ON HEEGNER POINTS FOR PRIMES OF ADDITIVE REDUCTION RAMIFYING IN THE BASE FIELD

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with an Appendix by Marc Masdeu

Abstract. Let E be a rational elliptic curve, and K be an imaginary quadratic field. In this article we give a method to construct Heegner points when E has a prime bigger than 3 of additive reduction ramifying in the field K. The ideas apply to more general contexts, like constructing Darmon points attached to real quadratic fields which is presented in the appendix.

Introduction

Heegner points play a crucial role in our nowadays understanding of the Birch and Swinnerton-Dyer conjecture, and are the only instances where non-torsion points can be constructed in a systematic way for elliptic curves over totally real fields (assuming some still unproven modularity hypotheses). Although Heegner points were heavily studied for many years, most applications work under the so called "Heegner hypothesis" which gives a sufficient condition for an explicit construction to hold. In general, if E is an elliptic curve over a number field F and K/F is any quadratic extension, the following should be true.

Conjecture: If $\operatorname{sign}(E,K) = -1$, then there is a non-trivial Heegner system attached to (E,K).

This is stated as Conjecture 3.16 in [Dar04]. When $F=\mathbb{Q}$, E is an elliptic curve of square-free conductor N and K is an imaginary quadratic field whose discriminant is prime to N, this is true and explained in Darmon's book ([Dar04]) using both the modular curve $X_0(N)$ and other Shimura curves. In [Zha01] it is explained how to relax the hypothesis to: if p divides N and it ramifies in K, then $p^2 \nmid N$.

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When the curve is not semistable at some prime p the situation is quite more delicate. An interesting phenomenon is that in this situation, the local root number at p has no relation with the factorization of p in K. Still the problem has a positive answer in full generality, due to the recent results of [YZZ13a], where instead of working with the classical group $\Gamma_0(N)$, they deal with more general arithmetic groups. The purpose of this article is to give "explicit" constructions of Heegner points for pairs (E, K) as above. Here by explicit we mean that we can compute numerically the theoretical points in the corresponding ring class field, which restricts us to working only with unramified quaternion algebras (since the modular parametrization is hard to compute for Shimura curves). For computational simplicity we will also restrict the base field to the field of rational numbers.

Let $\chi : K^\times \backslash K_{\mathbb{A}}^\times \to \mathbb{C}^\times$ be a finite order Hecke character, and η be the character corresponding to the quadratic extension $K/\mathbb{Q}$. In order to construct a Heegner point attached to χ in a matrix algebra, for each prime number p the following condition must hold

$$\epsilon(\pi_p, \chi_p) = \chi_p(-1)\eta_p(-1),$$

where π is the automorphic representation attached to E, and $\epsilon(\pi_p, \chi_p)$ is the local root number of $L(s, \pi, \chi)$ (see [YZZ13a, Section 1.3.2]). If we impose the extra condition $\gcd(\operatorname{cond}(\chi), N \operatorname{cond}(\eta)) = 1$, then at primes dividing the conductor of E/K the equation becomes

$$\varepsilon_p(E/K) = \eta_p(-1),$$

where $\varepsilon_p(E/K)$ is the local root number at p of the base change of E to K (it is equal to $\varepsilon_p(E)\varepsilon_p(E \otimes \eta)$). This root number is easy to compute if $p \neq 2, 3$ (see [Pac13]):

- If p is unramified in K, then $\eta_p(-1) = 1$ and

$$\varepsilon_p(E/K) = \begin{cases} 1 & \text{if } v_p(N) = 0, \\ \left(\frac{p}{\operatorname{disc}(K)}\right) & \text{if } v_p(N) = 1, \\ 1 & \text{if } v_p(N) = 2, \end{cases}$$

  where $v_p(N)$ denotes the valuation of N at p.

- If p is ramified in K then $\eta_p(-1) = \left(\frac{-1}{p}\right)$ and

$$\varepsilon_p(E/K) = \left(\frac{-1}{p}\right) \cdot \begin{cases} 1 & \text{if } v_p(N) = 0, \\ \varepsilon_p(E) & \text{if } v_p(N) = 1, \\ \varepsilon_p(E_p) & \text{if } v_p(N_{E_p}) = 1, \\ 1 & \text{if } E \text{ is P.S.}, \\ -1 & \text{if } E \text{ is S.C.}, \end{cases}$$

where E_p denotes the quadratic twist of E by the unique quadratic extension of $\mathbb{Q}$ unramified outside p; E is P.S. if the attached automorphic representation is a ramified principal series (which is equivalent to the condition that E acquires good reduction over an abelian extension of $\mathbb{Q}_p$) and E is S.C. if the attached automorphic representation is supercuspidal at p (which is equivalent to the condition that E acquires good reduction over a non-abelian extension).

Let $E/\mathbb{Q}$ be an elliptic curve. We call it Steinberg at a prime p if E has multiplicative reduction at p (and denote it by St.). In Table 1 we summarize the above equations for $p \neq 2, 3$, where the sign corresponds to the product $\varepsilon_p(E/K)\eta_p(-1)$.

	p is inert	p splits	p ramifies
St	-1	1	$\varepsilon_p(E)$
St $\otimes\chi_p$	1	1	$\varepsilon_p(E_p)$
P.S.	1	1	1
Sc.	1	1	-1

Table 1. Signs Table

Our goal is to give an explicit construction in all cases where the local sign of Table 1 equals $+1$. The cells colored in light grey correspond to the classical construction, and the ones colored with dark grey are considered in the article [KP15]. In the present article we will consider the following cases:

- E has additive but potentially multiplicative reduction, and $\varepsilon_p(E_p) = +1$.
- E has additive but potentially good reduction over an abelian extension.

Remark. The situation for $p = 2$ and $p = 3$ is more delicate, although most cases can be solved with the same ideas. For the rest of this article we assume $p > 3$.

The strategy is construct from E/K an abelian variety (in general of dimension greater than 1) and use a classical Heegner construction on such variety so that we can transfer the Heegner points back to our original elliptic curve. To clarify the exposition, we start assuming that there is only one prime p ramifying in K where our curve has additive reduction, and every other prime q dividing N is split in K. The geometric object we consider is the following:

- If E has potentially multiplicative reduction, we consider the elliptic curve E_p of conductor N/p which is the quadratic twist of E by the unique quadratic character ramified only at p.
- If E has potentially good reduction over an abelian extension, then we consider an abelian surface of conductor N/p, which is attached to a pair $(g, \bar{g})$, where g is the newform of level N/p corresponding to a twist of the weight 2 modular form E_f attached to E.

