The adjoint homology of the free 2-step nilpotent Lie algebra

Leandro Cagliero *
and
Paulo Tirao *

CIEM - FaMAF
Universidad Nacional de Córdoba

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Abstract

In this paper we determine the homology of the free 2-step nilpotent complex Lie algebra, with adjoint coefficients, as a module over the general linear group. This module is not multiplicity free. We give an explicit formula for the multiplicities and we compute the total dimension.

1 Introduction

In the celebrated paper [4], Kostant computed the cohomology $H^*(\mathfrak{n}, \pi)$ for all nilpotent radicals $\mathfrak{n}$ of parabolic subalgebras $\mathfrak{p}$ of complex semisimple Lie algebras $\mathfrak{g}$, where π is the restriction to $\mathfrak{n}$ of a representation of $\mathfrak{g}$. If $\mathfrak{g}_1$ is the semisimple part of $\mathfrak{p}$, then $\mathfrak{n}$ and $H^*(\mathfrak{n}, \pi)$ are $\mathfrak{g}_1$-modules. Kostant's result describes the $\mathfrak{g}_1$-module structure of $H^*(\mathfrak{n}, \pi)$. The adjoint representation of $\mathfrak{n}$, when $\mathfrak{n}$ is non-abelian, is never such a restriction, and the determination of $H^*(\mathfrak{n}, \mathfrak{n})$ is still an open problem.

The free 2-step nilpotent complex Lie algebra of rank r is $\mathcal{L}(r) = V \oplus \Lambda^2 V$, where V is an r dimensional vector space over $\mathbb{C}$. The only non-zero Lie brackets are for $v, w \in V$, in which case $[v, w] = v \wedge w \in \Lambda^2 V$. The center of $\mathcal{L}(r)$ is $\Lambda^2 V$. $\mathcal{L}(r)$ is the free object for 2-step nilpotent complex

*CONICET (Argentina) postdoctoral fellow.

Lie algebras with r generators. Moreover, any 2-step Lie algebra is of the form $\mathfrak{n} = \mathcal{L}(r)/W = V \oplus \Lambda^2 V/W$ for some r, where W is a subspace of $\Lambda^2 V$. The group $GL(V)$ acts polynomially on both, V and $\Lambda^2 V$, and hence so it does on $\mathcal{L}(r)$. This action is particulary interesting since $\mathcal{L}(r)$ is a model for $GL(V)$. That is, in the symmetric algebra $S(V \oplus \Lambda^2 V)$ every irreducible polynomial representation of $GL(V)$ occurs and it does with multiplicity 1. This remarkable fact is a classical result that can be found for example in [1].

We point out that $\mathcal{L}(r)$ falls into the class considered by Kostant. In fact, $\mathcal{L}(r)$ is the nilpotent radical of a parabolic subalgebra of $\mathfrak{so}(2r+1, \mathbb{C})$ where the corresponding $\mathfrak{g}_1$ is $\mathfrak{gl}(V)$. Its homology can be deduced from [4]; see Appendix 1 of [3]. In [6] Sigg gave an explicit description of the homology of $\mathcal{L}(r)$ with trivial coefficients as a $GL(V)$-module, for every $r \geq 2$. As it is in all Kostant's cases, this space is multiplicity free.

The main purpose of this paper is to describe the $GL(V)$-structure of the adjoint homology space. We proceed as follows. In the short exact sequence of $\mathcal{L}(r)$-modules

$$0 \longrightarrow \Lambda^2 V \longrightarrow \mathcal{L}(r) \longrightarrow V \longrightarrow 0,$$

$\Lambda^2 V$ and V are trivial and therefore the connecting homomorphism in the induced homological long exact sequence is

$$\delta : H_*(\mathcal{L}(r)) \otimes V \longrightarrow H_*(\mathcal{L}(r)) \otimes \Lambda^2 V$$

and

$$H_p(\mathcal{L}(r), \mathcal{L}(r)) \cong \ker \delta_p \oplus \operatorname{coker} \delta_{p+1}$$

as $GL(V)$-modules.

In §3 we recall the description of $H_*(\mathcal{L}(r))$ given in [6]. From this description it is not difficult to see that if an irreducible $GL(V)$-module occurs simultaneously in $H_p(\mathcal{L}(r)) \otimes V$ and $H_{p-1}(\mathcal{L}(r)) \otimes \Lambda^2 V$, then it occurs in both of them with multiplicity 1. In §4 we identify the connecting homomorphism δ and we prove that if E_p and F_{p-1} are isomorphic $GL(V)$-submodules of $H_p(\mathcal{L}(r)) \otimes V$ and $H_{p-1}(\mathcal{L}(r)) \otimes \Lambda^2 V$ respectively, then $\delta_p(E_p) = F_{p-1}$. There is no ambiguity here because the irreducible submodules of E_p and F_{p-1} have multiplicity 1. The proof is based on an explicit construction of a highest weight vector for the irreducible modules decomposing E_p. Our main result is the following theorem.

Theorem. *Let $\mathcal{L}(r) = V \oplus \Lambda^2 V$ be the free 2-step nilpotent complex Lie algebra of rank r and let $H_p(\mathcal{L}(r))$ be the p-homology space of $\mathcal{L}(r)$ with trivial coefficients, considered as a $GL(V)$-module. Let E_p and F_{p-1} be the maximal isomorphic submodules of $H_p(\mathcal{L}(r)) \otimes V$ and $H_{p-1}(\mathcal{L}(r)) \otimes \Lambda^2 V$, respectively. Then, for all p,*

$$H_p(\mathcal{L}(r), \mathcal{L}(r)) \cong \frac{H_p(\mathcal{L}(r)) \otimes \Lambda^2 V}{F_p} \oplus \frac{H_p(\mathcal{L}(r)) \otimes V}{E_p}$$

as $GL(V)$-modules.

This theorem proves a conjecture stated by Sigg in [5].

In contrast to the case of trivial coefficients, the adjoint homology space is not multiplicity free. Theorem 4.5 gives an explicit formula for the multiplicities.

In §5 we determine explicitly the total dimension of $H_*(\mathcal{L}(r), \mathcal{L}(r))$ by computations that are similar to those in [3].

The cohomology of $\mathcal{L}(r)$ with adjoint coefficients can be deduced from our result since the $GL(V)$-structures of the homology and cohomology spaces are related by the formula

$$H_{n-i}(\mathcal{L}(r), \mathcal{L}(r)) \cong H^i(\mathcal{L}(r), \mathcal{L}(r)) \otimes \det{}^r,$$

where $n = \dim \mathcal{L}(r)$.

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2 The Littlewood-Richardson rule

We include here some background in representation theory, we fix some notation and make some conventions. We assume all representations to be of finite dimension. Proofs, as well as the general theory, can be found in any standard book, e.g. [2].

Besides these preliminaries we give a constructive way to decompose the tensor product of any representation of $GL(V)$ with an exterior power of the canonical representation into irreducible subrepresentations The Littlewood-Richardson rule gives this decomposition up to isomorphism. We include the proofs since they may be of independent interest.

From now on, we fix a basis $\mathcal{B} = \{e_1, \ldots, e_r\}$ of V and we denote by $\{E_{ij} : i, j = 1, \ldots, r\}$ the canonical basis of $\mathrm{End}(V)$. Let $\mathfrak{gl}(V)$ be the Lie algebra of $GL(V)$ and fix the triangular decomposition $\mathfrak{gl}(V) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}^+$ are the subalgebras consisting of endomorphisms whose matrices in the basis $\mathcal{B}$ are respectively strictly lower triangular, diagonal and strictly upper triangular. Now $\{E_{11}, \dots, E_{rr}\}$ is a basis of a Cartan subalgebra $\mathfrak{h}$. We will denote the corresponding dual basis by $\{\epsilon_1, \dots, \epsilon_r\}$. In particular $\{\epsilon_i - \epsilon_j : i < j\}$ is the set of *positive roots* corresponding to the triangular decomposition chosen above.

A linear functional λ on $\mathfrak{h}$ is called a *weight* if it takes integer values on the vectors $E_{ii} - E_{jj}$ for all $i < j$. A weight is said to be a *dominant weight* if $\lambda(E_{ii} - E_{jj}) \geq 0$. A partition of length r is an r-tuple of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_1 \geq \dots \geq \lambda_r$. Any partition of length r defines a dominant weight, which we will denote again by λ, given by $\sum \lambda_i \epsilon_i$. By a polynomial representation we mean a finite dimensional representation of $GL(V)$ such that the matrix entries are given by polynomial functions. It is well known that every polynomial representation of $GL(V)$ can be decomposed as a sum of irreducible polynomial subrepresentations.

