ANTICYCLOTOMIC p-ADIC L-FUNCTIONS FOR ELLIPTIC CURVES AT SOME ADDITIVE REDUCTION PRIMES

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Abstract. Let E be a rational elliptic curve and let p be an odd prime of additive reduction. Let K be an imaginary quadratic field and fix a positive integer c prime to the conductor of E. The main goal of the present article is to define an anticyclotomic p-adic L-function $\mathscr{L}$ attached to E/K when $E/\mathbb{Q}_p$ attains semistable reduction over an abelian extension. We prove that $\mathscr{L}$ satisfies the expected interpolation properties; namely, we show that if χ is an anticyclotomic character of conductor cp^n then $\chi(\mathscr{L})$ is equal (up to explicit constants) to $L(E,\chi,1)$ or $L'(E,\chi,1)$.

Introduction

The theorem of Mordell states that the rank of a rational elliptic curve E is finite. It is a hard and interesting problem to determine it and, furthermore, to compute a set of generators for $E(\mathbb{Q})$. By Weil's generalization of Mordell's result, the rank is still finite over number fields L. Although the rank cannot be bounded over arbitrary algebraic extensions, sometimes this is still the case, for example, Mazur ([Maz72]) proved that if Σ is a finite set of primes then $E(\mathbb{Q}_\Sigma^{ab})$ is finitely generated, where $\mathbb{Q}_\Sigma^{ab}$ denotes the maximal abelian extension of $\mathbb{Q}$ unramified outside Σ.

The techniques used to bound the rank of E/L involve a detailed analysis of the Selmer group. If L is the $\mathbb{Z}_p$-extension of $\mathbb{Q}$, that is, a Galois extension with Galois group isomorphic to $\mathbb{Z}_p$, a deep conjecture of Iwasawa relates the dual of the p-primary part of this Selmer group to a p-adic analytic object called the cyclotomic p-adic L-function of E. The study and definition of such p-adic L-function was considered by many authors ([MSD74, AV75, Viš76, MTT86]).

A natural variation of the problem is to start with a base field K, and study the rank of E over a $\mathbb{Z}_p$-extension L/K. When K is an imaginary quadratic field, any such extension is contained in the compositum of the $\mathbb{Z}_p$-cyclotomic extension (lying inside the extension obtained by adjoining the p^n-th roots of unity for every $n \in \mathbb{N}$) and the so called $\mathbb{Z}_p$-anticyclotomic extension (a generalized dihedral extension of

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$\mathbb{Q}$). These two extensions are the only ones that are Galois over $\mathbb{Q}$. A good reason to study the anticyclotomic $\mathbb{Z}_p$- extension is that if χ is an anticyclotomic character then the L-function $L(E,\chi,s)$ satisfies a functional equation and its central value holds important arithmetic information. The p-adic L-function $\mathscr{L}$ is a p-adic analytic function that should encode the central values $L(E,\chi,1)$ (or its derivative $L'(E,\chi,1)$) for finite order anticyclotomic characters χ.

The study of the rank behavior over the anticyclotomic extension, and the generalization of Iwasawa's conjecture to this setting was pioneered by Bertolini and Darmon (see for example the breakthrough papers [BD96, BD05]), where they prove (among many important properties) one divisibility of the anticyclotomic Iwasawa's main conjecture. The strategy in this setting is to construct special geometric objects (CM points) arising from orders in the imaginary quadratic field K satisfying compatibility relations. More precisely, let N be the conductor of E, let c be a positive integer prime to N, and let $G_n := \mathrm{Gal}(H_n/K)$, where H_n denotes the ring class field of conductor cp^n. The special points allow to construct a p-adic measure on the Galois group $G_\infty := \varprojlim G_n$ (such measure is naturally defined in the characteristic functions of the sets G_n for each n and extended by continuity to locally constant p-adic functions). To ensure the additive property of the measure a suitable normalization of the geometric points is needed. In [MTT86] a normalization is presented using the action of the U_p operator and its eigenvalues. This imposes an extra condition at p, namely the curve must be semistable ordinary at p (the supersingular case was considered by Pollack [Pol03] and Darmon-Iovita [DI08] in the cyclotomic and anticyclotomic setting respectively).

Perrin-Riou ([PR94]) gave a very general construction of the p-adic L-function once a local condition at p is imposed (see Theorem 16.4 of [Kat04]) from the data of an Euler system and Kato constructed such an Euler system for modular forms. The local condition at p for the p-adic L-function can be understood as choosing a "canonical" direction to project such cohomological classes. In the multiplicative reduction case one can take the submodule given by the line fixed by inertia, while in the good ordinary reduction case the natural choice is to take the same submodule of the p-stabilized form attached to E. The problem is that when $p^2 \mid N$, there is no canonical choice! This obstruction continues to hold in the anticyclotomic scenario considered by Bertolini-Darmon. Nevertheless, even when E has additive reduction at p, there are some instances where a natural normalization can be taken, namely when $E/\mathbb{Q}_p$ attains *semistable reduction over an abelian extension* (SRAE) of $\mathbb{Q}_p$. This approach was carried over by Delbourgo [Del98] in the cyclotomic case and the main contribution of this article is to make an analogous construction in the anticyclotomic scenario.

To keep the statement as simple as possible, we state our main result with some extra hypotheses: let E be an elliptic curve of conductor N, with p a SRAE prime of additive reduction which is not a quadratic twist of an elliptic curve semistable at p, and let χ be a family of anticyclotomic characters of conductor cp^n, with $n \geq 1$. The sign of the functional equation of $L(E, \chi, s)$ is constant on this family; suppose it equals $+1$.