In both cases the classical Heegner hypothesis is satisfied (eventually for dimension greater than one), and the resulting abelian varieties are isogenous to our curve or to a product of the curve with itself over some field extension. Such isogeny is the key to relate the classical construction to the new cases considered. Each case has a different construction/proof (so they will be treated separately), but both follow the same idea. In all cases considered we will construct points on $(E(H_c) \otimes \mathbb{C})^\chi$. These points will be non-torsion if an only if $L'(E/K, \chi, 1) \neq 0$ as expected by the results of Gross-Zagier [GZ86] and Zhang [Zha01].

Our construction has interests on its own, and can be used to move from a delicate situation to a not so bad one (reducing the conductor of the curve at the cost of adding a character in some cases). So, despite we focus on classical modular curves, the methods of this article can be easily applied to a wide variety of contexts, for example more general Shimura curves.

In recent years, following a breakthrough idea of Darmon there has been a lot of work in the direction of defining and computing p-adic Darmon points, which are points defined over certain ring class fields of real quadratic extensions using p-adic methods. For references to this circle of ideas the reader can consult [Dar04], [Dar01], [BD09], [BD07]. These construction are mostly conjectural (but see [BD09]), and there has been a lot of effort to explicitly compute p-adic approximations to these points in order to gather numerical evidence supporting these conjectures. The interested reader might consult [DP06], [Gre09], [GM15], [GMŞ15], [GMŞ16].

In order to illustrate the decoupling of our techniques from the algebraic origin of the points, in an appendix by Marc Masdeu it is shown how these can be applied to the computation of p-adic Darmon points.

The article is organized as follows: in the first section we treat the case of a curve having potentially multiplicative reduction, and prove the main result in such case. In the second section we prove our main result in the case that we have potentially good reduction over an abelian extension. In the third section, we explain how to extend the result to general conductors and in the fourth section we finish the article with some explicit examples in the modular curves setting, including Cartan non-split curves, as in [KP15]. Lastly, we include the aforementioned appendix.

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1. The potentially multiplicative case

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $p^2 \cdot m$ where p is an odd prime and $gcd(p, m) = 1$. Suppose that E has potentially multiplicative reduction at the prime p. Let K be any imaginary quadratic field satisfying the Heegner hypothesis at all the primes dividing m and such that p is ramified in K. Let $p^* = \left(\frac{-1}{p}\right) p$ and let E_p be the quadratic twist of E by $\mathbb{Q}(\sqrt{p^*})$. We have an isomorphism $\phi : E_p \to E$ defined over $\mathbb{Q}(\sqrt{p^*})$. The elliptic curve E_p has conductor $p \cdot m$ and $\operatorname{sign}(E, K) = \operatorname{sign}(E_p, K)\varepsilon_p(E_p)$.

Recall that to have explicit constructions, we need to work with a matrix algebra so we impose the condition $\varepsilon_p(E_p) = 1$ (see Table 1). Then, $\operatorname{sign}(E, K) = \operatorname{sign}(E_p, K) = -1$ and the pair (E_p, K) satisfies the Heegner condition. Therefore, we can find Heegner points on E_p and map them to E with ϕ. More precisely, let c be a positive integer relatively prime to $N \cdot \operatorname{disc}(K)$ and let H_c be the ring class field associated to the order of conductor c in the ring of integers of K. Let $\chi : \operatorname{Gal}(H_c/K) \to \mathbb{C}^\times$ be any character and let χ_p be the quadratic character associated to $\mathbb{Q}(\sqrt{p^*})$ via class field theory. Take a Heegner point $P_c \in E_p(H_c)$ and consider the point

$$P_c^{\chi\chi_p} = \sum_{\sigma \in \operatorname{Gal}(H_c/K)} \bar{\chi}\bar{\chi}_p(\sigma) P_c^\sigma \in (E_p(H_c) \otimes \mathbb{C})^{\chi\chi_p}.$$

Theorem 1.1. *The point $\phi(P_c^{\chi\chi_p})$ belongs to $(E(H_c) \otimes \mathbb{C})^\chi$ and it is non-torsion if and only if $L'(E/K, \chi, 1) \neq 0$.*

Proof. The key point is that since $p \mid \operatorname{disc}(K)$, $\mathbb{Q}(\sqrt{p^*}) \subset H_c$ (by genus theory). For $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we have $\phi^\sigma = \chi_p(\sigma)\phi$, hence,

$$\phi(P_c^{\chi\chi_p}) = \sum_\sigma \bar{\chi}(\sigma)\phi(P_c)^\sigma \in (E(H_c) \otimes \mathbb{C})^\chi.$$

Finally note that by the Theorems of Gross-Zagier [GZ86] and Zhang [Zha01] the point $P_c^{\chi\chi_p}$ is non-torsion if and only if $L'(E_p/K, \chi\chi_p, 1) = L'(E/K, \chi, 1) \neq 0$. Since ϕ is an isomorphism the result follows.

□

2. The potentially good case (over an abelian extension)

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $p^2 \cdot m$ where p is an odd prime and $gcd(p, m) = 1$. For simplicity assume that E does not have complex multiplication. We recall some generalities on elliptic curves with additive but potentially good reduction over an abelian extension. Although such results can be stated and explained using the theory of elliptic curves, we believe that a representation theoretical approach is more general and clear. Let f_E denote the weight 2 newform corresponding to E.

Let $W(\mathbb{Q}_p)$ be the Weil group of $\mathbb{Q}_p$, and ω_1 be the unramified quasi-character giving the action of $W(\mathbb{Q}_p)$ on the roots of unity. Using the normalization given by Carayol ([Car86]), at the prime p the Weil-Deligne representation corresponds to a principal series representation on the automorphic side and to a representation

$$\rho_p(f) = \psi \oplus \psi^{-1}\omega_1^{-1},$$

on the Galois side for some quasi-character $\psi : W(\mathbb{Q}_p)^{ab} \to \mathbb{C}^\times$. Note that since the trace lies in $\mathbb{Q}$, ψ satisfies a quadratic relation, hence its image lies in a quadratic field contained in a cyclotomic extension (since ψ has finite order). This gives the following possibilities for the order of inertia of ψ: 1, 2, 3, 4 or 6.

- Clearly ψ cannot have order 1 (since otherwise the representation is unramified at p).
- If ψ has order 2, ψ must be the (unique) quadratic character ramified at p. Then E is the twist of an unramified principal series, i.e., E_p has good reduction at p.
- If ψ has order 3, 4 or 6, there exists a newform $g \in S_2(\Gamma_0(p \cdot m), \varepsilon)$, where $\varepsilon = \psi^{-2}$, such that $f_E = g \otimes \psi$. In particular ε has always order 2 or 3.

In the last case, the form has inner twists, since the Fourier coefficients satisfy that $\overline{a_p} = a_p \varepsilon^{-1}(p)$ (see for example [Rib77, Proposition 3.2]).