In each irreducible polynomial representation W there is a unique (up to scalars) non-zero vector v such that $\mathfrak{n}^+.v = 0$ and $H.v = \lambda(H)v$ where λ is a dominant weight. Such a vector is called a *highest weight vector* of weight λ, and W is called an irreducible representation of highest weight λ. In addition, $W = \mathcal{U}(\mathfrak{n}^-).v$, where $\mathcal{U}(\mathfrak{n}^-)$ is the enveloping algebra of $\mathfrak{n}^-$.

The isomorphism classes of irreducible polynomial representations of $GL(V)$ are in one-to-one correspondence with the partitions of length r. Given λ, $\mathcal{W}_\lambda$ will denote an irreducible representation of highest weight λ. If W is a (polynomial) representation of $GL(V)$ and λ is a partition of length r, $W(\lambda)$ is the λ-*isotypic component of* W, that is, the sum of all the irreducible subrepresentations of W isomorphic to $\mathcal{W}_\lambda$. Given a polynomial representation W of $GL(V)$ and a partition λ of length r, we say that λ, or $\mathcal{W}_\lambda$, *occurs* in W if the λ-isotypic component of W, $W(\lambda)$, is different from zero.

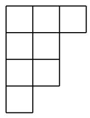
A partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is often represented by its Young diagram $Y(\lambda)$, a graphical arrangement of λ_i boxes in the i-th row starting in the first column. The conjugate partition λ' of λ is, by definition, the partition whose Young diagram $Y(\lambda')$ is obtained by reflecting $Y(\lambda)$ along the diagonal.

Another way to denote a partition λ is due to Frobenius. Let d_λ be the number of diagonal boxes of $Y(\lambda)$. For $i = 1, \dots, d_\lambda$, let α_i to be the number of boxes in the i-th row to the right of and on the diagonal. Let β_i to be the number of boxes in the i-th column below and on the diagonal. Then one writes $\lambda = (I; J)$ where $I = \{\alpha_1, \dots, \alpha_{d_\lambda}\}$ and $J = \{\beta_1, \dots, \beta_{d_\lambda}\}$. Notice that $\alpha_1 > \dots > \alpha_d \geq 1$ and $\beta_1 > \dots > \beta_d \geq 1$, so the sets I and J

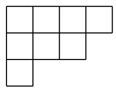
do determine the sequences α_i and β_i. It must be observed that there are different conventions on the form of Frobenius notation and, in particular, [6] uses a slightly different one. We finally remark that if $\lambda = (I; J)$ then $\lambda' = (J; I)$.

To compute the Frobenius notation directly from the standard notation $\lambda = (\lambda_1, \ldots, \lambda_r)$, set $d_\lambda = \#\{i : \ \lambda_i \geq i\}$, $\alpha_i = \lambda_i - i + 1$ and $\beta_i = \lambda'_i - i + 1$. As an example, we give the standard and the Frobenius notations of a partition λ and of its conjugate λ'. We also show their Young diagrams.

$$\lambda = (3, 2, 2, 1) = (\{3, 1\}; \{4, 2\}) \qquad Y(\lambda) =$$



$$\lambda' = (4, 3, 1) = (\{4, 2\}; \{3, 1\}) \qquad Y(\lambda') =$$



Definition 2.1. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r) = (I; J)$ we define $|\lambda| = \sum_{i \in I} \lambda_i$ and $\mathbf{\Sigma}(\lambda) = \sum_{i \in I} i + \sum_{j \in J} j$. We remark that $\mathbf{\Sigma}(\lambda) = |\lambda| + d_\lambda$.

For $k \leq r$, let 1^k be the partition $\underbrace{(1, \ldots, 1)}_{k}$. It is known that $W_{1^k} \cong \Lambda^k V$.

The decomposition, up to isomorphism, of the tensor product of two irreducible polynomial representations into irreducibles can be obtained via the Littlewood-Richardson rule (see [2], Lecture 6). The following proposition describes the case when one of the representations is W_{1^k}.

Proposition 2.2. *Let λ be a partition of length r and let $k \leq r$. Let $\Lambda_k(\lambda) = \{\mu : Y(\mu)$ is obtained from $Y(\lambda)$ by adding k boxes in k different rows$\}$. Then*

$$\mathcal{W}_\lambda \otimes W_{1^k} \cong \sum_{\mu \in \Lambda_k(\lambda)} \mathcal{W}_\mu.$$

The proof of this proposition is based on the explicit construction of the highest weight vectors in $\mathcal{W}_\lambda \otimes W_{1^k}$. In particular, the expression of the highest weight vectors for $k = 1$ will be necessary in §4.

Let $[\![r]\!]$ denote the set of integers from 1 to r. For each non-empty subset $I = \{i_1, \ldots, i_k\} \subseteq [\![r]\!]$, with $i_1 < \cdots < i_k$, let $U_I \in \mathcal{U}(\mathfrak{n}^-)$ be given by,

$$U_I = \begin{cases} E_{i_2 i_1} E_{i_3 i_2} \ldots E_{i_k i_{k-1}}, & \text{if } k > 1; \\ 1, & \text{if } k = 1. \end{cases}$$

Let us denote $I_{<a} = I \cap \{1, \dots, a-1\}$, $I_{>a} = I \cap \{a+1, \dots, r\}$, $I_{\leq a} = I \cap \{1, \dots, a\}$ and $I_{\geq a} = I \cap \{a, \dots, r\}$.

Lemma 2.3. *Let I be a non-empty subset of $[\![r]\!]$ and let $a < r$. Then the commutator $[E_{a(a+1)}, U_I] \in \mathcal{U}(\mathfrak{gl}(V))$, the enveloping algebra of $\mathfrak{gl}(V)$, is given according to the following table.*

$a \in I$ *and* $a+1 \in I$	$I_{>a+1} \neq \emptyset$	$U_{I_{\leq a}} U_{I_{\geq a+1}} (E_{aa} - E_{(a+1)(a+1)} + 1)$	*(1)*
	$I_{>a+1} = \emptyset$	$U_{I_{\leq a}} (E_{aa} - E_{(a+1)(a+1)})$	*(2)*
$a \in I$ *and* $a+1 \notin I$	$I_{>a+1} \neq \emptyset$	$-U_{I_{\leq a}} U_{\{a+1\} \cup I_{>a+1}}$	*(3)*
	$I_{>a+1} = \emptyset$	0	*(4)*
$a \notin I$ *and* $a+1 \in I$	$I_{<a} \neq \emptyset$	$U_{I_{<a} \cup \{a\}} U_{I_{\geq a+1}}$	*(5)*
	$I_{<a} = \emptyset$	0	*(6)*
$a, a+1 \notin I$		0	*(7)*

Proof. These relations are direct consequences of the commutation relations

$$[E_{ab}, E_{cd}] = \begin{cases} E_{ad}, & \text{if } b = c \text{ and } a \neq d; \\ -E_{cb}, & \text{if } b \neq c \text{ and } a = d; \\ E_{aa} - E_{bb}, & \text{if } b = c \text{ and } a = d; \\ 0, & \text{if } b \neq c \text{ and } a \neq d. \end{cases}$$

□

Proposition 2.4. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of length r. Assume that either $s = 1$ or $\lambda_{s-1} > \lambda_s$ for some $1 < s \leq r$. Let $A^{(s)} = \{I \subseteq [\![r]\!] : \max(I) = s\}$. For $i \leq s$ let $\sigma_i = \lambda_s - \lambda_i - (s-i) + 1$ and for $I \in A^{(s)}$ set $\sigma_I = \prod_{i \in I} \sigma_i$. If v is a non-zero highest weight vector of $\mathcal{W}_\lambda$ then*

$$w = \sum_{I \in A^{(s)}} \frac{1}{\sigma_I} U_I.v \otimes e_{\min(I)} \tag{2.1}$$

is a non-zero highest weight vector of $\mathcal{W}_\lambda \otimes V$ corresponding to a subrepresentation isomorphic to $\mathcal{W}_{\lambda+\epsilon_s}$.

Proof. For $I \in A^{(s)}$ let $S_I = \sigma_I^{-1} U_I.v \otimes e_{\min(I)}$. We note that $S_{\{s\}} = v \otimes e_s$ is linearly independent with the remaining terms in (2.1) since $\min(I) \neq s$ if $I \neq \{s\}$. Hence $w \neq 0$ and it is clear that it is a weight vector of weight $\lambda + \epsilon_s$.