Theorem. With the above hypotheses, there exists an antyciclotomic p-adic L-function $\mathscr{L} \in \mathbb{Z}[G_\infty]$ which satisfies the following interpolation properties:

$$\chi(\mathscr{L}) = \frac{p^n}{\alpha^{2n}} \cdot \frac{L(1, E, \chi)}{\Omega'_E} \cdot \frac{u_n^2 \sqrt{D} c}{2^{-\#\Sigma_D}},$$

where Ω'_E is a period, Σ_D is the set of places dividing both N/p^2 and the discriminant D of K, u_n is half the number of units of the order of conductor cp^n and α is a p-adic unit which depends only on E.

We prove a stronger result valid for any elliptic curve E for which p is a SRAE prime and including a slightly more general class of characters χ. Furthermore, when the functional equation sign in the family equals -1, we have a similar theorem, but replacing $L(E, \chi, 1)$ with the special values of the derivative $L'(E, \chi, 1)$. See Theorems 3.6 and 4.1 for the precise statements.

Our strategy is as follows: the modularity of rational elliptic curves (due to Wiles et al. [Wil95, BCDT01]) implies that there exists an automorphic representation π_E of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character whose L-series coincides with that of E. The SRAE at p hypothesis (for $p \geq 3$) is equivalent to π_E being a Steinberg representation or a ramified principal series at p. Then there exists a Dirichlet character ψ and an automorphic form π_g whose level has valuation at most 1 at p (with non-trivial Nebentypus in general) such that $\pi_g \otimes \psi = \pi_E$. Following the general philosophy, the restriction of the p-adic Galois representation attached to π_g (by Deligne) to the local Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ does have a stable line (hence a natural submodule).

Concretely, the form π_g has an abelian surface A_g attached to it (of GL_2-type, whose endomorphism ring $\mathrm{End}_\mathbb{Q}(A_g) \otimes \mathbb{Q}$ isomorphic to $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$) [Koh17, Section 2.1] where we make a *classical* construction of CM points on A_g (as we did in [KP18] for constructing Heegner points for SRAE primes ramifying in K) and use them to define the p-adic L-function of E. Clearly, the p-adic L-function of E and that of A_g should be related by a "shift" on the analytic functions space (corresponding to the twist by ψ). The main novelty of the present article is that the special points used to construct the p-adic L-function of E are in A_g (not in E); still their existence and properties are enough to define the p-adic L-function.

The second goal of the article is to prove the interpolation properties of the p-adic L-function. In order to prove it we make heavy use of the fact that the CM points used to define the p-adic L-function have heights related to central values, as proved by Waldspurger and by Gross-Zagier (in our setting the explicit formulas are due to Cai-Shu-Tian [CST14]). Note that special values $L(E,\chi,1)$ are related to $L(A_g,\psi\chi,1)$, which justifies working with A_g instead of E. The results we obtained are similar in spirit to the ones by Chida-Hsieh [CH] and Van Order [Van17] but they only consider the case where the reduction at p is semistable.

In addition, Disegni on [Dis17] deals with a much more general situation but under the hypothesis that the prime p splits in K (in that case our result can be obtained by plugging the corresponding test vector in his formula). We want to stress that we do not make any assumptions on the factorization of p in K: it could be split, inert or ramified. The ramified case is of special interest as it is widely overlooked in the literature (in the semistable case see the very recent preprint of Longo-Pati [LP17]). In a sequel article, we will use the present construction to prove one divisibility of Iwasawa's main conjecture.

To ease the exposition, we assume that the level of π_g is divisible by p (i.e. E is not the quadratic twist of an elliptic curve with good reduction at p). In the last section we explain the changes needed to handle this case.

The method described in the present article can be used to handle the case of newforms in $S_k(\Gamma_0(N))$, for arbitrary weights k, whose level N is exactly divisible by p^2 with the conditions:

(1) The local component at p is not supercuspidal,
(2) The L-series $L(f,\chi,s)$ has functional equation sign $+1$ (so as to work with definite quaternion algebras).

The techniques are developed in [CH] in the semistable case, and our technique can be applied with the natural modifications.

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Setting and notation

We fix the following hypotheses and notation throughout the article:

- Let p be a fixed odd prime number.
- Let $E/\mathbb{Q}$ be an elliptic curve of conductor N with SRAE at p. Let π_E be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to E.
- As explained in the introduction, π_g denotes an automorphic representation with $v_p(\mathrm{cond}(\pi_g)) \leq 1$, and ψ denotes a character of conductor p such that $\pi_g \otimes \psi = \pi_E$. We assume in all sections but the last one that $v_p(\mathrm{cond}(\pi_g)) = 1$.

- Let K be an imaginary quadratic field and let η be the quadratic Hecke character in correspondence with K via class field theory.
- Let c be a positive integer relatively prime to N (in particular $p \nmid c$).
- For $d \in \mathbb{N}$, let $\mathcal{O}_d := \mathbb{Z} + d\mathcal{O}_K$ be the order in K of conductor d.
- Let H_n be the ring class field of conductor cp^n and let $\widetilde{H}_n = H_n(\overline{\mathbb{Q}}^{\ker(\psi)})$. We define the Galois groups $G_n := \mathrm{Gal}(H_n/K)$ and $\widetilde{G}_n := \mathrm{Gal}(\widetilde{H}_n/K)$ and their respective limits $G_\infty := \varprojlim G_n$, $\widetilde{G}_\infty := \varprojlim \widetilde{G}_n$.
- χ will denote a finite order anticyclotomic character of K, i.e. $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ denotes a finite order Hecke character whose restriction to $\mathbb{A}_\mathbb{Q}^\times$ is trivial.
- For Σ a finite set of places and $L^{(\Sigma)}(E, \chi, s)$ denotes the classical L-series, with the factors at primes in Σ removed.
- $L^\epsilon(E, \chi, s)$ denotes the L-series for $\epsilon = 0$ and its derivative for $\epsilon = 1$.
- If M is a $\mathbb{Z}$-module, we denote by M_p the extension of scalars to $\mathbb{Z}_p$, namely $M_p = M \otimes_\mathbb{Z} \mathbb{Z}_p$.
- $\widehat{\mathbb{Z}}$ denotes the profinite integers, namely $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p$. If M is a $\mathbb{Z}$-module we write $\widehat{M} := M \otimes \widehat{\mathbb{Z}}$.
- B will denote a rational quaternion algebra, and $\hat{B} = B \otimes_\mathbb{Q} \hat{\mathbb{Q}}$.
- R will denote an order in B, and consistently $\hat{R} = R \otimes_\mathbb{Z} \hat{\mathbb{Z}}$.
- If B is a rational quaternion algebra split at p, and $M \in \mathrm{M}_2(\mathbb{Q}_p) \cong B_p$, $M^{(p)}$ denotes the element in $\widehat{B}$ whose p-th entry equals M and the others equal 1.

