak{B}(V) = \oplus_{n \geq 0} \mathfrak{B}^n(V)$
graded algebra with extra structure

- $\mathfrak{B}^0(V) = \mathbb{C}$, $\mathfrak{B}^1(V) = V$.
- $\mathfrak{B}(V)$ generated by V as algebra.
- $\mathfrak{B}(V)$ is a braided Hopf algebra.
- $P(\mathfrak{B}(V)) = V$.

rank of $\mathfrak{B}(V) = \dim V$

$\mathfrak{B}(V) = T(V)/J$, but J not explicit!

Γ finite abelian group

Braided vector space of diagonal type.

$\exists$ basis $v_1, \dots, v_\theta$, $(q_{ij})_{1 \le i,j \le \theta}$ in $\mathbb{C}^\times$:

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \forall i, j$$

Theorem. $1 \neq q_{ii}$ roots of 1. $\Rightarrow \dim \mathfrak{B}(V) < \infty$ classified.

I. Heckenberger, *Classification of arithmetic root systems*,
http://arxiv.org/abs/math.QA/0605795.

Braided vector space of Cartan type.

$\exists\ (a_{ij})_{1\le i,j\le\theta}$ generalized Cartan matrix

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}.$$

Theorem. (V,c) Cartan type, $1 \neq q_{ii}$ root of 1.
$\dim \mathfrak{B}(V) < \infty \iff (a_{ij})$ of finite type.

N. A. & H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), 1-45.

I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Invent. Math. **164**, 175–188 (2006).

II. Pointed Hopf algebras with non-abelian group.

H pointed, $G = G(H)$ not abelian

Braided vector spaces attached to G

$\mathcal{C}$ a conjugacy class in G; fix $s \in \mathcal{C}$; let (ρ, V) irred. repr. of G^s.

$$M(\mathcal{C}, \rho) = \mathrm{Ind}_{G^s}^{G} = \mathbb{C}\mathcal{C} \otimes V.$$

Since $s \in Z(G^s)$, by the Schur Lemma,

s acts by a scalar q_{ss} on V.

The braided vector spaces attached to G are direct sums of different $M(\mathcal{C}_i, \rho_i)$'s (Dijkgraaf, Pasquier, Roche).

Problem. To classify finite dim. pointed Hopf algs. with $G(H) = G$, the first step is

when $\dim \mathfrak{B}(M(\mathcal{C}_1, \rho_1) \oplus \cdots \oplus M(\mathcal{C}_N, \rho_N)) < \infty$?

Example: $\mathcal{C}$ = transpositions in $G = \mathbb{S}_n$, $s = (12)$, $\rho = \mathrm{sgn}$

n	**rk**	**Relations**	$\dim \mathfrak{B}(V)$	**top**
3	3	5 relations in degree 2	$12 = 3.2^2$	$4 = 2^2$
4	6	16 relations in degree 2	576	12
5	10	45 relations in degree 2	8294400	40

$\mathbb{S}_3$, $\mathbb{S}_4$: A. Milinski and H.-J. Schneider, Contemp. Math. **267** (2000), pp. 215–236.

S. Fomin and K. Kirillov, Progr. Math. **172**, Birkhauser, (1999), 146–182.

$\mathbb{S}_5$: [FK], plus web page of M. Graña. http://mate.dm.uba.ar/~matiasg/

$\mathbb{S}_n$, $n \geq 6$: open!

Example: quadratic approximation $\mathfrak{B}_2(V)$

$\mathfrak{B}_2(V) = \mathbb{C}\langle x_{ij} = x_{ji} : (ij) \in \mathbb{S}_6 \rangle$
with relations:

$$\begin{aligned} x_{ij}^2 &= 0, \\ x_{ij}x_{kl} + x_{kl}x_{ij} &= 0, \quad \#\{i,j,k,l\} = 4, \\ x_{ij}x_{ik} + x_{jk}x_{ij} + x_{ik}x_{jk} &= 0, \quad \#\{i,j,k\} = 3. \end{aligned}$$

Problem: Is dim $\mathfrak{B}_2(V) < \infty$?