For each $1 \le b \le s$ consider the set

$$A^{(s)}_{(b)} = \{I \in A^{(s)} : \min(I) = b\}.$$

To prove that w is a highest weight vector it suffices to check that $E_{a(a+1)}.w = 0$, for all $a < r$. We consider separately the cases $a \ge s$, $a = s-1$ and $a < s-1$.

(i) $a \ge s$. The claim follows directly from statements (4) and (7) of Lemma 2.3.

(ii) $a = s-1$. For $b \ne s-1$ let I be any set in $A^{(s)}_{(b)}$ such that $s-1 \notin I$ and let $I^{s-1} = I \cup \{s-1\}$. Notice that

$$A^{(s)} = \bigcup_{b \ne s-1} \{I, I^{s-1} : s-1 \notin I\}.$$

Then $E_{(s-1)s}.(S_I + S_{I^{s-1}}) = 0$ for all $I \in A^{(s)}_{(b)}$ with $s-1 \notin I$ and $b \ne s-1$. In fact, if $b = s$ the only set in $A^{(s)}_{(b)}$ is $I = \{s\}$ and

$$\begin{aligned} E_{(s-1)s}.(S_I + S_{I^{s-1}}) &= E_{(s-1)s}.(v \otimes e_s + \frac{1}{\sigma_{s-1}} E_{s(s-1)}.v \otimes e_{s-1}) \\ &= v \otimes e_{s-1} + \frac{\lambda_{s-1} - \lambda_s}{\sigma_{s-1}} v \otimes e_{s-1} \\ &= v \otimes e_{s-1} - v \otimes e_{s-1} \\ &= 0. \end{aligned}$$

If $b < s-1$, it follows from statements (5) and (2) of Lemma 2.3 that

$$\begin{aligned} E_{(s-1)s}.(S_I + S_{I^{s-1}}) &= \frac{1}{\sigma_I} U_{I_{<s-1} \cup \{s-1\}}.v \otimes e_b + \frac{\lambda_{s-1} - \lambda_s}{\sigma_{s-1}\sigma_I} U_{I^{s-1}_{\le s-1}}.v \otimes e_b \\ &= \frac{1}{\sigma_I} U_{I^{s-1}_{\le s-1}}.v \otimes e_b - \frac{1}{\sigma_I} U_{I^{s-1}_{\le s-1}}.v \otimes e_b \\ &= 0. \end{aligned}$$

(iii) $a < s-1$. For $b \ne a$ and $b \ne a+1$ let I be any set in $A^{(s)}_{(b)}$ such that $a \notin I$ and $a+1 \notin I$ and let $I^a = I \cup \{a\}$, $I^{a+1} = I \cup \{a+1\}$ and $I^{a,a+1} = I \cup \{a, a+1\}$. Notice that

$$A^{(s)} = \bigcup_{b \ne a, a+1} \{I, I^a, I^{a+1}, I^{a,a+1} : a, a+1 \notin I\}.$$

Statement (7) of Lemma 2.3 says that

$$E_{a(a+1)}.S_I = 0, \qquad \text{for all } I \text{ such that } a, a+1 \notin I.$$

If $b > a+1$, statements (3), (6) and (1) of Lemma 2.3 imply that

$$\begin{aligned}
E_{a(a+1)}.S_{I^a} &= -\frac{1}{\sigma_a\sigma_I} U_{\{a+1\}\cup I^a_{>a+1}}.v\otimes e_a,\\
E_{a(a+1)}.S_{I^{a+1}} &= \frac{1}{\sigma_{a+1}\sigma_I} U_{I^{a+1}}.v\otimes e_a,\\
E_{a(a+1)}.S_{I^{a(a+1)}} &= \frac{\lambda_a-\lambda_{a+1}+1}{\sigma_a\sigma_{a+1}\sigma_I} U_{I^{a(a+1)}_{\geq a+1}}.v\otimes e_a.
\end{aligned}$$

Therefore

$$\begin{aligned}
&E_{a(a+1)}.(S_{I^a}+S_{I^{a+1}}+S_{I^{a,a+1}}) =\\
&\quad= \left(-\frac{1}{\sigma_a\sigma_I}+\frac{1}{\sigma_{a+1}\sigma_I}+\frac{\lambda_a-\lambda_{a+1}+1}{\sigma_a\sigma_{a+1}\sigma_I}\right) U_{I^{a(a+1)}_{\geq a+1}}.v\otimes e_a\\
&\quad= 0,
\end{aligned}$$

since $-\sigma_{a+1}+\sigma_a+\lambda_a-\lambda_{a+1}+1=0$.

If $b < a$, statements (3), (5) and (1) of Lemma 2.3 imply that

$$\begin{aligned}
E_{a(a+1)}.S_{I^a} &= -\frac{1}{\sigma_a\sigma_I} U_{I^a_{\leq a}} U_{\{a+1\}\cup I^a_{>a+1}}.v\otimes e_b,\\
E_{a(a+1)}.S_{I^{a+1}} &= \frac{1}{\sigma_{a+1}\sigma_I} U_{I^{a+1}_{<a}\cup\{a\}} U_{I^{a+1}_{\geq a+1}}.v\otimes e_b,\\
E_{a(a+1)}.S_{I^{a(a+1)}} &= \frac{\lambda_a-\lambda_{a+1}+1}{\sigma_a\sigma_{a+1}\sigma_I} U_{I^{a(a+1)}_{\leq a}} U_{I^{a(a+1)}_{\geq a+1}}.v\otimes e_b.
\end{aligned}$$

Therefore

$$\begin{aligned}
&E_{a(a+1)}.(S_{I^a}+S_{I^{a+1}}+S_{I^{a,a+1}}) =\\
&\quad= \left(-\frac{1}{\sigma_a\sigma_I}+\frac{1}{\sigma_{a+1}\sigma_I}+\frac{\lambda_a-\lambda_{a+1}+1}{\sigma_a\sigma_{a+1}\sigma_I}\right) U_{I^{a(a+1)}_{\leq a}} U_{I^{a(a+1)}_{\geq a+1}}.v\otimes e_b\\
&\quad= 0,
\end{aligned}$$

since $-\sigma_{a+1}+\sigma_a+\lambda_a-\lambda_{a+1}+1=0$. □

Now we can prove Proposition 2.2.

Proof of Proposition 2.2. If $v=\sum v_i\otimes w_i$ is a highest weight vector in $\mathcal{W}_\lambda\otimes\mathcal{W}_{1^k}$, and if $\{v_i\}$ and $\{w_i\}$ are linearly independent in $\mathcal{W}_\lambda$ and $\mathcal{W}_{1^k}$ respectively, then it is not difficult to see that v_i is a highest weight vector of $\mathcal{W}_\lambda$ for some i. This implies that the only highest weights that may occur in the tensor product $\mathcal{W}_\lambda\otimes\mathcal{W}_{1^k}$ are the dominant weights of the form λ plus some weight of $\mathcal{W}_{1^k} \cong \Lambda^k V$, in other words those in $\Lambda_k(\lambda)$. Moreover, they could only occur with multiplicity one because every weight space in $\mathcal{W}_{1^k}$ is one dimensional. Therefore we only need to show that all of them occur. We proceed by induction on k. Proposition 2.4 is the case $k = 1$. Let us assume the result true for $k-1$, and let $\mu \in \Lambda_k(\lambda)$, that is $Y(\mu)$ is obtained from $Y(\lambda)$ by adding k boxes in k different rows. Let $\{s_1, \dots, s_k\}$ be these rows. By the inductive hypothesis, there is a copy of $\mathcal{W}_{\mu-\epsilon_{s_k}}$ in $\mathcal{W}_\lambda \otimes \mathcal{W}_{1^{k-1}}$ and it occurs with multiplicity one. In particular, if v is a non-zero highest weight vector in it, it must be of the form

$$v = v_\lambda \otimes e_{s_1} \wedge \dots \wedge e_{s_{k-1}} + \sum_i w_i \otimes \alpha_i$$

with v_λ a non-zero highest weight vector in $\mathcal{W}_\lambda$ and α_i of weight $\epsilon_{s_1} + \dots + \epsilon_{s_{k-1}} +$ *a sum of positive roots.* Again by Proposition 2.4 we can find a non-zero highest weight vector $v' \in \mathcal{W}_{\mu-\epsilon_{s_k}} \subset \mathcal{W}_\lambda \otimes \mathcal{W}_{1^{k-1}} \otimes V$ of weight μ and

$$v' = v_\lambda \otimes e_{s_1} \wedge \dots \wedge e_{s_{k-1}} \otimes e_{s_k} + \sum_i w_i' \otimes \beta_i \otimes e_i$$

with $\beta_i \otimes e_i$ of weight $\epsilon_{s_1} + \dots + \epsilon_{s_k} +$ *a sum of positive roots.* Now this implies that the projection of v' under the map $\mathcal{W}_\lambda \otimes \mathcal{W}_{1^{k-1}} \otimes V \to \mathcal{W}_\lambda \otimes \mathcal{W}_{1^k}$ is non-zero, thus giving the desired vector. □