Remark 2.1. The newform g can be taken to be the same for E and E_p.

2.1. The case ψ has order 2. This case is very similar to the one treated in the previous section. The curve E_p has good reduction at p, and is isomorphic via ϕ to E. It is quite easy to see that under these conditions $\text{sign}(E, K) = \text{sign}(E_p, K) = -1$. Exactly as before we can construct Heegner points on E_p and transfer them to E.

2.2. The case ψ has order 3, 4 or 6. Let d be the order of ψ. Let $g \in S_2(\Gamma_0(p \cdot m), \varepsilon)$ as before. Suppose its q-expansion at the infinity cusp is given by $g = \sum a_n q^n$. Following [Rib04], we define the coefficient field $K_g := \mathbb{Q}(\{a_n\})$.

Remark 2.2. K_g is an imaginary quadratic field generated by the values of ψ. It is equals to $\mathbb{Q}(i)$ if $d=4$ and to $\mathbb{Q}(\sqrt{-3})$ if $d=3$ or $d=6$.

There is an abelian variety A_g defined over $\mathbb{Q}$ attached to g via the Eichler-Shimura construction, with an action of K_g on it, i.e. there is an embedding $\theta : K_g \hookrightarrow (\mathrm{End}_{\mathbb{Q}}(A_g)\otimes\mathbb{Q})$. The variety A_g can be defined as the quotient $J_1(p\cdot m)/I_g J_1(p\cdot m)$ where I_g is the annihilator of g under the Hecke algebra acting on the Jacobian. Moreover, the L-series of A_g satisfies the relation

$$L(A_g/\mathbb{Q},s)=L(g,s)L(\overline{g},s).$$

The variety A_g has dimension $[K_g:\mathbb{Q}]=2$ and is $\mathbb{Q}$-simple. However, it is not absolutely simple. The variety A_g is isogenous over $\overline{\mathbb{Q}}$ to a power of an absolutely simple abelian variety B_g (called a building block for A_g, see [GL01] for the general theory).

Under our hypotheses we have an explicit description. Let $L=\overline{\mathbb{Q}}^{\ker(\varepsilon)}$ (which is the splitting field of A_g). It is a cubic extension if $d=3,6$ (and in particular $p\equiv 1 \pmod 3$) and the quadratic extension $\mathbb{Q}(\sqrt{p})$ if $d=4$ (which implies $p\equiv 1 \pmod 4$). Let M be the extension $\overline{\mathbb{Q}}^{\ker(\psi)}$.

Proposition 2.3.
- *There exists an elliptic curve $\tilde{E}/L$ and an isogeny, defined over L, $\omega : A_g \to \tilde{E}^2$. Furthermore, if $d=3$ (resp. $d=6$) $\tilde{E}=E$ (resp. $\tilde{E}=E_p$) while if $d=4$, $\tilde{E}$ is the quadratic twist of $E/\mathbb{Q}(\sqrt{p})$ by the unique quadratic extension unramified outside p.*
- *In any case, there exists an isogeny $\varphi : A_g \to E^2$ defined over M.*

Proof. $A_g \simeq E^2$ over M because (on the representation side) the twist becomes trivial while restricted to M, so the L-series of A_g becomes the square of that of E (over such field) and by Falting's isogeny Theorem there exists an isogeny (defined over M). If $d=3$, $\varepsilon=\psi^2$ and $M=L$, while if $d=6$, starting with E_p (whose character has order 3) gives the result. If $d=4$, it is clear (on the representation side) that $L(A_g,s)=L(\tilde{E},s)$ over L, where $\tilde{E}$ is the twist of E (while looked over $\bar{\mathbb{Q}}^{\ker(\varepsilon)}=\mathbb{Q}(\sqrt{p})$) by the quadratic character ψ^2. Then Falting's isogeny Theorem proves the claim. $\square$

Proposition 2.4. *Let $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then $\varphi^\sigma : A_g \to E^2$ is equal to $\varphi\kappa(\sigma|_M)$, where κ is some character of* $\mathrm{Gal}(M/\mathbb{Q})$ *of order* $[M:\mathbb{Q}]$.

Proof. Since φ and φ^σ are isogenies of the same degree there exists an element $a_\sigma\in \mathrm{End}(A_g)\otimes\mathbb{Q}=K_g$ of norm 1 such that $\varphi^\sigma=\varphi a_\sigma$. The map $\kappa(-|_M):\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to K_g^\times$, given by sending $\kappa(\sigma|_M)\mapsto a_\sigma$ is a character, since the endomorphism a_σ is defined over $\mathbb{Q}$. Clearly κ has the predicted order since otherwise the isogeny φ

could be defined over a smaller extension (given by the fixed field of its kernel), which is not possible. □

In order to explicitly compute Heegner points it is crucial to have a better understanding of the isogenies ω and φ. Let us recall some basic properties of Atkin-Li operators for modular forms with nebentypus, as explained in [AL78]. Let N be a positive integer, and let $P \mid N$ be such that $\gcd(P, N/P) = 1$. Let $N' = \frac{N}{P}$ and decompose $\varepsilon = \varepsilon_P \varepsilon_{N'}$, where each character is supported in the set of primes dividing the sub-index.

Theorem 2.5. *Assuming the previous hypotheses, there exists an operator* $W_P : S_2(\Gamma_0(N), \varepsilon) \to S_2(\Gamma_0(N), \overline{\varepsilon_P}\varepsilon_{N'})$ *which satisfies the following properties:*

- $W_P^2 = \varepsilon_P(-1)\overline{\varepsilon_{N'}}(P)$.
- *If* g *is an eigenvector for* T_q *for some prime* $q \nmid N$ *with eigenvalue* a_q*, then* $W_p(g)$ *is an eigenvector for* T_q *with eigenvalue* $\overline{\varepsilon_P}(q)a_q$.
- *If* $g \in S_2(\Gamma_0(N), \varepsilon)$ *is a newform, then there exists another newform* $h \in S_2(\Gamma_0(N), \overline{\varepsilon_P}\varepsilon_N)$ *and a constant* $\lambda_P(g)$ *such that* $W_P(g) = \lambda_P(g)h$.
- *The number* $\lambda_P(g)$ *is an algebraic number of absolute value* 1*. Furthermore, if* a_P*, the* P*-th Fourier coefficient of the newform* g*, is non-zero then*
$$\lambda_P(g) = G(\varepsilon_P)/a_P,$$
where $G(\chi)$ *denotes the Gauss sum of the character* χ.

The number $\lambda_P(g)$ is called the pseudo-eigenvalue of W_P at g.