Example: $G = \mathbb{D}_n$, $\rho =$ sgn

$n =$**rk**	**Relations**	$\dim \mathfrak{B}(V)$	**top**
3	5 relations in degree 2	$12 = 3.2^2$	$4 = 2^2$
5	10 relations in degree 2 1 relation in degree 4	$1280 = 5.4^4$	$16 = 4^2$
7	21 relations in degree 2 1 relation in degree 6	$326592 = 7.6^6$	$36 = 6^2$

A few more examples:

M. Graña, J. Algebra **231** (2000), pp. 235-257.

N. A. and M. Graña, Adv. in Math. **178**, 177–243 (2003).

No general approach up to now! But ...

III. Discarding infinite-dimensional pointed Hopf algebras.

Strategy. *Given $(\mathcal{C}, \rho)$, find a braided subspace U of $M(\mathcal{C}, \rho)$ of diagonal type. Check if* $\dim \mathfrak{B}(U)$ *is infinite using the above mentioned results. If so, then* $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$.

M. Graña, Contemp. Math. **267** (2000), pp. 111–134.

If $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$, then

- $\deg \rho > 2$ implies $q_{ss} = -1$.
- $\deg \rho = 2$ implies $q_{ss} = -1$, ω_3 or ω_3^2.

Lemma. (A.– S. Zhang) Assume that there exists $\sigma \in G$ such that

$$\sigma s \sigma^{-1} = s^{-1}.$$

If $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$ then $q_{ss} = -1$, $\operatorname{ord} s$ even.

Corollary. (A.– S. Zhang) Let W be the Weyl group of a finite-dimensional semisimple Lie algebra.
If $\pi \in W$ has odd order then $\dim \mathfrak{B}(\mathcal{C}_\pi, \rho) = \infty$ for any $\rho \in \widehat{W^\pi}$.

Definition. $M(\mathcal{C}, \rho)$ is negative if

for all $s, t \in \mathcal{C}$ s. t. $st = ts$, $c(sv \otimes tw) = -tw \otimes sv$.

Theorem. Let $\pi \in \mathbb{S}_n$ and $\mathcal{C}$ the conjugacy class of π. Then for any $\rho \in \widehat{\mathbb{S}_n^\pi}$, either $\dim \mathfrak{B}(\mathcal{C}_\pi, \rho) = \infty$, or $M(\mathcal{C}, \rho)$ is negative.

N. A. and Shouchuan Zhang, *On pointed Hopf algebras associated to some conjugacy classes in* $\mathbb{S}_n$, Proc. Amer. Math. Soc. **135** (2007), 2723-2731.

N. A. and F. Fantino, *On pointed Hopf algebras associated to some conjugacy classes in* $\mathbb{S}_n$, J. Math. Phys 48, 033502 (2007).

N. A., F. Fantino and Shouchuan Zhang, *in preparation.*

Theorem. Let $\pi \in \mathbb{A}_n$ and $\rho \in \widehat{\mathbb{A}_n^\pi}$. Assume that π is neither $(1\,2\,3)$ nor $(1\,3\,2)$ in $\mathbb{A}_4$. If $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) < \infty$, then π has even order and $q_{\pi\pi} = -1$.

Proposition. Let π in $\mathbb{A}_4$ of type (2^2). Then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, for every ρ in $\widehat{\mathbb{A}_4^\pi}$.

Theorem. H finite-dimensional complex pointed Hopf algebra with $G(H) \simeq \mathbb{A}_5 \implies H \simeq \mathbb{C}\mathbb{A}_5$.

Remark. $\pi \in \mathbb{A}_4$ of type $(1, 3)$; $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) =?$, if ord $\rho = 3$.

N. A. and F. Fantino, *On pointed Hopf algebras associated with alternating and dihedral groups*, Rev. Un. Math. Arg. `http://arxiv.org/abs/math.QA/0702559`

Orbit	**Isotropy group**	**Rep.**	$\dim \mathfrak{B}(V)$
e	$\mathbb{D}_n$	any	∞
$\mathcal{O}_{y^h} = \{y^{\pm h}\}$, $h \neq 0$, $\mid \mathcal{O}_{y^h} \mid = 2$	$\mathbb{Z}_n \simeq \langle y \rangle$	any	∞
$\mathcal{O}_x = \{xy^h : 0 \leq h \leq n-1\}$, $\mid \mathcal{O}_x \mid = n$	$\mathbb{Z}_2 \simeq \langle x \rangle$	ε	∞
		sgn	negative braiding

TABLE 1. Nichols algebras of irreducible Yetter-Drinfeld modules over $\mathbb{D}_n$, n odd.