3 The Lie algebra $\mathcal{L}(r)$ and its homology

We recall that the homology of a Lie algebra $\mathfrak{n}$ with trivial coefficients, $H_*(\mathfrak{n})$, is the homology of the exterior algebra complex $(\Lambda^*\mathfrak{n}, \partial_0)$, where

$$\partial_0(x_1 \wedge \dots \wedge x_p) = \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \dots \widehat{x_i} \dots \widehat{x_j} \dots \wedge x_p \ . \tag{3.1}$$

The homology of $\mathfrak{n}$ with adjoint coefficients, $H_*(\mathfrak{n}, \mathfrak{n})$, is the homology of the complex $(\Lambda^*\mathfrak{n} \otimes \mathfrak{n}, \partial)$ where $\partial = \partial_0 \otimes \mathrm{id} + \partial_1$ and

$$\partial_1(x_1 \wedge \dots \wedge x_p \otimes x) = \sum_{i=1} (-1)^{i+1} x_1 \wedge \dots \widehat{x_i} \dots \wedge x_p \otimes [x_i, x] \ . \tag{3.2}$$

The free 2-step nilpotent complex Lie algebra of rank r is $\mathcal{L}(r) = V \oplus \Lambda^2 V$, where V is an r dimensional vector space over $\mathbb{C}$. The only non-zero Lie brackets are for $v, w \in V$, in which case $[v, w] = v \wedge w \in \Lambda^2 V$. The center of

$\mathcal{L}(r)$ is $\Lambda^2 V$. The group $GL(V)$ acts polynomially on both, V and $\Lambda^2 V$, and hence so it does on $\mathcal{L}(r)$. This action commutes with ∂_0 and with ∂_1 and therefore we obtain a polynomial representation of $GL(V)$ on the homology spaces $H_*(\mathcal{L}(r))$ and $H_*(\mathcal{L}(r), \mathcal{L}(r))$.

It turns out that $\mathcal{L}(r)$ with the $GL(V)$ action can be viewed as the nilpotent radical of a parabolic subalgebra of a simple Lie algebra with the corresponding Levi subalgebra acting in the following way. Let $\mathfrak{g} = \mathfrak{so}(2r+1, \mathbb{C})$. If $\{\varepsilon_i\}$ is an orthonormal basis of the dual of a Cartan subalgebra $\mathfrak{h}_\mathfrak{g}$ of $\mathfrak{g}$, then we may choose

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j\}_{1 \le i < j \le r} \cup \{\varepsilon_i\}_{1 \le i \le r}$$

as the set of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}_\mathfrak{g}$. Consider the parabolic subalgebra $\mathfrak{p} = \sum_{i>j} \mathfrak{g}_{\varepsilon_i - \varepsilon_j} \oplus \mathfrak{h}_\mathfrak{g} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. One can check that the Levi decomposition $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$ is given by

$$\begin{aligned} \mathfrak{g}_1 &= \mathfrak{h}_\mathfrak{g} \oplus \sum_{i<j} \mathfrak{g}_{\varepsilon_i - \varepsilon_j} \oplus \sum_{i>j} \mathfrak{g}_{\varepsilon_i - \varepsilon_j}, \\ \mathfrak{n} &= \sum_i \mathfrak{g}_{\varepsilon_i} \oplus \sum_{i<j} \mathfrak{g}_{\varepsilon_i + \varepsilon_j}, \end{aligned}$$

and we have $\mathfrak{g}_1 \cong \mathfrak{gl}(V)$ and $\mathfrak{n} \cong \mathcal{L}(r)$ as $\mathfrak{gl}(V)$-modules.

Now, the decomposition of $H_*(\mathcal{L}(r))$ as a $GL(V)$-module is a particular case of one of the main results in [4] and is given very explicitly in [6] (see also [3]).

Theorem 3.1 (Kostant-Sigg).

$$H_*(\mathcal{L}(r)) \cong \bigoplus_{I \subseteq [\![r]\!]} \mathcal{W}_{\lambda_I} \;, \tag{3.3}$$

where $\lambda_I = (I; I)$. Furthermore, the homology grading of $\mathcal{W}_{\lambda_I}$ is $p = \frac{\boldsymbol{\Sigma}(\lambda)}{2}$.

Remark 3.2. The highest weight vectors of the irreducible summands of $H^*(\mathcal{L}(r))$ are given in [4]. The cohomology and homology of $\mathcal{L}(r)$ are related by the formula

$$H_{n-i}(\mathcal{L}(r)) \cong H^i(\mathcal{L}(r)) \otimes \det{}^r.$$

From this it follows that if $\{f_{ij} = e_i \wedge e_j : i < j\}$ is a basis of $\Lambda^2 V$ and $I = \{a_1, \ldots, a_{d_\lambda}\}$ with $a_1 > \cdots > a_{d_\lambda}$, then the class of the vector

$$\begin{aligned} v_I = e_1 \wedge f_{12} \wedge \ldots \wedge f_{1a_1} \wedge e_2 \wedge f_{23} \wedge \ldots \wedge f_{2(a_2+1)} \wedge \ldots \\ \ldots \wedge e_{d_\lambda} \wedge f_{d_\lambda(d_\lambda+1)} \wedge \ldots \wedge f_{d_\lambda(a_{d_\lambda}+d_\lambda-1)} \end{aligned}$$

($f_{d_\lambda(d_\lambda+1)}\wedge\ldots\wedge f_{d_\lambda(a_{d_\lambda}+d_\lambda-1)}$ is omitted if $a_{d_\lambda}=1$) in $H_p(\mathcal{L}(r))$ is a highest weight vector of the unique subrepresentation of $H(\mathcal{L}(r))$ isomorphic to $\mathcal{W}_{\lambda_I}$.

The computation of $H_*(\mathcal{L}(r),\mathcal{L}(r))$ is not contained in Kostant's results because the adjoint representation of $\mathcal{L}(r)$ is not the restriction of a representation of $\mathfrak{so}(2r+1,\mathbb{C})$. This is a particular instance of the following general fact.

Proposition 3.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and let $\mathfrak{n}$ be the nilpotent radical of a parabolic subalgebra $\mathfrak{p}\subseteq\mathfrak{g}$. Let $\mathrm{ad}_{\mathfrak{n}}$ be the adjoint representation of $\mathfrak{n}$. If there exist π, a representation of $\mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{n}}=\pi|_{\mathfrak{n}}$, then $\mathfrak{n}$ is abelian.*

Proof. Let $\mathfrak{g}=\oplus\mathfrak{g}_i$, $\mathfrak{g}_i$ simple, and let $\mathfrak{n}_i=\mathfrak{n}\cap\mathfrak{g}_i$. From the structure of parabolic subalgebras of a semisimple Lie algebra, it follows that $\mathfrak{n}=\oplus\mathfrak{n}_i$ (see, for instance, [4]). For each i take z_i in the center of $\mathfrak{n}_i$. Since $\pi(z_i)=\mathrm{ad}_{\mathfrak{n}_i}(z_i)=0$, then, by simplicity, $\pi(\mathfrak{g}_i)=0$ and in particular $\mathrm{ad}_{\mathfrak{n}_i}=0$ for all i. Therefore $\mathrm{ad}_{\mathfrak{n}}=0$ and $\mathfrak{n}$ is abelian. □

4 The adjoint homology of $\mathcal{L}(r)$

Let $H_p=H_p(\mathcal{L}(r))$ and let $H_p^{\mathrm{ad}}=H_p(\mathcal{L}(r),\mathcal{L}(r))$. The short exact sequence of $\mathcal{L}(r)$-modules

$$0\longrightarrow\Lambda^2V\longrightarrow\mathcal{L}(r)\longrightarrow V\longrightarrow 0,$$

induces the long exact sequence of $GL(V)$-modules,

$$H_{p+1}(\mathcal{L}(r),V)\xrightarrow{\delta_{p+1}}H_p(\mathcal{L}(r),\Lambda^2V)\xrightarrow{i}H_p(\mathcal{L}(r),\mathcal{L}(r))\xrightarrow{\pi}$$
$$H_p(\mathcal{L}(r),V)\xrightarrow{\delta_p}H_{p-1}(\mathcal{L}(r),\Lambda^2V).$$

Since Λ^2V and V are trivial $\mathcal{L}(r)$-modules, this long exact sequence becomes

$$H_{p+1}\otimes V\xrightarrow{\delta_{p+1}}H_p\otimes\Lambda^2V\xrightarrow{i}H_p^{\mathrm{ad}}\xrightarrow{\pi}H_p\otimes V\xrightarrow{\delta_p}H_{p-1}\otimes\Lambda^2V. \tag{4.1}$$

Therefore, $H_p^{\mathrm{ad}}\cong\ker\delta_p\oplus\operatorname{coker}\delta_{p+1}$.