Proof. See [AL78, Propositions 1.1, 1.2, and Theorems 1.1 and 2.1]. □

In our setting $N = p \cdot m$, $P = p$, $\varepsilon_{N'}$ is trivial, and W_p is an involution (i.e. $W_p^2 = 1$) acting on the differential forms of A_g.

If η is an endomorphism of $J(\Gamma_1(N))$ (or one of its quotients), we denote η^* the pullback it induces on the differential forms. Given an integer u let α_u be the endomorphism of $J(\Gamma_1(N))$ corresponding to the action of the matrix $\left(\begin{smallmatrix} 1 & u/p \\ 0 & 1 \end{smallmatrix}\right)$ on differential forms. Such endomorphism is defined over the cyclotomic field of p-th roots of unity.

Let $\tau \in \operatorname{Gal}(K_g/\mathbb{Q})$ denote complex conjugation. Recall that $\tau a_q = a_q\varepsilon^{-1}(q)$ for all positive integers q prime to $p \cdot m$. Following [Rib80] we define
$$\eta_\tau = \sum_{u \pmod p} \varepsilon(u)\alpha_u.$$
Since $\varepsilon(u) \in \mathcal{O}_{K_g}$, via the map θ we think of η_τ as an element in $\operatorname{End}_L(A_g)$. To normalize η_τ we follow [GL01]. Let $a_p \in K_g$ be the p-th Fourier coefficient of g.

Lemma 2.6. *The element* a_p *has norm* p.

Proof. Looking at the curve E over $\mathbb{Q}_p$, the coefficient a_p is one of the roots of the characteristic polynomial attached to the Frobenius element in the minimal (totally ramified) extension where E acquires good reduction (see for example Section 3 of [DD11]). Since the norm of the local uniformizer in such extension is p (because the extension ramifies completely) the result follows. $\square$

We then consider the normalized endomorphism $\frac{\eta_\tau}{a_p}$.

Remark 2.7. Our choice is a particular case of the one considered in [GL01], since our normalization corresponds to the splitting map $\beta : \mathrm{Gal}(K_g/\mathbb{Q}) \to K_g^\times$ given by $\beta(\tau) = a_p$.

Theorem 2.8. *The operator W_p coincides with $\left(\frac{\eta_\tau}{a_p}\right)^*$.*

Proof. It is enough to see how it acts on the basis $\{g, \overline{g}\}$ of differential forms of A_g. By Theorem 2.5 (since a_p is non-zero), $W_p(g) = \lambda_p \overline{g}$, where $\lambda_p = G(\varepsilon)/a_p$. On the other hand, $\eta_\tau(f) = G(\varepsilon)\overline{f}$, by [GL01, Lemma 2.1]. Exactly the same argument applies to $\overline{g}$, using the fact that $\overline{G(\varepsilon)} = G(\overline{\varepsilon})$, since ε is an even character. $\square$

Corollary 2.9. *The Atkin-Li operator W_p is defined over L, i.e. it corresponds to an element in $\mathrm{End}_L(A_g)\otimes\mathbb{Q}$. Its action decomposes as a direct sum of two 1-dimensional spaces.*

Let
$$\omega : A_g \to (W_p + 1)A_g \times (W_p - 1)A_g.$$
Then both terms are 1-dimensional, and the isogeny ω gives a splitting as in Proposition 2.3.

Remark 2.10. The explicit map ω satisfies the first statement of Proposition 2.3. In order to get the second statement we need to eventually compose it the isomorphism between $\tilde{E}$ and E. Recall that $E = \tilde{E}$ if $d = 3$ and $\tilde{E}$ is a quadratic twist of it otherwise, so in any case the isomorphism is easily computed.

2.3. **Heegner points.** This section follows Section 4 of [DRZ12], so we suggest the reader to look at it first. Keeping the notations of the previous sections, let $\varepsilon : (\mathbb{Z}/p)^\times \to \mathbb{C}^\times$ be a Dirichlet character. Extend the character to $(\mathbb{Z}/p\cdot m)^\times$ by composing with the canonical projection $(\mathbb{Z}/p\cdot m)^\times \to (\mathbb{Z}/p)^\times$ and define
$$\Gamma_0^\varepsilon(p\cdot m) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p\cdot m) : \varepsilon(a) = 1 \right\}.$$
Let $X_0^\varepsilon(p\cdot m)$ be the modular curve obtained as the quotient of the extended upper half plane $\mathcal{H}^*$ by this group. This modular curve has a model defined over $\mathbb{Q}$ and it coarsely represents the moduli problem of parameterizing quadruples $(E, Q, C, [s])$ where

- E is an elliptic curve over $\mathbb{C}$,
- Q is a point of order m on $E(\mathbb{C})$,
- C is a cyclic subgroup of $E(\mathbb{C})$ of order p,
- $[s]$ is an orbit in $C \setminus \{0\}$ under the action of $\ker(\varepsilon) \subset (\mathbb{Z}/p)^\times$.

Remark 2.11. There is a canonical map $\Phi : X_0^\varepsilon(p \cdot m) \to X_0(p \cdot m)$ which is the forgetful map in the moduli interpretation. This map has degree $\operatorname{ord}(\varepsilon)$.

As in the classical case, there exists a modular parametrization

$$\begin{tikzcd}
X_0^\varepsilon(p \cdot m) \arrow[r, "\Psi"] \arrow[d, "\Phi"'] \arrow[rr, bend right=10, "\Psi_g"'] & \operatorname{Jac}(X_0^\varepsilon(p \cdot m)) \arrow[r, "\pi"] & A_g \\
X_0(p \cdot m) \arrow[urr, dotted] & &
\end{tikzcd}$$

where $\Psi(P) = (P) - (\infty)$ (the usual immersion of the curve in its Jacobian) and π is the Eichler-Shimura projection onto A_g. These maps are defined over $\mathbb{Q}$, as the cusp ∞ is rational. Our strategy is to construct Heegner points on $X_0^\varepsilon(p \cdot m)$ and push them through the modular parametrization Ψ_g to the abelian variety A_g and finally project them onto the elliptic curve E. To construct points on $X_0^\varepsilon(p \cdot m)$, we consider the canonical map

$$\Phi : X_0^\varepsilon(p \cdot m) \to X_0(p \cdot m),$$

and look at preimages of classical Heegner points on $X_0(p \cdot m)$.