1. Quaternion algebras and CM points

From now on, we will let E be a fixed elliptic curve of conductor N with SRAE at p, and π_g the automorphic representation with $v_p(\mathrm{cond}(\pi_g)) = 1$. Let K be an imaginary quadratic field, corresponding via class field theory to a quadratic character η.

Let χ be an anticyclotomic character of K whose conductor divides cp^n; this corresponds to a character of $\mathrm{Gal}(K^{ab}/K)$ factoring through G_n. The anticyclotomic assumption implies that the twisted L-function $L(\pi_E, \chi, s)$ satisfies a functional equation

$$L(\pi_E, \chi, s) = \varepsilon(\pi_E, \chi, s) L(\pi_E, \chi, 2 - s),$$

where $\varepsilon(\pi_E, \chi, s)$ is the so called *epsilon factor* (for definitions and facts regarding such L-series, consult [Jac72, Chapter IV]). The global root number $\varepsilon(\pi_E, \chi, 1)$ can be computed as the product of local root numbers $\varepsilon(\pi_{Ev}, \chi_v, 1)$ each of them being ± 1 (see [Del73]). Consider the set

$$S(\chi) := \{v : \ \varepsilon(\pi_{Ev}, \chi_v, 1) \neq \chi_v(-1)\eta_v(-1)\}.$$

By Theorem 1.3 of [YZZ13], $\varepsilon(\pi_E, \chi, 1) = (-1)^{\#S(\chi)}$ and thus the parity of the size of $S(\chi)$ determines the parity of the order of vanishing of $L(\pi_E, \chi, s)$ at $s = 1$.

Proposition 1.1. *The set $S(\chi)$ satisfies the following properties:*
(1) *The archimedean prime ∞ belongs to $S(\chi)$.*
(2) *If $v \neq p$ is a non-archimedean prime, then the condition "$v \in S(\chi)$" depends only on K, i.e. is independent of χ.*
(3) *The prime p does not belong to $S(\chi)$ if either*
  - *The local Weil-Deligne representation of E at p is a principal series.*
  - *The prime p splits in K.*
  - *The prime p is inert in K and χ_p is not equal to the quadratic character modulo p.*
  - *The prime p is ramified in K and χ_p is not trivial.*

Proof. The first statement follows from [Gro88, Proposition 6.5], while the second one follows from the assumption that $\gcd(c, N) = 1$. Regarding the last one, the assumption on p being a SRAE prime implies that the local representation of E at p is either a twist of Steinberg or a principal series. The result then follows from [Tun83, Propositions 1.6 and 1.7]. $\square$

In particular, for all but finitely many characters χ of conductor cp^n (with $p \nmid c$), the set $S(\chi)$ is constant. Let S denote such generic common set. Let $\epsilon \in \{0, 1\}$ be such that $\epsilon \equiv \#S \pmod 2$. By $L^\epsilon(\pi_E, \chi, s)$ we denote the L-series $L(\pi_E, \chi, 1)$ if $\epsilon = 0$ and its derivative $L'(\pi_E, \chi, s)$ if $\epsilon = 1$. Our main goal is to interpolate the special values $L^\epsilon(\pi_E, \chi, 1)$.

To relate central values of π_E to those of π_g, let

$$\widetilde{\chi} := \chi \cdot (\psi \circ Nm_{\mathbb{A}_K^\times/\mathbb{A}_{\mathbb{Q}}^\times}) : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times.$$

Since $\widetilde{\chi}\mid_{\mathbb{A}_{\mathbb{Q}}^\times} = \psi^2$, $L(\pi_g, \widetilde{\chi}, s)$ is self dual and clearly $L(\pi_E, \chi, s) = L^{(\{p\})}(\pi_g, \widetilde{\chi}, s)$.

Definition 1.2. The character χ is *good* if the conductor of $\widetilde{\chi}$ is divisible by p.

If $\widetilde{\chi}$ has conductor cp^n (with $p \nmid c$), we will see that the central value $L^\epsilon(\pi_g, \widetilde{\chi}, 1)$ is related to the height of a linear combination of CM points of conductor cp^n. Varying the character's conductor, involves constructing CM points of different conductors and good characters correspond to good CM points in the sense of Cornut-Vatsal [CV07, Definition 1.6], that will give the distribution relations needed to define a p-adic measure. Note that Proposition 1.1 implies that if χ is good, $p \notin S(\chi)$. From now on, we will only work with good characters.

Let $B/\mathbb{Q}$ be the quaternion algebra ramified at the places of S if $\epsilon = 0$ (the *definite case*) and at all places of S but the infinite one if $\epsilon = 1$ (the *indefinite case*).

Lemma 1.3. *There exists an embedding $\iota : K \mapsto B$.*

Proof. Proposition 1.1 implies that if v splits in K, then $v \notin S$. The result then follows from Theoreme 3.8 of [Vig80]. $\square$

1.1. **Quaternionic level.** Given a good character χ as before we seek for an order R in B and an embedding $\iota : K \to B$ with the properties that π_g transfers to an automorphic form of level R and, at the same time, R contains CM points.