Let us identify the connecting homomorphism δ_p.

Notation 4.1. Let $v=\sum a_i\otimes b_i$ be an element of $\Lambda\mathcal{L}(r)\otimes\mathcal{L}(r)$. We will denote $[v]=\sum[a_i]\otimes b_i$ in $H_*(\mathcal{L}(r))\otimes\mathcal{L}(r)$. We remark that $[v]$ is not the homology class of v in $H_*(\mathcal{L}(r),\mathcal{L}(r))$.

Lemma 4.2. *Let $\delta_p : H_p \otimes V \longrightarrow H_{p-1} \otimes \Lambda^2 V$ be the connecting morphism defined by (4.1), and let $v \in \Lambda\mathcal{L}(r) \otimes V$. Then*

$$\delta_p([v]) = [\partial_1(v)]$$

Proof. Consider the commutative diagram of $GL(V)$-modules

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Lambda^{p-1}\mathcal{L}(r) \otimes \Lambda^2 V & \longrightarrow & \Lambda^{p-1}\mathcal{L}(r) \otimes \mathcal{L}(r) & \longrightarrow & \Lambda^{p-1}\mathcal{L}(r) \otimes V & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \partial_0 \otimes \mathrm{id}} & & \uparrow{\scriptstyle \partial_0 \otimes \mathrm{id} + \partial_1} & & \uparrow{\scriptstyle \partial_0 \otimes \mathrm{id}} & & \\
0 & \longrightarrow & \Lambda^{p}\mathcal{L}(r) \otimes \Lambda^2 V & \longrightarrow & \Lambda^{p}\mathcal{L}(r) \otimes \mathcal{L}(r) & \longrightarrow & \Lambda^{p}\mathcal{L}(r) \otimes V & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}$$

The map δ_p is defined by taking images and preimages in this diagram. Let $[v] \in H_p \otimes V$, that is $v \in \Lambda^p\mathcal{L}(r) \otimes V$ and $\partial_0 \otimes \mathrm{id}(v) = 0$. This implies that $\partial(v) = \partial_1(v)$. From the definition of ∂_1 (see (3.2)) it is clear that $\partial_1(v) \in \Lambda^{p-1}\mathcal{L}(r) \otimes \Lambda^2 V$. Therefore $[\partial_1(v)]$ is the image of $[v]$ by δ_p. □

4.1 The submodules E_p and F_{p-1}

Let C_p be the collection of all partitions μ that occur simultaneously in $H_p \otimes V$ and $H_{p-1} \otimes \Lambda^2 V$ and let E_p and F_{p-1} be, respectively, the sum of all the corresponding submodules in $H_p \otimes V$ and $H_{p-1} \otimes \Lambda^2 V$. It is immediate that, for all p,

$$\frac{H_p \otimes \Lambda^2 V}{F_p} \oplus \frac{H_p \otimes V}{E_p}$$

is isomorphic to a submodule of H_p^{ad}. Moreover, this submodule coincides with H_p^{ad}, for all p, if and only if

$$\delta(E_p) = F_{p-1}, \qquad \text{for all } p. \tag{4.2}$$

Sigg conjectured in [5] that (4.2) holds, and he checked that

$$H_p^{\mathrm{ad}} \cong \frac{H_p \otimes \Lambda^2 V}{F_p} \oplus \frac{H_p \otimes V}{E_p}, \qquad \text{for } p \leq 2.$$

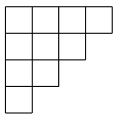
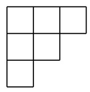
Let

$$\mathcal{S} = \{\lambda_I = (I; I) : I \subseteq [\![r]\!]\}$$

be the set of all self-conjugate partitions of length r and set

$$\mathcal{S}_o = \{\lambda_I \in \mathcal{S} : \min(I) > 1\} \quad \text{and} \quad \mathcal{S}_1 = \{\lambda_I \in \mathcal{S} : \min(I) = 1\}.$$

In terms of Young diagrams, a partition λ_I is in $\mathcal{S}_o$ if it is not possible to remove the last diagonal box of $Y(\lambda_I)$ to obtain a new Young diagram. On the other hand a partition λ_I is in $\mathcal{S}_1$ if, by removing the last diagonal box of $Y(\lambda_I)$, one obtains a new Young diagram. Thus, for example, $\lambda_I = (\{4,2\}; \{4,2\}) \in \mathcal{S}_0$ and $\lambda_J = (\{3,1\}; \{3,1\}) \in \mathcal{S}_1$ as it is clear from their diagrams.

$Y(\lambda_I) =$  $\qquad Y(\lambda_J) =$ 

Theorem 3.1 says that λ occurs in H_p if and only if $\lambda \in \mathcal{S}$ and $\mathbf{\Sigma}(\lambda) = 2p$. According to Proposition 2.2, a partition μ occurs in $H_p \otimes V$ if and only if $\mu = \lambda + \epsilon_s$ for some $1 \leq s \leq r$ with $\lambda \in \mathcal{S}$ and $\mathbf{\Sigma}(\lambda) = 2p$. We split the collection of these μ's into two classes:

(I) $\mu = \lambda + \epsilon_s$, $\lambda \in \mathcal{S}$ with $s \neq d_\lambda + 1$ if $\lambda \in \mathcal{S}_o$.

These μ's are all non self-conjugate and $\mathbf{\Sigma}(\mu) = 2p + 1$.

(II) $\mu = \lambda + \epsilon_{d_\lambda + 1}$, $\lambda \in \mathcal{S}_o$.

These μ's are all self-conjugate and $\mathbf{\Sigma}(\mu) = 2p + 2$.

Similarly, a partition μ occurs in $H_{p-1} \otimes \Lambda^2 V$ if and only if $\mu = \lambda + \epsilon_{s_1} + \epsilon_{s_2}$ for some $s_1 \neq s_2$, $\lambda \in \mathcal{S}$ and $\mathbf{\Sigma}(\lambda) = 2p - 2$. We split the collection of these μ's into two classes:

(III) $\mu = \lambda + \epsilon_{s_1} + \epsilon_{s_2}$, $\lambda \in \mathcal{S}$, $s_1 \neq s_2$, and both s_1 and s_2 different from $d_\lambda + 1$ if $\lambda \in \mathcal{S}_o$.

For these μ's $\mathbf{\Sigma}(\mu) = 2p$.

(IV) $\mu = \lambda + \epsilon_s + \epsilon_{d_\lambda + 1}$, $\lambda \in \mathcal{S}_o$, $s \neq d_\lambda + 1$.

These μ's are all non self-conjugate and $\mathbf{\Sigma}(\mu) = 2p + 1$.

From this picture it is clear that if a partition μ occurs simultaneously in $H_p \otimes V$ and in $H_{p-1} \otimes \Lambda^2 V$ it must be in classes (I) and (IV). Hence C_p is the intersection of the sets of partitions in class (I) and in class (IV). Moreover, we claim that any partition in class (IV) is in class (I) and therefore C_p coincides with class (IV). Indeed, given $\mu = \lambda' + \epsilon_s + \epsilon_{d_\lambda+1}$ in class (IV) consider $\lambda = \lambda' + \epsilon_{d_\lambda+1} \in \mathcal{S}_1$. It is immediate that $\mathbf{\Sigma}(\lambda) = 2p$ and hence $\mu = \lambda + \epsilon_s$ is in class (I). Notice that only those μ in class (I) coming from $\lambda \in \mathcal{S}_1$ and $s \neq d_\lambda$ are in C_p.

We must remark that for μ in either class (I) or (IV), the corresponding expressions $\mu = \lambda + \epsilon_s$ or $\mu = \lambda' + \epsilon_s + \epsilon_{d_{\lambda'}+1}$ are uniquely determined by μ. In particular, for $\mu \in C_p$, the multiplicity of $\mathcal{W}_\mu$ in E_p and F_{p-1} equals 1, and therefore the μ-isotypic components $E_p(\mu)$ and $F_{p-1}(\mu)$ are irreducible submodules of E_p and F_{p-1} respectively.