Orbit	**Isotropy group**	**Rep.**	$\dim \mathfrak{B}(V)$
e	$\mathbb{D}_n$	any	∞
$\mathcal{O}_{y^m} = \{y^m\}$, $\mid \mathcal{O}_{y^m} \mid = 1$	$\mathbb{D}_n$	$(V,\rho) \in \widehat{\mathbb{D}_n},\ \rho(y^m) = 1$	∞
		$(V,\rho) \in \widehat{\mathbb{D}_n},\ \rho(y^m) = -1$	$2^{\dim V}$
$\mathcal{O}_{y^h} = \{y^{\pm h}\}$, $h \neq 0, m$, $\mid \mathcal{O}_{y^h} \mid = 2$	$\mathbb{Z}_n \simeq \langle y \rangle$	χ^j, $\omega^{hj} = -1$	4
		χ^j, $\omega^{hj} \neq -1$	∞
$\mathcal{O}_x = \{xy^{2h} : 0 \leq h \leq m-1\}$ $\mid \mathcal{O}_x \mid = m$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \langle x \rangle \oplus \langle y^m \rangle$	$\varepsilon \otimes \varepsilon$, $\varepsilon \otimes$ sgn	∞
		sgn $\otimes$ sgn, sgn $\otimes \varepsilon$	negative braiding
$\mathcal{O}_{xy} = \{xy^{2h+1} : 0 \leq h \leq m-1\}$ $\mid \mathcal{O}_x \mid = m$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \langle xy \rangle \oplus \langle y^m \rangle$	$\varepsilon \otimes \varepsilon$, $\varepsilon \otimes$ sgn	∞
		sgn $\otimes$ sgn, sgn $\otimes \varepsilon$	negative braiding

TABLE 1. Nichols algebras of irreducible Yetter-Drinfeld modules over $\mathbb{D}_n$, $n = 2m$ even.

Let $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ be the algebra presented by generators e_t, $t \in T := \{(12), (23)\}$, and a_σ, $\sigma \in \mathcal{O}_2^3$; with relations

$$e_t e_s e_t = e_s e_t e_s, \quad e_t^2 = 1, \quad s \neq t \in T; \tag{1}$$
$$e_t a_\sigma = -a_{t\sigma t} e_t \qquad t \in T, \sigma \in \mathcal{O}_2^3; \tag{2}$$
$$a_\sigma^2 = 0, \qquad \sigma \in \mathcal{O}_2^3; \tag{3}$$
$$a_{(12)}a_{(23)} + a_{(23)}a_{(13)} + a_{(13)}a_{(12)} = \lambda(1 - e_{(12)}e_{(23)}); \tag{4}$$
$$a_{(12)}a_{(13)} + a_{(13)}a_{(23)} + a_{(23)}a_{(12)} = \lambda(1 - e_{(23)}e_{(12)}). \tag{5}$$

Set $e_{(13)} = e_{(12)}e_{(23)}e_{(12)}$. Then $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ is a Hopf algebra of dimension 72 with comultiplication determined by

$$\Delta(a_\sigma) = a_\sigma \otimes 1 + e_\sigma \otimes a_\sigma, \quad \Delta(e_t) = e_t \otimes e_t, \qquad \sigma \in \mathcal{O}_2^3, t \in T. \tag{6}$$

Theorem. (N. A., I. Heckenberger, H.-J. Schneider., using previous work with Milinski, Graña, Zhang).

Let H be a finite dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_3$. Then either $H \simeq \mathbb{C}\mathbb{S}_3$, or $H \simeq \mathfrak{B}(\mathcal{O}_2^3, \mathrm{sgn})\#\mathbb{C}\mathbb{S}_3$ or $H \simeq \mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, 1)$.

The proof is based on the *Weyl groupoid*, cf. Schneider's talk at the AMS meeting.