Definition 1.4. Let $\iota : K \to B$ be an embedding, $\mathfrak{n}$ a positive integer divisible by p with $\gcd(\mathfrak{n}, N/p^2) = 1$ and $R \subset B$ be an order. We say that R is *admissible* for $(\pi_g, \mathfrak{n}, \iota)$ if R_p is an Eichler order of level $p\mathbb{Z}_p$ and ι is an optimal embedding of $\mathcal{O}_{\mathfrak{n}}$ into R, that is, $\iota(K) \cap R = \iota(\mathcal{O}_{\mathfrak{n}})$.

Remark 1.5. If $\widetilde{\chi}$ is a character whose conductor is divisible by p, then our admissibility condition for $(\pi_g, \mathrm{cond}(\widetilde{\chi}), \iota)$ implies admissibility in the sense of [CST14, Definition 1.3].

Given an embedding ι and a good character $\widetilde{\chi}$, there always exists an admissible order R for $(\pi_g, \mathrm{cond}(\widetilde{\chi}), \iota)$ by [Gro88, Propositions 3.2, 3.4], [CST14, Lemma 3.2] and the local-global principle. Still, for explicit computations, it is useful to choose R such that its completion R_p matches the standard Eichler order. This can be achieved allowing to change the embedding to an equivalent one.

Lemma 1.6. *Let c be a positive integer prime to p. Then, there exists an embedding $\iota : K \to B$ and an order $\mathscr{R} \subset B$ which is admissible for (π_g, cp, ι) whose completion at p is the standard Eichler order of level $p\mathbb{Z}_p$.*

Proof. Let R be any admissible order for (π_g, cp, ι). Locally, R_p is conjugate to the standard Eichler order, but by weak approximation we can find a global element that sends this order to the standard one. Conjugating both ι and the order the result follows. $\square$

Fix once and for all $\mathscr{R}$ and ι as in the lemma. For $n \geq 1$, let $\delta_n := \left(\begin{smallmatrix} p^{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{(p)} \in \widehat{B}^\times$ (see the notations section).

Lemma 1.7. *Let $n \geq 1$ be a positive integer. The order $\mathscr{R}_n := \delta_n \widehat{\mathscr{R}} \delta_n^{-1} \cap B$ is admissible for (π_g, cp^n, ι).*

Proof. Let $\omega' \in K$ be such that $\mathcal{O}_c = \mathbb{Z} + \omega'\mathbb{Z}$. Then, $\mathcal{O}_{cp} = \mathbb{Z} + \omega\mathbb{Z}$, where $\omega := p\omega'$. Since the order $\mathscr{R}$ is admissible for (π_g, cp, ι), the p-th component of the image of ω under ι is a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p)$ such that p divides a, c, d and does not divide b. Moreover, $\mathcal{O}_{cp^n} = \mathbb{Z} + p^{n-1}\omega\mathbb{Z}$ and

$$\left(\begin{smallmatrix} p^{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} p^{n-1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} p^{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

is a matrix whose entries are p-integers and its $(1,2)$ entry is not divisible by p. This shows that ι is an optimal embedding of $\mathcal{O}_{cp^n}$ into the order $\mathscr{R}_n$ as stated. $\square$

Let U be an open compact subgroup of $\widehat{B}^\times$. If B is definite, let

$$X_U := B^\times \backslash \widehat{B}^\times / U,$$

where U acts on $\widehat{B}^\times$ by right multiplication and $B^\times$ acts on $\widehat{B}^\times$ by left multiplication. If B is indefinite, let

$$X_U := B^\times \backslash (\mathbb{C} - \mathbb{R}) \times \widehat{B}^\times / U,$$

where U acts trivially on $\mathbb{C} - \mathbb{R}$ while $B^\times$ acts on $\mathbb{C} - \mathbb{R}$ by Möbius transformations under the identification $B_\infty \cong \mathrm{M}_2(\mathbb{R})$. If R is an order in B we write $X_R := X_{\widehat{R}^\times}$.

Remark 1.8. In the definite case, the curves X_U are 0-dimensional (i.e. are finite sets) while in the indefinite case, they have dimension 1. In the latter case, we denote by J_U its Jacobian variety.

Let $X := \varinjlim_U X_U$ and $J := \varinjlim_U J_U$, where the limit is induced by the natural projection arising from the inclusion of level structures. Since $\mathscr{R}_p \subset B_p \cong M_2(\mathbb{Q}_p)$ we can regard ψ as a character on $\widehat{\mathscr{R}}^\times$ by the reduction modulo p of the $(2,2)$-entry of $\mathscr{R}_p$. Recall that in the introduction we defined the abelian variety $A_g/\mathbb{Q}$ associated to π_g (see Section 6.6 of [DS05] for more details, in particular Theorem 6.6.6)

Theorem 1.9 (Jacquet-Langlands)**.** *With the same notations as before, there is an automorphic transfer of the form π_g to the algebra $\widehat{B}^\times$. Furthermore,*

(1) *if B is definite, there exists an automorphic form $g_B : B^\times \backslash \widehat{B}^\times \to \mathbb{C}$, such that*

$$r \cdot g_B(x) := g_B(xr) = \psi^{-2}(r) g_B(x) \text{ for all } r \in \widehat{\mathscr{R}}^\times.$$

(2) *If B is indefinite, there exists $g_B \in Hom(J, A_g) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that*

$$r \cdot g_B = \psi^{-2}(r) g_B \text{ for all } r \in \widehat{\mathscr{R}}^\times,$$

where $Hom(J, A_g) \otimes_{\mathbb{Z}} \mathbb{Q}$ is endowed with the right Hecke action of $\widehat{B}^\times$ inherited from X.