Summarizing, for $\mu \in C_p$ there exist unique $\lambda \in \mathcal{S}_1$, $\lambda' \in \mathcal{S}_o$ and $s \neq d_{\lambda'} + 1$ such that:

$$\lambda = \lambda' + \epsilon_{d_{\lambda'}+1} \qquad d_\lambda = d_{\lambda'} + 1; \tag{4.3}$$

$$\mu = \lambda + \epsilon_s \qquad d_\mu = d_\lambda; \tag{4.4}$$

$$\mu = \lambda' + \epsilon_{d_{\lambda'}+1} + \epsilon_s \qquad d_\mu = d_{\lambda'} + 1. \tag{4.5}$$

Our goal is to prove that $\delta_p(E_p(\mu)) = F_{p-1}(\mu)$.

4.2 The main theorem

We first describe the highest weight vectors in $E_p(\mu)$ and $F_{p-1}(\mu)$ explicitly enough to show then that, up to a non-zero scalar, δ_p maps one onto the other.

Choose $v_0 \in \Lambda^p \mathcal{L}(r) \otimes V$ such that $[v_0] \in H_p \otimes V$ is a non-zero highest weight vector of $E_p(\mu)$. Therefore $\partial_0 \otimes \mathrm{id}\,(v_0) = 0$ and

$$v_0 \in \big(\Lambda^p \mathcal{L}(r) \otimes V\big)(\mu).$$

More precisely, let $\lambda \in \mathcal{S}_1$ be such that $\mu = \lambda + \epsilon_s$ as in (4.4). Remark 3.2 implies that we may have chosen v_0 so that

$$v_0 \in \Big(\big(\Lambda^{d_\mu} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu)\big)(\lambda) \otimes V\Big)(\mu) \tag{4.6}$$

with $\nu = \lambda - (\epsilon_1 + \cdots + \epsilon_{d_\mu})$. From the definition we know that ∂_1 is the identity in the factor $\Lambda^{p-d_\mu}(\Lambda^2 V)$. Thus

$$\partial_1(v_0) \in \Big(\Lambda^{d_\mu - 1} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu) \otimes \Lambda^2 V\Big)(\mu). \tag{4.7}$$

Proposition 2.2 implies that we can decompose the μ-isotypic component (4.7) as a direct sum of irreducible subrepresentations in the following way:

$$\bigoplus_{\tau} \Big(\big(\Lambda^{d_\mu - 1} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu)\big)(\tau) \otimes \Lambda^2 V \Big)(\mu). \tag{4.8}$$

The sum is taken over all partitions τ such that:

(1) $Y(\tau)$ is obtained from $Y(\nu)$ by adding $d_\mu - 1$ boxes in $d_\mu - 1$ different rows, and

(2) $Y(\tau)$ is obtained from $Y(\mu)$ by removing two boxes from two different rows.

In other words, the collection of all partitions τ occurring in (4.8), $T_{\nu,\mu}$, is contained in either the set $\{\tau_{i,j} : i < j\}$ if $s > d_\mu$ or in the set $\{\tau_i\}$ if $s < d_\mu$, where

$$\tau_{i,j} = \begin{cases} \nu + \epsilon_1 + \cdots + \widehat{\epsilon_i} + \cdots + \widehat{\epsilon_j} + \cdots + \epsilon_{d_\mu} + \epsilon_s, & 1 \le i < j \le d_\mu; \\ \nu + \epsilon_1 + \cdots\cdots + \widehat{\epsilon_i} + \cdots\cdots\cdots + \epsilon_{d_\mu}, & 1 \le i \le d_\mu,\ j = s; \end{cases}$$
$$\tau_i = \nu + \epsilon_1 + \cdots\cdots + \widehat{\epsilon_i} + \cdots\cdots\cdots + \epsilon_{d_\mu}, \qquad 1 \le i \le d_\mu,\ i \ne s;$$

Recall that $s \ne d_\mu$ ($d_\mu = d_\lambda$) and notice that $\tau_{i,j}$ (or τ_i) is not necessary a partition and in fact it is a partition if and only if it is in $T_{\nu,\mu}$.

Let

$$\tau_o = \begin{cases} \tau_{d_\mu, s}, & \text{if } s > d_\mu; \\ \tau_{d_\mu}, & \text{if } s < d_\mu. \end{cases}$$

Since $\lambda = \nu + \epsilon_1 + \cdots + \epsilon_{d_\mu} \in \mathcal{S}_1$, τ_o is a partition and hence it is in $T_{\nu,\mu}$.

For $\tau \in T_{\nu,\mu}$, let $v_\tau \in \Big(\big(\Lambda^{d_\mu - 1} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu)\big)(\tau) \otimes \Lambda^2 V \Big)(\mu)$ be a non-zero highest weight vector. Recall that this space is irreducible. Then

$$\partial_1(v_0) = \sum_{\tau \in T_{\nu,\mu}} a_\tau v_\tau, \quad a_\tau \in \mathbb{C}. \tag{4.9}$$

From Theorem 3.1 we know that $[v_\tau]$, is zero for all $\tau \in T_{\nu,\mu}$ except for $\tau = \tau_o$. Therefore, $\delta_p(E_p(\mu)) = F_{p-1}(\mu)$ if and only if $a_{\tau_o} \ne 0$.

To distinguish τ_o from all other partitions in $T_{\nu,\mu}$, consider the linear maps

$$F_{i,j} : \Lambda\mathcal{L}(r) \otimes \Lambda^2 V \longrightarrow \Lambda\mathcal{L}(r)$$

defined for $i \neq j$ and $a < b$ by

$$F_{i,j}(u \otimes f_{ab}) = \begin{cases} 0, & \text{if } \{a,b\} \neq \{i,j\}; \\ u, & \text{if } (a,b) = (i,j) \text{ and } i < j; \\ -u, & \text{if } (a,b) = (j,i) \text{ and } i > j. \end{cases}$$

Lemma 4.3. *Let $\tau \in T_{\nu,\mu}$. If $s > d_\mu$ assume that $\tau = \tau_{ij}$ with $i < j$. If $s < d_\mu$ assume that $\tau = \tau_i$ with $i \neq s$. Then*

$$v_\tau = \begin{cases} \displaystyle\sum_{a=1}^{i} \sum_{b=a+1}^{j} w_{a,b} \otimes f_{ab}, & \textit{if } s > d_\mu; \\ \displaystyle\sum_{a=1}^{m=\min\{i,s\}} \sum_{b=a+1}^{M=\max\{i,s\}} w_{a,b} \otimes f_{ab}, & \textit{if } s < d_\mu; \end{cases}$$

where $w_{a,b} \in \big(\Lambda^{d_\mu - 1} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu)\big)(\tau)$ and $w_{i,j} \neq 0$ if $s > d_\mu$, $w_{m,M} \neq 0$ if $s < d_\mu$. In particular, $F_{d_\mu,s}(v_\tau) \neq 0$ if and only if $\tau = \tau_o$.

Proof. Since $\{f_{ab} : a < b\}$ is a basis of $\Lambda^2 V$, there exists a unique $w_{a,b} \in \big(\Lambda^{d_\mu - 1} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu)\big)(\tau)$ such that $v_\tau = \sum_{a<b\leq r} w_{a,b} \otimes f_{ab}$. Then $w_{a,b}$ is a weight vector of weight $\mu - \epsilon_a - \epsilon_b$ where μ is the weight of v_τ.

If w_0 is a highest weight vector in $\big(\Lambda^{d_\mu - 1} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu)\big)(\tau)$, then there exists $U_{a,b} \in U(\mathfrak{n}^-)$ such that $w_{a,b} = U_{a,b}.w_0$. This implies that the weight of $U_{a,b}$ is $\mu - \tau - \epsilon_a - \epsilon_b$. It must be either zero or a sum of negative roots. For $s > d_\mu$ we have

$$\mu - \tau - \epsilon_a - \epsilon_b = \epsilon_i + \epsilon_j - \epsilon_a - \epsilon_b$$

and hence $a \leq i$ and $b \leq j$ because $i < j$ and $a < b$. On the other hand, if $s < d_\mu$

$$\mu - \tau - \epsilon_a - \epsilon_b = \epsilon_i + \epsilon_s - \epsilon_a - \epsilon_b$$

and therefore $a \leq m = \min\{i, s\}$ and $b \leq M = \max\{i, s\}$.

It remains to show that $w_{i,j}$ (or $w_{m,M}$) is not zero. Since $v_\tau = \sum w_{a,b} \otimes f_{ab}$ is a non-zero highest vector and $\{f_{ab} : a < b\}$ is a linearly independent set, it is not difficult to see that one $w_{a,b}$ must be a non-zero multiple of w_0. From the weight of $w_{a,b}$ it is clear that the only possible candidate is $w_{i,j}$ (or $w_{m,M}$) and therefore it is not zero. □

We are now in a position to prove our main result.