Since the conductor $p \cdot m$ satisfies the classical Heegner hypothesis with respect to K there is a cyclic ideal $\mathfrak{n}$ of norm $p \cdot m$. Let c be a positive integer such that $gcd(c, p \cdot m) = 1$. Then, a classical Heegner point on $X_0(p \cdot m)$ corresponds to a triple $P_{\mathfrak{a}} = (\mathscr{O}_c, \mathfrak{n}, [\mathfrak{a}]) \in X_0(p \cdot m)(H_c)$, where $[\mathfrak{a}] \in \operatorname{Pic}(\mathscr{O}_c)$. Such point is represented by the elliptic curve $E_{\mathfrak{a}} = \mathbb{C}/\mathfrak{a}$ and its $\mathfrak{n}$ torsion points $E_{\mathfrak{a}}[\mathfrak{n}]$ (which are isomorphic to $(\mathfrak{a}\mathfrak{n}^{-1}/\mathfrak{a})$) are defined over H_c.

The action of $\operatorname{Gal}(\overline{\mathbb{Q}}/H_c)$ on $E_{\mathfrak{a}}[\mathfrak{n}]$ gives a map $\operatorname{Gal}(\overline{\mathbb{Q}}/H_c) \to (\mathfrak{a}\mathfrak{n}^{-1}/\mathfrak{a})^\times$. Composing such map with the character ε gives

$$\rho : \operatorname{Gal}(\overline{\mathbb{Q}}/H_c) \to (\mathfrak{a}\mathfrak{n}^{-1}/\mathfrak{a})^\times \xrightarrow{\varepsilon} \mathbb{C}^\times.$$

Its kernel corresponds to an extension $\tilde{H}_c$ of degree $\operatorname{ord}(\varepsilon)$ of H_c. Let $\tilde{H}_c = H_c M$.

Proposition 2.12. *The* $\operatorname{ord}(\varepsilon)$ *points* $\Phi^{-1}(P_{\mathfrak{a}})$ *lie in* $X_0^\varepsilon(p \cdot m)(\tilde{H}_c)$ *and are permuted under the action of* $\operatorname{Gal}(\tilde{H}_c/H_c)$.

Proof. By complex multiplication $\tilde{H}_c$ lies in the composition of H_c and the ray class field $K_{\mathfrak{p}}$, where $\mathfrak{p}$ is the unique prime of K dividing p. The composition $H_c K_{\mathfrak{p}}$ equals $H_c(\xi_p)$, where ξ_p is a p-th root of unity. Note that $\mathbb{Q}(\sqrt{p^*}) \subset H_c$ and the extension

H_c/K is unramified at p. Therefore, the unique extension of degree $ord(\varepsilon)$ of H_c lying inside $H(\xi_p)$ is given by $H_c\bar{\mathbb{Q}}^{ker(\psi)} = H_cM$. $\square$

Using the aforementioned moduli interpretation, points on $X_0^{\varepsilon}(p \cdot m)$ represent quadruples $(\mathscr{O}_c, \mathfrak{n}, [\mathfrak{a}], [t])$ where $[t]$ is an orbit under $\ker(\varepsilon)$ inside $(\mathscr{O}_c/(\mathfrak{n}/\mathfrak{p}))^{\times}$.

Remark 2.13. Let $\sigma \in \mathrm{Gal}(\tilde{H}_c/K)$. Its action on Heegner points is given by

$$\sigma \cdot (\mathscr{O}_c, \mathfrak{n}, [\mathfrak{a}], [t]) = (\mathscr{O}_c, \mathfrak{n}, [\mathfrak{a}\mathfrak{b}^{-1}], [dt]),$$

where $\sigma |_{H_c} = \mathrm{Frob}_{\mathfrak{b}}$, and $d = \rho(\sigma) \in \mathscr{O}_c/(\mathfrak{n}/\mathfrak{p})^{\times}$.

2.4. **Zhang's formula.**

Theorem 2.14 (Tian-Yuan-Zhang-Zhang)**.** *Let K be an imaginary quadratic field satisfying the Heegner hypothesis for $p \cdot m$ and let $\tilde{\chi} : \mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ be a finite order Hecke character such that $\tilde{\chi} |_{\mathbb{A}_{\mathbb{Q}}^{\times}} = \varepsilon^{-1}$. Then $L(g, \tilde{\chi}, s)$ vanishes at odd order at $s = 1$. Moreover, if such order equals 1, $(A_g(\tilde{H}_c)) \otimes \mathbb{C}))^{\tilde{\chi}}$ has rank one over $K_g \otimes \mathbb{C}$.*

More precisely, consider the Heegner point $([\mathfrak{a}], \mathfrak{n}, 1) \in X_0^{\varepsilon}(p \cdot m)(\tilde{H}_c)$ and denote by P_c its image under the modular parametrization Ψ_g. Then

$$P^{\tilde{\chi}} = \sum_{\sigma \in \mathrm{Gal}(\tilde{H}_c/K)} \bar{\tilde{\chi}}(\sigma) P_c^{\sigma} \in (A_g(\tilde{H}_c) \otimes \mathbb{C})^{\tilde{\chi}}$$

generates a rank one subgroup over $K_g \otimes \mathbb{C}$.

Proof. See [TZ03, Theorem 4.3.1], [Zha10], and [YZZ13b, Theorem 1.4.1]. $\square$

Let c be a positive integer relatively prime to $\mathrm{disc}(K) \cdot p \cdot m$, and let χ be a ring class character of $\mathrm{Gal}(H_c/K)$. Since $\bar{\kappa}^2 = \varepsilon^{\pm 1}$, the character $\tilde{\chi} : \mathrm{Gal}(\tilde{H}_c/K) \to \mathbb{C}^{\times}$ given by $\tilde{\chi} = \chi\bar{\kappa}$ satisfies the hypothesis of Theorem 2.14 (for both g or its conjugate $\bar{g}$). Summing up, we get the following theorem:

Theorem 2.15. *The point $\varphi(P^{\chi\bar{\kappa}})$ belongs to $(E^2(H_c \otimes \mathbb{C}))^{\chi}$. In addition, it is non-torsion if and only if $L'(E/K, \chi, 1) \neq 0$.*

Proof. By definition and Proposition 2.4.

$$\varphi(P^{\chi\bar{\kappa}}) = \sum_{\sigma \in \mathrm{Gal}(\tilde{H}_c/K)} \bar{\chi}(\sigma)\varphi(\kappa(\sigma)P^{\sigma}) = \sum_{\sigma \in \mathrm{Gal}(\tilde{H}_c/K)} \bar{\chi}(\sigma)\varphi(P)^{\sigma},$$

so it lies in the right space. Since $\mathrm{ord}(\kappa) = \mathrm{ord}(\psi)$ and $\bar{\kappa}^2 = \varepsilon^{\pm 1}$ we get $\bar{\kappa} = \psi^{\pm 1}$. We know that $g \otimes \psi = \bar{g} \otimes \psi^{-1} = f_E$, therefore, using g or $\bar{g}$, we obtain $L(g, \tilde{\chi}, s) = L(E, \chi, s)$. Theorem 2.14 and the previous result imply that $\varphi(P^{\chi\bar{\kappa}}) \in (E^2(H_c \otimes \mathbb{C}))^{\chi}$ is non-torsion if and only if $L'(E/K, \chi, 1) \neq 0$. $\square$

Once we construct a non-torsion point on $E \times E$ we can project it to some coordinate in order to obtain a non-torsion point on E.