Moreover, if all primes $q \neq p$ such that $q^2 \mid N$ are unramified in K, the form g_B is unique up to a constant.

Proof. The existence of the form g_B and its uniqueness follow from [GP91, Proposition 2.6] and [CST14, Propositions 3.7 and 3.8]) combined with the Jacquet-Langlands philosophy.

$\square$

1.2. **CM points.** The embedding $\iota : K \hookrightarrow B$ induces an embedding $\widehat{\iota} : \widehat{K}^\times \hookrightarrow \widehat{B}^\times$. If B is indefinite, let z_0 be the unique fixed point on the upper half plane under the action of $K^\times$. Define

$$P = [(z_0, 1)] \in X,$$

where if B is definite, abusing notation, the point $[(z_0, b)]$ denotes the class of $b \in \widehat{B}^\times$ in X. Let

$$\mathscr{U} := \left\{ (x_\ell)_\ell \in \widehat{\mathscr{R}}^\times : x_p \equiv \left(\begin{smallmatrix} * & * \\ 0 & a \end{smallmatrix}\right) \bmod p \text{ with } \psi^2(a) = 1 \right\}.$$

Note that from Theorem 1.9 we immediately obtain that the form g_B is invariant under the action of $\mathscr{U}$ (so we can think of g_B as a form with "trivial Nebentypus" with respect to the level $\mathscr{U}$). The inclusion $\mathscr{U} \subseteq \widehat{\mathscr{R}}^\times$ induces a quotient map $\beta : X_{\mathscr{U}} \to X_{\mathscr{R}}$.

Definition 1.10.

- A *CM point of conductor* cp^n on $X_{\mathscr{R}}$ is a pair $[z_0, b] \in X_{\mathscr{R}}$, where $b \in \widehat{B}^\times$ is such that ι is an optimal embedding of $\mathcal{O}_{cp^n}$ into $b\widehat{\mathscr{R}}b^{-1} \cap B$.
- For $n \geq 1$, the *CM points of conductor* cp^n on $X_{\mathscr{U}}$ are the preimages under β of CM points of conductor cp^n on $X_{\mathscr{R}}$.

Let

$$\mathfrak{z}_n := \delta_n \cdot P = \left[\left(z_0, \left(\begin{smallmatrix} p^{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{(p)} \right) \right] \in X_{\mathscr{R}}.$$

Proposition 1.11. *The points $\mathfrak{z}_n$ (for $n \geq 1$) are CM points of conductor cp^n on $X_{\mathscr{R}}$. In particular their preimages under β are CM points on $X_{\mathscr{U}}$.*

Proof. This follows immediately from Lemma 1.7. $\square$

There is a natural action of $\mathrm{Gal}(K^{ab}/K) \cong K^\times \backslash \widehat{K}^\times$ on CM points given by

$$\mathfrak{a} \cdot [(z_0, b)] := [(z_0, \widehat{\iota}(\mathfrak{a}) b)].$$

In the indefinite case the Galois action is the natural one on algebraic points. However, in the definite scenario we do not have a similar interpretation.

Consider the operator U_p whose action on both $\mathrm{Div}(X_{\mathscr{R}})$ and $\mathrm{Div}(X_{\mathscr{U}})$ is given by

$$U_p([(z_0, b)]) = \sum_{i=0}^{p-1} [(z_0, b \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)^{(p)})].$$

The interplay between the Galois action and the U_p action on CM points is as follows.

Proposition 1.12. *Let $n \geq 1$.*

1. *If B is definite $\mathfrak{z}_n \in H^0(\mathrm{Gal}_{\widetilde{H}_n}, X_{\mathscr{U}})$ and if B is indefinite $\mathfrak{z}_n \in X_{\mathscr{U}}(\widetilde{H}_n)$.*
2. $\sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{n+1}/\widetilde{H}_n)} \mathfrak{z}_{n+1}^\sigma = U_p(\mathfrak{z}_n)$.

Proof. This is essentially proved by Longo and Vigni in [LV11, Propositions 3.2, 3.3, 3.4 and Section 4.4], with the remark that for $n \geq 1$, the second condition in their definition of Heegner points (Ibid Definition 3.1) is redundant, hence it coincides with our definition of CM points. The only difference is that they work with a full $\Gamma_1(p)$ structure and thus their points are defined over the extension $H_n(\mu_p)$. But proceeding as in [KP18, Proposition 2.12] we see that the points for $\mathscr{U}$ are defined over $\widetilde{H}_n$. □

2. Waldspurger and Gross-Zagier formulas

The CM points defined in the previous section are related to the central values of $L^{\epsilon}(\pi_g, \widetilde{\chi}, s)$ via the Waldspurger formula for $\epsilon = 0$ (the definite case) and the Gross-Zagier formula for $\epsilon = 1$ (the indefinite case). We follow the more general formulas by Yuan-Zhang-Zhang [YZZ13] and the explicit formulation given by Cai-Shu-Tian in [CST14].

Recall the choice of the ramification algebra B and the ramification set $S(\chi)$ given in Section 1. By results of Tunnel and Saito ([Tun83, Propositions 1.6 and 1.7] and [Sai93, Propositions 6.3 and 6.5]) the space $\mathrm{Hom}_{K^\times}(\pi_{g_B}, \widetilde{\chi})$ is 1-dimensional.

Definition 2.1. A vector $v \in \pi_{g_B}$ is a called a *test vector* for $\widetilde{\chi}$ if $\ell_{\widetilde{\chi}}(v) \neq 0$ for any nonzero $\ell_{\widetilde{\chi}} \in \mathrm{Hom}_{K^\times}(\pi_{g_B}, \widetilde{\chi})$.