Theorem 4.4. *Let $\mathcal{L}(r) = V \oplus \Lambda^2 V$ be the free 2-step nilpotent complex Lie algebra of rank r. Let $H_p(\mathcal{L}(r))$ be the p-homology space of $\mathcal{L}(r)$ with trivial coefficients, considered as a $GL(V)$-module. Let E_p and F_{p-1} be the maximal isomorphic submodules of $H_p(\mathcal{L}(r)) \otimes V$ and $H_{p-1}(\mathcal{L}(r)) \otimes \Lambda^2 V$ respectively. Then, for all p, the sequence of $GL(V)$-modules*

$$0 \longrightarrow \frac{H_p(\mathcal{L}(r)) \otimes \Lambda^2 V}{F_p} \longrightarrow H_p(\mathcal{L}(r), \mathcal{L}(r)) \longrightarrow \frac{H_p(\mathcal{L}(r)) \otimes V}{E_p} \longrightarrow 0$$

obtained from (4.1) is exact. In particular

$$H_p(\mathcal{L}(r), \mathcal{L}(r)) \cong \frac{H_p(\mathcal{L}(r)) \otimes \Lambda^2 V}{F_p} \oplus \frac{H_p(\mathcal{L}(r)) \otimes V}{E_p}.$$

Proof. As we already noticed the result will follow if we prove that $\delta(E_p) = F_{p-1}$ for all p, and this is equivalent to proving that $a_{\tau_0} \neq 0$ in equation (4.9). In view of Lemma 4.3, it suffices to show that $F_{d_\mu,s}(\partial_1(v_0)) \neq 0$. We use the explicit expression of v_0 given by Proposition 2.4 to compute $F_{d_\mu,s}(\partial_1(v_0))$. Recall that $\lambda \in \mathcal{S}_1$ and assume that $\lambda = (I; I)$ with $I = \{a_1, a_2, \dots, a_{d_\mu - 1}, a_{d_\mu} = 1\}$. Set

$$\begin{aligned} \alpha &= e_1 \wedge \dots \wedge e_{d_\mu - 1} \ ; \\ \beta &= f_{12} \wedge f_{13} \wedge \dots \wedge f_{1a_1} \wedge f_{23} \wedge f_{24} \wedge \dots \wedge f_{2(a_2+1)} \wedge \\ &\qquad\qquad \wedge f_{(d_\mu - 1)d_\mu} \wedge \dots \wedge f_{(d_\mu-1)(a_{d_\mu - 1} + d_\mu - 2)} \ . \end{aligned}$$

According to Remark 3.2, since $a_{d_\mu} = 1$,

$$\alpha \wedge e_{d_\mu} \wedge \beta \in \big(\Lambda^{d_\mu} V \otimes \Lambda^{p-d_\mu}(\Lambda^2 V)(\nu)\big)(\lambda)$$

is a non-zero highest weight vector. By Proposition 2.4 we may thus assume that

$$v_0 = \sum_{I \in A^{(s)}} \frac{1}{\sigma_I} U_I.(\alpha \wedge e_{d_\mu} \wedge \beta) \otimes e_{\min(I)}.$$

When applying $F_{d_\mu,s} \circ \partial^1$ to v_0, all the summands are killed except those corresponding to sets $I \in A^{(s)}$ such that either $\min(I) = s$ or $\min(I) = d_\mu$. The only set $I \in A^{(s)}$ such that $\min(I) = s$ is $I = \{s\}$. If $s < d_\mu$ there is no set $I \in A^{(s)}$ with $\min(I) = d_\mu$. Therefore

$$\begin{aligned} F_{d_\mu,s} \circ \partial^1(v_0) &= F_{d_\mu,s} \circ \partial^1(\alpha \wedge e_{d_\mu} \wedge \beta \otimes e_s) \\ &= (-1)^{d_\mu - 1} \alpha \wedge \beta \end{aligned}$$

and we are done. If $s > d_\mu$, then

$$
\begin{aligned}
F_{d_\mu,s}&\circ\partial^1(v_0) = \\
&= F_{d_\mu,s} \circ \partial^1 \left(\alpha \wedge e_{d_\mu} \wedge \beta \otimes e_s + \sum_{I \in A^{(s)}_{(d_\mu)}} \frac{1}{\sigma_I} U_I.(\alpha \wedge e_{d_\mu} \wedge \beta) \otimes e_{d_\mu} \right) \\
&= (-1)^{d_\mu - 1} \alpha \wedge \beta + F_{d_\mu,s} \circ \partial^1 \left(\sum_{I \in A^{(s)}_{(d_\mu)}} \frac{1}{\sigma_I} \alpha \wedge U_I.(e_{d_\mu} \wedge \beta) \otimes e_{d_\mu} \right) \\
&= (-1)^{d_\mu - 1} \alpha \wedge \beta + F_{d_\mu,s} \circ \partial^1 \left(\frac{1}{\sigma_{\{d_\mu,s\}}} \alpha \wedge U_{\{d_\mu,s\}}.(e_{d_\mu} \wedge \beta) \otimes e_{d_\mu} \right) \\
&= (-1)^{d_\mu - 1} \alpha \wedge \beta + F_{d_\mu,s} \circ \partial^1 \left(\frac{1}{\sigma_{\{d_\mu,s\}}} \alpha \wedge e_s \wedge \beta \otimes e_{d_\mu} \right) \\
&= (-1)^{d_\mu - 1} \alpha \wedge \beta + \frac{(-1)^{d_\mu}}{\lambda_s - \lambda_{d_\mu} - s + d_\mu + 1} \alpha \wedge \beta \\
&= (-1)^{d_\mu - 1} \frac{\lambda_s - \lambda_{d_\mu} - s + d_\mu}{\lambda_s - \lambda_{d_\mu} - s + d_\mu + 1} \, \alpha \wedge \beta
\end{aligned}
$$

This last vector is not zero and hence the theorem is proved. □

Finally we give an explicit description of the $GL(V)$-module structure of $H_*(\mathcal{L}(r), \mathcal{L}(r))$ determining the multiplicity of each irreducible $GL(V)$-module that occurs.

It will be convenient to ignore the distinction between a partition μ and the corresponding Young diagram $Y(\mu)$.

In addition to the class $\mathcal{S}$ of self-conjugate diagrams (of length r) already defined, let we consider these two classes:

- $\mathcal{S}'$: the set of all diagrams that fail to be self-conjugate by one extra box, such that is not possible to get a new diagram by removing the last diagonal box and the box immediately below it (if present);
- $\mathcal{S}''$: the set of all diagrams that fail to be self-conjugate by two extra boxes, such that these two boxes are not in the same row.

Theorem 4.5. *Let $\mathcal{L}(r) = V \oplus \Lambda^2 V$ be the free 2-step nilpotent complex Lie algebra of rank r. Then*

$$H_*(\mathcal{L}(r), \mathcal{L}(r)) = \bigoplus_{\mu \in \mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}''} m(\mu) \, \mathcal{W}_\mu$$

where $m(\mu) = 1$ *if* $\mu \in \mathcal{S}' \cup \mathcal{S}''$ *or*

$$m(\mu) = \#\{i : \alpha_i > \alpha_{i+1} + 1,\ 1 \le i < n\} + 1$$

if $\mu = (I; I) \in \mathcal{S}$, $I = \{\alpha_1, \ldots, \alpha_n\}$. *Moreover, for* $\mu \in \mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}''$ *the homology grading is* $p = \left[\frac{\boldsymbol{\Sigma}(\mu)-1}{2}\right]$.

Proof. Recall the description of $H_p \otimes V$ and $H_p \otimes \Lambda^2 V$ from 4.1. Notice that we have to consider classes (III) and (IV) for p rather than for $p-1$.

It follows from Theorem 4.4 that $H_p(\mathcal{L}(r), \mathcal{L}(r))$ is isomorphic to the direct sum of all diagrams in classes (II) and (III) plus some diagrams in class (I). It is convenient to further split class (III) into two sub-classes (IIIa) and (IIIb) consisting of self-conjugate and non-self-conjugate diagrams respectively.

It is clear that μ is in class (IIIb) if and only if $Y(\mu) \in \mathcal{S}''$ and $\boldsymbol{\Sigma}(\mu) = 2p+2$. Every diagram $Y(\mu)$ in $\mathcal{S}''$ determines, by definition, a unique self-conjugate diagram $Y(\lambda)$. Since the multiplicity of λ in $H_*(\mathcal{L}(r))$ is 1, so it is the multiplicity of μ in $H_*(\mathcal{L}(r), \mathcal{L}(r))$.