2.5. **Heegner systems.** As in the classical case, the family of Heegner points constructed using different orders satisfy certain compatibilities. We will not state all these relations between points but content ourselves with stating some conditions that are enough for proving a Kolyvagin-type theorem. Assume for simplicity that the character χ is trivial.

Proposition 2.16. *Let ℓ be a prime such that $\ell \nmid N$ and ℓ is inert in K. Then for every Heegner point $P_{c\ell} \in A_g(\tilde{H}_{c\ell})$ there exists a Heegner point $P_c \in A_g(\tilde{H}_c)$ with*

$$\operatorname{Tr}_{\tilde{H}_{c\ell}/\tilde{H}_c} P_{c\ell} = \theta(a_\ell) P_c, \tag{1}$$

where a_ℓ is the ℓ-th Fourier coefficient of g.

Proof. The proof mimics the classical case one (see [Gro91, Proposition 3.7]). □

To construct a point on E, we first apply the isogeny φ to a point in A_g and then project onto one of the coordinates (call π_i the pojection to the i-th corrdinate). But K_g does not act on E! To overcome this problem, we restrict to primes ℓ which split completely in L. Let $Q_c := \pi_i(\operatorname{Tr}_{\tilde{H}_c/H_c} \varphi(P_c)) \in E(H_c)$.

Proposition 2.17. *Let ℓ be a prime such that $\ell \nmid N$, ℓ is inert in K and ℓ splits completely in L. Then for every Heegner point $Q_{c\ell} \in E(H_{c\ell})$ there exists a Heegner point $Q_c \in E(H_c)$ such that*

$$\operatorname{Tr}_{H_{c\ell}/H_c} Q_{c\ell} = a_\ell Q_c.$$

Proof. Applying $\pi_i(\operatorname{Tr}_{\tilde{H}_c/H_c} \varphi)$ to equation (1), since φ commutes with the trace and $a_\ell \in \mathbb{Q}$ (because ℓ splits completely in L) we get

$$\pi_i(\operatorname{Tr}_{\tilde{H}_c/H_c} \operatorname{Tr}_{\tilde{H}_{c\ell}/\tilde{H}_c} \varphi(P_{c\ell})) = a_\ell Q_c.$$

Also

$$\pi_i(\operatorname{Tr}_{\tilde{H}_c/H_c} \operatorname{Tr}_{\tilde{H_{c\ell}}/\tilde{H}_c} \varphi(P_{c\ell})) = \pi_i(\operatorname{Tr}_{H_{c\ell}/H_c} \operatorname{Tr}_{\tilde{H_{c\ell}}/H_{c\ell}} \varphi(P_{c\ell})),$$

but since π_i is defined over $\mathbb{Q}$, this expression equals $\operatorname{Tr}_{H_{c\ell}/H_c} Q_{c\ell}$ as claimed. □

3. General case

While considering the case of many primes ramifying in K, it is clear that the potentially multiplicative case works the same. Some extra difficulties arise in the other cases. To make the exposition/notation easier, we start considering the following two cases:

Case 1: Suppose that the conductor of E equals $p_1^2 \cdots p_r^2 \cdot m$ where:

- E has potentially good reduction at all p_i's over an abelian extension,
- all characters ψ_{p_i} have the same order,
- all p_i's are ramified in K,
- m satisfies the classical Heegner hypothesis.

Let $P = \prod_{i=1}^{r} p_i$. There are 2^r newforms of level $P\cdot m$ which are twists of f (obtained, following the previous section notation, by twisting f_E by all possible combinations of $\{\psi_{p_i}, \overline{\psi_{p_i}}\}$). Working with all of them implies considering an abelian variety of dimension 2^r, but the coefficient field has degree 2 so such variety is not simple over $\mathbb{Q}$.

Instead, take "any" newform $g \in S_2(\Gamma_0(P\cdot m), \varepsilon)$, and consider the abelian surface A_g attached to it by Eichler-Shimura. The only Atkin-Li operator acting on (the space of holomorphic differentials of) such variety is the operator W_P, which again is an involution, so we can split the space in the ± 1 part and proceed as in the previous case considered (where the splitting map is determined by $\beta(\tau) = \prod_{i=1}^{r} a_{p_i}$).

The ambiguity on the choice of g is due to the following: the operators W_{p_i} act transitively on the set of all newforms g. In particular they "permute" the different abelian surfaces (note that such operators are not involutions, but have eigenvalues in the coefficient field K_g which is independent of g). Although surfaces attached to different choices of g are in general not isomorphic (the traces of the Galois representations are different), they become isomorphic over M hence all of them gives the same Heegner points construction.

Case 2: Suppose the conductor of E equals $p^2 \cdot q^2 \cdot m$, where

- E has potentially good reduction at p and q over an abelian extension,
- the order of ψ_p equals 4 and that of ψ_q equals 3,
- both p and q ramify in K,
- m satisfies the classical Heegner hypothesis.

With such assumptions the coefficient field K_g equals $\mathbb{Q}(\sqrt{-1}, \sqrt{-3})$. Let $g \in S_2(\Gamma_0(pqm), \varepsilon)$ be any twist of f, obtained by choosing local characters ψ_p at p and ψ_q at q (so $\varepsilon = \psi_p^2\psi_q^2$). By Eichler-Shimura there exists a 4 dimensional abelian variety A_g defined over $\mathbb{Q}$ (attached to g) and an embedding $K_g \hookrightarrow \mathrm{End}(A_g) \otimes \mathbb{Q}$. The Atkin-Li operators W_p and W_q do act on the differential forms of A_g although not necessarily as involutions. Since their eigenvalues lie in K_g, we can diagonalize them.

Let σ_i denote the Galois automorphism of K_g which fixes $\sqrt{-3}$ and $\sigma_{\sqrt{-3}}$ be the one fixing $\sqrt{-1}$ (so their composition is complex conjugation). We have the following analogue of Theorem 2.8.

Theorem 3.1. *With the previous notations:*

1. *the operator* W_p *coincides with* $\left(\frac{\eta_{\sigma_i}}{a_p}\right)^*$,
2. *the operator* W_q *coincides with* $\left(\frac{\eta_{\sigma_{\sqrt{-3}}}}{a_q}\right)^*$,
3. *the operator* W_{pq} *coincides with* $\left(\frac{\eta_{\sigma_i\sigma_{\sqrt{-3}}}}{a_pa_q}\right)^*$.