Proposition 2.2. *Suppose that for every prime $q \neq p$ such that $q^2 \mid N$, q is unramified in K. Let χ be a good character such that $\widetilde{\chi}$ is of conductor cp^n. Then the space ${\pi_{g_B}}^{\delta_n^{-1}\mathscr{U}\delta_n}$ is one dimensional. Moreover, every non-zero vector of it is a test vector for $\widetilde{\chi}$.*

Proof. The follows from [GP91, Proposition 2.6] and [CST14, Propositions 3.7 and 3.8]. □

Remark 2.3. In the case when there are primes q ramified in K such that $q^2 \mid N$, the local space $(\pi_{g_B})_q^{\mathscr{R}_q}$ has dimension 2, but there is a canonical fixed line to consider, as explained in [GP91, Remark 2.7]. For the general construction, we take an element in such line as the test vector g_B. Note that this small technical issue is not important as we will only be varying the test vectors at the prime p which is different from any such q.

For $n \geq 1$, consider the vector $\phi_n := \delta_n \cdot g_B \in \pi_{g_B}$.

Lemma 2.4. *The vector ϕ_n is a non-zero test vector for $\widetilde{\chi}$. The complex conjugate of ϕ_n viewed as an element of $\pi_{g_B}^\vee$ is a non-zero test vector for $\widetilde{\chi}^{-1}$.*

Proof. The statement follows from the fact that $g_B \in \pi_{g_B}$ is invariant under the action of $\mathscr{U}$. □

Let $\mathcal{Z}$ be $\mathbb{Z}$ in the definite case and $A_g(\overline{\mathbb{Q}})$ in the indefinite one. The projection of $P = [(z_0, 1)]$ to the $\widetilde{\chi}$-isotypical component in $\mathcal{Z}$ is given by

$$P_{\widetilde{\chi}}(\phi_n) := \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_n/K)} \phi_n(P^\sigma)\widetilde{\chi}(\sigma) = \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_n/K)} g_B(\mathfrak{z}_n^\sigma)\widetilde{\chi}(\sigma) \in (\mathcal{Z} \otimes \mathbb{C})^{\widetilde{\chi}}.$$

Let Σ_D be the set of places dividing both N/p^2 and the discriminant D of K and let $u_n := \#\mathcal{O}_{cp^n}{}^\times/2$. Let $\langle -, - \rangle$ denote the natural pairing in $\mathcal{Z}$, i.e. multiplication in the definite case and the Néron-Tate pairing in the indefinite one. We are now able to state the explicit version of Gross-Zagier and Waldspurger formulas.

Theorem 2.5. *Let χ a good character, and let cp^n be the conductor of $\widetilde{\chi}$. Then*

$$L^{\epsilon,\{p\}}(1, \pi_g, \widetilde{\chi}) := \frac{2^{-\#\Sigma_D} 8\pi^2 \langle g, \overline{g} \rangle_{U_0(N/p)}}{u_n^2 \sqrt{D} cp^n} \frac{\left\langle P_{\widetilde{\chi}}(\phi_n), P_{\widetilde{\chi}^{-1}}(\overline{\phi_n}) \right\rangle}{\left\langle \phi_n, \overline{\phi_n} \right\rangle_{\delta_n^{-1} \mathscr{U} \delta_n}}.$$

Proof. See [CST14] Theorem 1.8 for $\epsilon = 0$ and Theorem 1.5 for $\epsilon = 1$. □

3. Anticyclotomic p-adic L-function

The p-adic L-function is a functional on locally constant functions attached to a p-adic measure μ_E, i.e. if h is a locally constant function, $\mathscr{L}_p(h) = \int h d\mu_E$. We will construct it using the CM points we defined. Once the p-adic L-function is defined, we will use the results of the previous sections to relate its values at characters χ with the values $L^\epsilon(\pi_E, \chi, 1)$.

A crucial hypothesis in the classical constructions is that π_E has an eigenvalue for the U_p operator with *small* slope. Since E has additive reduction at p its unique eigenvalue for U_p is 0. However, under our working assumptions E has SRAE at p so we can bypass this considering the abelian variety A_g. Let α be the eigenvalue of the U_p-operator acting on g_B. If f is Steinberg at p, $\alpha = \pm 1$; otherwise the coefficient field M of g_B is a quadratic extension of $\mathbb{Q}$ (either $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$) in which p splits ([Koh17, Section 2.1]), so there exists a prime $\mathfrak{p} \mid p$ such that $\mathfrak{p} \nmid \alpha$. Then $\alpha \in \mathcal{O}_{M_\mathfrak{p}}^\times \cong \mathbb{Z}_p^\times$. Since the space of modular forms has an integral basis and the modular form g has eigenvalues lying in $\mathbb{Z}_p$, we can always normalize g_B such that the images of the CM points lie in $\mathcal{Z}_p := \mathcal{Z} \otimes \mathbb{Z}_p$.

Definition 3.1. For $n \geq 1$ the *regularized* CM points on $\mathcal{Z}_p$ are

$$\zeta_n^\sigma := g_B(\mathfrak{z}_n^\sigma)\alpha^{-n}.$$

Proposition 3.2 (Distribution relation). *If $n \geq 1$, the regularized CM points satisfy the relation*

$$\sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{n+1}/\widetilde{H}_n)} \zeta_{n+1}^\sigma = \zeta_n.$$

Proof. This is an immediate consequence of Proposition 1.12. □

For $n \geq 1$ let

$$\widetilde{\theta}_n := \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_n/K)} \zeta_n^\sigma \sigma \in \mathcal{Z}_p[\widetilde{G}_n]. \tag{1}$$

The compatibility relation allows to attach a p-adic measure to g_B, since it gives a well defined element

$$\widetilde{\theta} := \varprojlim_n \widetilde{\theta}_n \in \mathcal{Z}_p[[\widetilde{G}_\infty]].$$

Its twisted version (that will give rise to the p-adic L-function of π_E) is defined by

$$\theta_n := \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_n/K)} \psi(\sigma)\zeta_n^\sigma \sigma \in \mathcal{Z}_p[\widetilde{G}_n], \tag{2}$$

where by class field theory, we can think of ψ as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ factoring through $\widetilde{H}_n$. It is clear from the definition that ψ is compatible with the natural map $\widetilde{G}_{n+1} \to \widetilde{G}_n$ hence we also get a well defined object

$$\theta := \varprojlim_n \theta_n \in \mathcal{Z}_p[[\widetilde{G}_\infty]].$$

Let $\mu_{g_B,\alpha}$ (respectively μ_E) denote the measure on $\widetilde{G}_\infty$ attached to $\widetilde{\theta}$ (resp. θ). Note that for $\sigma \in \widetilde{G}_\infty$, the two measures satisfy that

$$\psi(\sigma)\mu_{g_B,\alpha}(\sigma) = \mu_E(\sigma).$$

If χ is a good character such that $\widetilde{\chi}$ is of conductor cp^n, then

$$\widetilde{\chi}(\widetilde{\theta}) = \int_{\widetilde{G}_\infty} \widetilde{\chi}(g) d\mu_{g_B,\alpha}(g).$$

The character $\widetilde{\chi}$ factors through $\mathrm{Gal}(\widetilde{H}_n/K)$ so the integral equals the finite sum

$$\widetilde{\chi}(\widetilde{\theta}) = \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_n/K)} \widetilde{\chi}(\sigma)\zeta_n^\sigma.$$

Looking at the definitions of $\widetilde{\chi}$ and $\widetilde{\theta}$ it is clear that $\widetilde{\chi}(\widetilde{\theta}) = \chi(\theta)$. In particular, a similar formula holds for $\chi(\theta)$.

One should think of θ as the square root of the p-adic L-function. More precisely, let $*$ be the involution sending σ to σ^{-1} and let

$$\mathscr{L}_n := \theta_n \otimes \theta_n^* \in (\mathcal{Z}_p \otimes \mathcal{Z}_p)[\widetilde{G}_n].$$

Definition 3.3. The p-adic L-function attached to π_E is

$$\mathscr{L} := \varprojlim_n \mathscr{L}_n \in (\mathcal{Z}_p \otimes \mathcal{Z}_p)[[\widetilde{G}_\infty]].$$

Remark 3.4. If we change our compatible sequence of CM points $\{\mathfrak{z}_n\}_{n\geq 1}$ for another compatible sequence $\{\mathfrak{z}'_n\}_{n\geq 1}$ there must exist an element $\sigma_0 \in \widetilde{G}_\infty$ such that for every $n \geq 1$, $\mathfrak{z}_n^{\sigma_0} = \mathfrak{z}'_n$. Let θ' be the corresponding element associated to $\{\mathfrak{z}'_n\}_{n\geq 1}$. Then we have that

$$\theta' = \psi(\sigma^{-1})\sigma_0^{-1} \cdot \theta.$$

Similarly, working with θ^* and θ'^* we obtain

$$\theta'^* = \psi(\sigma^{-1})\sigma_0 \cdot \theta^*.$$

Putting these two equations together we get

$$\mathscr{L}' = \psi(\sigma_0^{-2})\mathscr{L}.$$

Thus $\mathscr{L}$ is more intrinsic that θ, as it depends very mildly on the sequence of compatible CM points. The reader should compare this with [BD05, Remark 1, p.12].

Once we fix an embedding of $\mathbb{Z}_p$ into $\mathbb{C}$, the pairing $\langle -, - \rangle$ induces

$$\langle -, - \rangle : \mathcal{Z}_p \otimes \mathcal{Z}_p \to \mathbb{C},$$

and we let $\mathscr{L}_\mathbb{C} \in \mathbb{C}[[\widetilde{G}_\infty]]$ be the image of $\mathscr{L}$ under such pairing.

Proposition 3.5. *Let χ be a good character such that $\widetilde{\chi}$ has conductor cp^n. Then*

$$\chi(\mathscr{L}_\mathbb{C}) = \sum_{\tau_1,\tau_2\in\widetilde{G}_n} \psi(\tau_1\tau_2)\langle \zeta_n^{\tau_1}, \zeta_n^{\tau_2}\rangle \chi(\tau_1\tau_2^{-1}).$$

Proof. By definition, $\chi(\mathscr{L}_\mathbb{C}) = \chi(\langle \theta_n, \theta_n^*\rangle)$. The result follows immediately replacing θ_n and θ_n^* by their definitions (2) and (1). □

We are now ready to prove the main result of this article.

Theorem 3.6 (Interpolation)**.** *There exists a constant Ω'_E that depends on E such that for every good character χ for which $\widetilde{\chi}$ has conductor cp^n, the following holds:*

$$\chi(\mathscr{L}_\mathbb{C}) = \frac{p^n}{\alpha^{2n}} \cdot \frac{L^\epsilon(1, E, \chi)}{\Omega'_E} \cdot \frac{u_n^2\sqrt{D}c}{2^{-\#\Sigma_D}}.$$

Proof. At the level of modular forms we have that $\pi_g \otimes \psi^2 = \pi_{\overline{g}}$. This induces the same relation under the Jacquet-Langlands transfer and we obtain that $g_B \otimes \psi^2 = \overline{g}_B$. Using the definition of $P_{\widetilde{\chi}}$ and the fact that $\widetilde{\chi} = \psi\chi$, we can write the last factor of the main formula of Theorem 2.5 as

$$\left\langle P_{\widetilde{\chi}}(\phi_n), P_{\widetilde{\chi}^{-1}}(\overline{\phi_n})\right\rangle = \sum_{\tau_1,\tau_2\in\widetilde{G}_n} \psi(\tau_1\tau_2)\chi(\tau_1\tau_2^{-1})\left\langle g_B(\mathfrak{z}_n^{\tau_1}), g_B(\mathfrak{z}_n^{\tau_2})\right\rangle.$$