Proof. The proof mimics that of Theorem 2.8. Consider the basis of differential forms given by $\{g, \overline{g}, h, \overline{h}\}$, where $h \in S_2(pqm, \overline{\varepsilon_p}\varepsilon_q)$ equals $\sigma_i(g)$. By Theorem 2.5:

$$W_p\, g = \frac{G(\varepsilon_p)}{a_p}\, h, \qquad W_p\, \overline{g} = \frac{\overline{G(\varepsilon_p)}}{\overline{a_p}}\, \overline{h}, \qquad W_p\, h = \frac{\overline{G(\varepsilon_p)}}{\overline{a_p}}\, g, \qquad W_p\, \overline{h} = \frac{G(\varepsilon_p)}{a_p}\, \overline{g}.$$

A splitting map is given by

$$\beta(\sigma_i) = a_p, \qquad \beta(\sigma_{\sqrt{-3}}) = a_q, \qquad \beta(\sigma_i\sigma_{\sqrt{-3}}) = a_p a_q \psi_p(q)\psi_q(p), \tag{2}$$

By [GL01, Lemma 2.1] we have

$$\left(\frac{\eta_{\sigma_i}}{a_p}\right)^* g = \frac{G(\varepsilon_p)}{a_p} h, \qquad \left(\frac{\eta_{\sigma_i}}{a_p}\right)^* \overline{g} = \frac{\overline{G(\varepsilon_p)}}{\overline{a_p}} \overline{h}, \qquad \left(\frac{\eta_{\sigma_i}}{a_p}\right)^* h = \frac{\overline{G(\varepsilon_p)}}{\overline{a_p}} g,$$
$$\left(\frac{\eta_{\sigma_i}}{a_p}\right)^* \overline{h} = \frac{G(\varepsilon_p)}{a_p} \overline{g}.$$

The same computations proves the second statement, and the last one follows from the fact that if χ, χ' are two characters of conductors N and N' with $(N : N') = 1$, then

$$G(\chi \cdot \chi') = \chi(N')\chi'(N)G(\chi)G(\chi'). \tag{3}$$

□

Then we can split A_g into four pieces over M as in the previous section.

Although we considered only two particular cases, the general construction follows easily from them. Just split the primes into three sets: the ones with potentially multiplicative reduction, the ones with potentially good reduction with characters of order 4 and the ones with potentially good reduction with characters of order 3 or 6. Treat each set as in Case 1, and use Case 2 to mix them. Note that in any case the abelian surface A_g has dimension 1, 2 or 4.

4. Examples

In this section we show some examples of our construction, which were done using [GP 14]. The potentially multiplicative case is straightforward since we only have to find the corresponding quadratic twist and then construct classical Heegner points. The potentially good case is a little more involved. We consider the following two cases:

- The case where $\mathrm{ord}(\psi_p) = 2$ works exactly the same as the previous one, since we only have to find the quadratic twist.
- In the case $\mathrm{ord}(\psi_p) = 3, 4$ or 6 we start by applying Dokchitser's algorithm [DD11] (see also the appendix in [KP15]) to find ψ_p as well as the corresponding Fourier

coefficient a_p (which give the q-expansion of g). We compute A_g using the Abel-Jacobi map, and then we split it following Section 2.2.

Each factor is isomorphic to E over M. To find the isomorphism explicitly, we compare the lattices of E and the one computed and find one $\alpha \in M$ sending one lattice to the other.

N	E	St	Ps	K	$\mathrm{ord}(\psi_p)$	a_p	τ	P
$5^2 \cdot 29$	.a1	$\{5, 29\}$	$\emptyset$	$\mathbb{Q}(\sqrt{-5})$			$\frac{45+\sqrt{-45}}{145}$	$[8, 8]$
$5^2 \cdot 23$	.e1	$\{23\}$	$\{5\}$	$\mathbb{Q}(\sqrt{-5})$	4	$2-i$	$\frac{15+\sqrt{-5}}{5 \cdot 23}$	$[\frac{-1637}{2^6}, \frac{-2^8-3\cdot5^2\cdot127\sqrt{-5}}{2^9}]$
$2^2 \cdot 7^2$	.b2	$\emptyset$	$\{7\}$	$\mathbb{Q}(\sqrt{-7})$	3	$\frac{-5+\sqrt{-3}}{2}$	$\frac{21+\sqrt{-7}}{7\cdot2^3}$	$[\frac{-139}{4}, \frac{581\sqrt{-7}}{8}]$
$2 \cdot 3^2 \cdot 7^2$	.a1	$\{2\}$	$\{7\}$	$\mathbb{Q}(\sqrt{-7})$	3	$\frac{-1+3\sqrt{-3}}{2}$	$\frac{21+\sqrt{-7}}{28}$	$[39, 15]$
$5^2 \cdot 7^2$	.d2	$\emptyset$	$\{5\}$	$\mathbb{Q}(\sqrt{-35})$	4	$1-2i$	$\frac{-35+\sqrt{-35}}{70}$	$[-15, \frac{15+175\sqrt{-35}}{2}]$
			$\{7\}$		3	$\frac{1-3\sqrt{-3}}{2}$		

TABLE 2. Examples of ramified primes

The computations are summarized in Table 2. The table is organized as follows: the first two columns contain the curve conductor and its label (following [LMF13] notation). The next two columns list the principal series and the Steinberg primes of the curve (following [Pac13] algorithm). The fifth column contains the imaginary quadratic field. For the computations we just considered the whole ring of integers. The sixth and seventh columns contain the order of the character and the number a_p for the principal series primes ramifying in K. Finally the last two columns show the Heegner points considered in the upper-half plane and the point constructed in $E(K)$.

Some remarks regarding the examples considered:

- The first example corresponds to a potentially multiplicative case. The class number of $\mathcal{O}_K$ is 2 and $H = \mathbb{Q}(\sqrt{5}, i)$. If χ_5 denotes the non-trivial character of the class group, we can trace with respect to it and get the point $[9, \frac{-9+15\sqrt{5}}{2}] \in E(H)^{\chi_5}$.
- The second and third examples correspond to elliptic curves with only one potentially good reduction prime ramifying in K. The former has $ord(\varepsilon) = 2$ while the latter has $ord(\varepsilon) = 3$
- The fourth example is quite interesting, since the prime 2 splits in K (so we use an Eichler order at 2), the prime 3 is inert in K (so we use a Cartan order at 3), and the prime 7 is ramified in K. This is a mixed case of the Cartan-Heegner hypothesis (as in [KP15]) and the present one. We compute the q-expansion of g (as explained in the aforementioned article) as a form in $S_2(\Gamma_0(2 \cdot 7^2) \cap \Gamma_{ns}(3))$ and then twist by the character ψ_7 (of order 3) to get

a form in $S_2(\Gamma_0^{\varepsilon}(2\cdot 7)\cap\Gamma_{ns}(3))$. The results of Section 2.2 apply to give the corresponding splitting.