Since $g_B(\mathfrak{z}_n^{\tau_1})\alpha^{-n} = \zeta_n^{\tau_1}$ and $L^{(p)}(1, \pi_{g_B}, \widetilde{\chi}) = L(1, E, \chi)$ we obtain the desired result using Proposition 3.5 and the fact that $\langle \phi_n, \overline{\phi_n} \rangle_{\delta_n^{-1}\mathscr{U}\delta_n}$ does not depend on n. $\square$

4. The good reduction twist case

In the case when π_g has good reduction at p, i.e. E is a quadratic twist of a curve with good reduction at p, the previous construction and results hold with some minor modifications. We focus in the case when the twisted curve is ordinary at p, to follow the classical construction. In the supersingular case, the same approach works (with the additional assumption that p splits in K), but instead of following the classical construction, one follows the one done by Pollack in [Pol03].

The choice of level $\mathscr{R} = \mathscr{U}$ is the same, but it will be maximal at p. Moreover, we can change the embedding ι in such a way that the CM point $P = \mathfrak{z}_0 = [z_0, 1]$ is of conductor c and $\mathfrak{z}_n := \delta_n \cdot P$ are of conductor cp^n in a similar way as we did in Lemma 1.7. The distribution relations are the following (see for example [BD96, p.433]):

- If $n \geq 1$, $\sum_{\sigma \in \mathrm{Gal}(H_{n+1}/H_n)} \mathfrak{z}_{n+1}^{\sigma} = U_p \mathfrak{z}_n - \mathfrak{z}_{n-1}$.
- If $n = 0$,

$$u_0 \cdot \sum_{\sigma \in \mathrm{Gal}(H_1/H_0)} \mathfrak{z}_1^{\sigma} = \begin{cases} (U_p - \sigma_{\mathfrak{p}_1} - \sigma_{\mathfrak{p}_2})\mathfrak{z}_0 & \text{if } p \text{ is split in } K, \\ (U_p - \sigma_{\mathfrak{p}_1})\mathfrak{z}_0 & \text{if } p \text{ is ramified in } K \\ U_p \mathfrak{z}_0 & \text{if } p \text{ is inert in } K, \end{cases}$$

  where $\sigma_{\mathfrak{p}_i}$ are the Frobenii of the primes above p in K.

If α denotes the p-adic unit root of the Frobenius polynomial at p, the normalized CM points are defined by

$$\zeta_n{}^{\sigma} := \begin{cases} (\alpha g_B(\mathfrak{z}_n^{\sigma}) - g_B(\mathfrak{z}_{n-1}^{\sigma})) \cdot \alpha^{-n-1} & \text{if } n \geq 1, \\ u_0^{-1}(1 - (\sigma_{\mathfrak{p}_1} + \sigma_{\mathfrak{p}_2})\alpha^{-1} + \alpha^{-2}) g_B(\mathfrak{z}_0^{\sigma}) & \text{if } n = 0 \text{ and } p \text{ splits in } K, \\ u_0^{-1}(1 - \sigma_{\mathfrak{p}_1}\alpha^{-1}) g_B(\mathfrak{z}_0^{\sigma}) & \text{if } n = 0 \text{ and } p \text{ is ramified in } K, \\ u_0^{-1}(1 - \alpha^{-2}) g_B(\mathfrak{z}_0^{\sigma}) & \text{if } n = 0 \text{ and } p \text{ is inert in } K. \end{cases}$$

The definition of the theta element and the p-adic L-function is the same. If we take χ such that $\widetilde{\chi}$ has conductor cp^n (with $n \in \mathbb{N} \cup \{0\}$), we can evaluate $\chi(\mathscr{L}_{\mathbb{C}})$ as we did before.

Theorem 4.1. *There exists a constant Ω'_E that depends on E such that for every character χ such that $\widetilde{\chi}$ has conductor cp^n the following holds:*

$$\chi(\mathscr{L}_{\mathbb{C}}) = \frac{p^n}{\alpha^{2n}} \cdot \frac{e_p(\widetilde{\chi})^2}{L_p(\pi_g, \widetilde{\chi}, 1)} \cdot \frac{L^{\epsilon}(1, E, \chi)}{\Omega'_E} \cdot \frac{u_n^2 c\sqrt{D}}{2^{-\#\Sigma_D}},$$

where the p-adic multiplier is given by

$$e_p(\widetilde{\chi}) := \begin{cases} 1 & \textit{if } n \geq 1. \\ (1 - \widetilde{\chi}(\sigma_{\mathfrak{p}_1})\alpha^{-1})(1 - \widetilde{\chi}(\sigma_{\mathfrak{p}_2})\alpha^{-1}) & \textit{if } n = 0 \textit{ and } p \textit{ splits in } K. \\ (1 - \widetilde{\chi}(\sigma_{\mathfrak{p}_1})\alpha^{-1}) & \textit{if } n = 0 \textit{ and } p \textit{ is ramified in } K. \\ (1 - \alpha^{-2}) & \textit{if } n = 0 \textit{ and } p \textit{ is inert in } K. \end{cases}.$$

Proof. When $n \geq 1$, if we expand the four terms in the pairing $\langle -, - \rangle$ we obtain that the only term that survives is the same as we had in the general case. The reason is that all the other terms involve a sum of the form $\sum_\sigma \widetilde{\chi}(\sigma) g_B(\mathfrak{z}_{n-1}^\sigma)$ which equals zero as the conductors of the test vector and the character do not agree. Finally, both Waldspurger and Gross-Zagier formulas (Theorem 2.5) for central values of the elliptic curve E_{π_g} (or its derivative) need to include the p-th Euler factor at p (which is trivial if $n \geq 1$ and non-vanishing in general) which gives the extra local factor to the formula. $\square$

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