- The last example corresponds to an elliptic curve with two primes of potentially good reduction ramifying in K, hence the coefficient field is $K_g = \mathbb{Q}(\sqrt{-1},\sqrt{-3})$.

Appendix A. Computation of a Darmon point (by Marc Masdeu)

Let E denote the elliptic curve [LMF13, 147.c2], of conductor $3\cdot 7^2$ which has potentially good reduction over an abelian extension at the prime 7. Let $K = \mathbb{Q}(\sqrt{35})$, which has class number 2. The prime 3 is inert in K, while 7 ramifies. It is easy to see that $\operatorname{sign}(E,K) = -1$.

Let $p = 3$ and consider the Dirichlet character χ of conductor 7 which maps $3\in(\mathbb{Z}/7\mathbb{Z})^\times$ to $\zeta_6 = e^{\pi i/3}$. Let Γ denote the group

$$\Gamma = \Gamma_0^{\chi}(7)[1/3] = \left\{ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2\left(\mathbb{Z}\left[1/3\right]\right) \mid c\in 7\mathbb{Z}[1/3],\ \chi(a)=1\right\}.$$

In the page `http://github.com/mmasdeu/` there is code available to make computations with such groups.

There is a 2-dimensional irreducible component in the plus-part of $H^1(\Gamma_0^{\chi}(21),\mathbb{Z})$, which corresponds to the abelian surface A_g. Let $\{g_1,g_2\}$ be an integral basis of this subspace, normalized such that its basis vectors are not multiples of other integral vectors. Following the constructions of [GMŞ15] with the non-standard arithmetic groups, each of these vectors yield a cohomology class

$$\varphi_E^{(i)}\in H^1(\Gamma,\Omega^1_{\mathcal{H}_3}),\quad i=1,2.$$

Here $\mathcal{H}_3$ denotes the 3-adic upper half-plane and $\Omega^1_{\mathcal{H}_3}$ is the module of rigid-analytic differentials with 3-adically bounded residues.

The ring of integers $\mathcal{O}_K$ of K embeds into $M_2(\mathbb{Z})$ via

$$\sqrt{35}\mapsto\psi(\sqrt{35}) = \begin{pmatrix} 15 & 10\\ -19 & -15\end{pmatrix}.$$

The fundamental unit of K is $u_K = \sqrt{5}+6$, which is mapped to the matrix

$$\psi(u_K) = \begin{pmatrix} 21 & 10\\ -19 & -9\end{pmatrix}.$$

In order to obtain an element of $\Gamma_0^{\chi}(7)$ we need to consider u_K^{14}, which maps to

$$\gamma_K = \psi(u_K)^{14} = \begin{pmatrix} -3057309462214237 & -4524404717310744\\ 2852342104391556 & 4221080735198699\end{pmatrix}\in\Gamma_0^{\chi}(7).$$

The matrix γ_K fixes a point τ_K in $\mathcal{H}_3$,

$$\tau_K = 680113883076491926203393 + 188920523076803312834276\,\alpha_3 + O(3^{50}),$$

where α_3 denotes a square root of 35 in K_3, the completion of K at 3.

We present the above groups using Farey symbols so as to solve the word problem for them. Although the homology class of $\gamma_K \otimes \tau_K$ might not lie in $H_1(\Gamma_0^{\mathcal{X}}(7), \mathrm{Div}^0\,\mathcal{H}_3)$, its projection into the A_g isotypical component is. It can be seen that such projection is given by the operator $(T_2^2 - 3T_2 + 3)(T_2 + 3)$, where T_2 is the 2-th Hecke operator (just by computing the characteristic polynomial of the Hecke operator T_2 in the whole space and computing its irreducible factors). This allows to represent $(T_2^2 - 3T_2 + 3)(T_2 + 3)(\gamma_K \otimes \tau_K)$ by a cycle of the form

$$\left(\begin{smallmatrix} -6 & 1 \\ -7 & 1 \end{smallmatrix}\right) \otimes D_1 + \left(\begin{smallmatrix} 15 & -4 \\ 49 & -13 \end{smallmatrix}\right) \otimes D_2 + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \otimes D_3 + \left(\begin{smallmatrix} 22 & -9 \\ 49 & -20 \end{smallmatrix}\right) \otimes D_4 + \left(\begin{smallmatrix} -13 & 5 \\ -21 & 8 \end{smallmatrix}\right) \otimes D_5,$$

where D_i are divisors of degree 0 obtained by the aforementioned code (each divisor has support consisting of more than a thousand points in $\mathcal{H}_3$).

This class was integrated against the cohomology classes $\varphi_E^{(i)}$ using an overconvergent lift as explained in [GMŞ15] giving a point in $A_g(\mathbb{C}_3)$ which can be projected onto $E(\mathbb{C}_3)$ by choosing an appropriate linear combination of the basis elements. In the generic case any projection would work. We have taken in this case the projection onto g_1. Concretely, the integral corresponding to $\varphi_E^{(1)}$ resulted in the 3-adic element

$$J = 2+(\alpha_3+2)\cdot 3+3^2+(2\cdot\alpha_3+1)\cdot 3^3+(\alpha_3+1)\cdot 3^5+(\alpha_3+2)\cdot 3^6+(\alpha_3+1)\cdot 3^7+\cdots+O(3^{120})$$

If we apply Tate's uniformization (at 3) to such point, we obtain a point in $E(K_3)$ which coincides up to the working precision of 3^{120} with

$$14\cdot 13\cdot P = 14\cdot 13\cdot\left(\frac{164850\sqrt{7}}{2809} + \frac{610894}{2809}, \frac{63872781\sqrt{35}\sqrt{7}}{297754} + \frac{96772060\sqrt{35}}{148877} - \frac{1}{2}\right).$$

Note that $P \in E(H)$, where $H = K(\sqrt{7}) = \mathbb{Q}(\sqrt{35}, \sqrt{7})$ is the Hilbert class field of K as would be predicted by the conjectures. The factor 14 appears because we took the 14th power of the fundamental unit, while the factor 13 is due to the fact that the point would naturally lie in the elliptic curve 147.c1, which is 13-isogenous to E.

Finally, if one takes the trace of P to K one obtains:

$$P_K = P + P^\sigma = \left(\frac{63367}{2000}, \frac{5823153}{200000}\sqrt{35} - \frac{1}{2}\right), \quad \mathrm{Gal}(H/K) = \langle\sigma\rangle,$$

and one can check that P_K is non-torsion and thus generates a subgroup of finite index in $E(K)$.

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