It is straightforward to see that a partition μ is in class (I) and not in class (IV) if and only if $Y(\mu)$ is in $\mathcal{S}'$ and $\boldsymbol{\Sigma}(\mu) = 2p+1$. It follows as in the previous case that the multiplicity of μ in $H_*(\mathcal{L}(r), \mathcal{L}(r))$ is 1.

Finally, if μ is a self-conjugate partition with $\boldsymbol{\Sigma}(\mu) = 2p+2$, then the diagram $Y(\mu)$ occurs in $H_*(\mathcal{L}(r), \mathcal{L}(r))$ several times. More precisely, its multiplicity is equal to the number of self-conjugate Young diagrams that can be obtained from $Y(\mu)$ by removing any two boxes or by removing the diagonal box. This gives $m(\mu)$. Notice that in the first case μ falls in class (IIIa) and in the second case μ falls in class (II). □

Example 4.6. Let us consider $\mathcal{L}(3)$. Recall that the trivial homology of $\mathcal{L}(3)$ is given by all the self-conjugate diagrams $Y(\lambda)$ with grading $p = \frac{\boldsymbol{\Sigma}(\lambda)}{2}$ as shown in the first column of the following table. In the second and third columns, the Young diagram decomposition of $H_p \otimes V$ and $H_p \otimes \Lambda^2 V$ is displayed. Using Theorem 4.4 it is straightforward to compute the adjoint homology of $\mathcal{L}(3)$. The resulting Young diagrams are shown in bold face.

p	H_p	$H_p \otimes V$	$H_p \otimes \Lambda^2 V$
0		**(1)**	(1,1)
1	(1)	**(2)**, (1,1)	**(2,1)**, **(1,1,1)**
2	(2,1)	**(3,1)**, **(2,2)**, **(2,1,1)**	(3,2), **(3,1,1)**, (2,2,1)
3	(2,2), (3,1,1)	(3,2), (2,2,1), **(4,1,1)**, **(3,2,1)**	**(3,3)**, **(3,2,1)**, (4,2,1), (3,2,2)
4	(3,2,1)	(4,2,1), **(3,3,1)**, (3,2,2)	**(4,3,1)**, **(4,2,2)**, **(3,3,2)**
5	(3,3,2)	**(4,3,2)**, **(3,3,3)**	**(4,4,2)**, (4,3,3)
6	(3,3,3)	(4,3,3)	**(4,4,3)**

Example 4.7. Even though the case of $\mathcal{L}(4)$ is analogous to that of $\mathcal{L}(3)$ we present it because it helps to illustrate an interesting duality. We will refer to it in Remark 4.8. In this case instead of displaying Young diagrams we list partitions. The partitions that occur in the adjoint homology groups are in bold face.

p	H_p	$H_p \otimes V$	$H_p \otimes \Lambda^2 V$
0		**(1,0,0,0)**	(1,1,0,0)
1	(1,0,0,0)	**(2,0,0,0)** (1,1,0,0)	**(2,1,0,0)** **(1,1,1,0)**
2	(2,1,0,0)	**(3,1,0,0)** **(2,2,0,0)** **(2,1,1,0)**	(3,2,0,0) **(3,1,1,0)** (2,2,1,0) **(2,1,1,1)**
3	(2,2,0,0) (3,1,1,0)	(3,2,0,0) (2,2,1,0) **(4,1,1,0)** **(3,2,1,0)** **(3,1,1,1)**	**(3,3,0,0)** **(3,2,1,0)** **(2,2,1,1)** (4,2,1,0) **(4,1,1,1)** (3,2,2,0) (3,2,1,1)
4	(3,2,1,0) (4,1,1,1)	(4,2,1,0) **(3,3,1,0)** (3,2,2,0) (3,2,1,1) **(5,1,1,1)** **(4,2,1,1)**	**(4,3,1,0)** **(4,2,2,0)** **(4,2,1,1)** **(3,3,2,0)** **(3,3,1,1)** **(3,2,2,1)** (5,2,1,1) (4,2,2,1)
5	(4,2,1,1) (3,3,2,0)	(5,2,1,1) **(4,3,1,1)** (4,2,2,1) **(4,3,2,0)** **(3,3,3,0)** **(3,3,2,1)**	**(5,3,1,1)** **(5,2,2,1)** **(4,3,2,1)** **(4,2,2,2)** **(4,4,2,0)** (4,3,3,0) **(4,3,2,1)** (3,3,3,1)
6	(4,3,2,1) (3,3,3,0)	**(5,3,2,1)** **(4,4,2,1)** **(4,3,3,1)** **(4,3,2,2)** (4,3,3,0) (3,3,3,1)	**(5,4,2,1)** (5,3,3,1) **(5,3,2,2)** (4,4,3,1) **(4,4,2,2)** (4,3,3,2) **(4,4,3,0)** **(4,3,3,1)**
7	(4,4,2,2) (4,3,3,1)	**(5,4,2,2)** **(4,4,3,2)** (5,3,3,1) (4,4,3,1) (4,3,3,2)	**(5,5,2,2)** (5,4,3,2) (4,4,3,3) **(5,4,3,1)** **(5,3,3,2)** **(4,4,4,1)** **(4,4,3,2)**
8	(4,4,3,2)	(5,4,3,2) **(4,4,4,2)** (4,4,3,3)	**(5,5,3,2)** **(5,4,4,2)** **(5,4,3,3)** **(4,4,4,3)**
9	(4,4,4,3)	**(5,4,4,3)** **(4,4,4,4)**	**(5,5,4,3)** (5,4,4,4)
10	(4,4,4,4)	(5,4,4,4)	**(5,5,4,4)**

Remark 4.8. By inspection we see that the number of irreducible representations in each adjoint homology group are in duality.
If $h_p = \#\{\text{irreducible representations in } H_p(\mathcal{L}(r), \mathcal{L}(r))\}$, then

$$h_i = h_{\dim \mathcal{L}(r) - i}.$$

Moreover, this duality follows from the remarkable fact that the number of partitions in $H_p \otimes V$ not in H_p^{ad} is the number of partitions in $H_{\dim \mathcal{L}(r)-p} \otimes \Lambda^2 V$ not in $H^{\text{ad}}_{\dim \mathcal{L}(r)-p}$.

5 Dimension

If λ is a partition of length r let us denote by λ^* the partition corresponding to the irreducible representation $\mathcal{W}_\lambda^* \otimes \det^r$ where $\mathcal{W}_\lambda^*$ is the dual representation of $\mathcal{W}_\lambda$. The Young diagrams $Y(\lambda)$ and $Y(\lambda^*)$ are complementary to each other in the $r \times r$ square diagram, provided that the conjugate of λ is of length r. The following proposition follows directly from Theorem 3.1.

Proposition 5.1. *The map* $^* : \mathcal{S}_o \to \mathcal{S}_1$ *is a bijection and hence*

$$\sum_{\lambda \in \mathcal{S}_o} \dim \mathcal{W}_\lambda = \sum_{\lambda \in \mathcal{S}_1} \dim \mathcal{W}_\lambda = \frac{1}{2} \dim H_*(\mathcal{L}(r)).$$

Recall that we have seen in 4.1 that,

$$\begin{aligned}\bigcup_p C_p &= \{\mu : \mu = \lambda + \epsilon_s : \lambda \in \mathcal{S}_1 \text{ and } s \neq d_\lambda\} \\ &= \{\mu : \mu \text{ occurs in } \sum_{\lambda \in \mathcal{S}_1} \mathcal{W}_\lambda \otimes V \text{ but } \mu \neq \lambda + \epsilon_{d_\lambda}\}.\end{aligned}$$

In particular

$$\dim \sum_p E_p = \dim \sum_p F_p = \frac{1}{2} r \dim H_*(\mathcal{L}(r)) - g_r$$

where $g_r = \dim \sum_{\lambda \in \mathcal{S}_1} \mathcal{W}_{\lambda + \epsilon_{d_\lambda}}$. Therefore, from Theorem 4.4 we obtain

$$\begin{aligned}\dim H_*(\mathcal{L}(r), \mathcal{L}(r)) &= \dim \sum_p \frac{H_p \otimes \Lambda^2 V}{F_p} \oplus \frac{H_p \otimes V}{E_p} \\ &= \dim H_*(\mathcal{L}(r)) \dim \mathcal{L}(r) - \dim \sum_p E_p - \dim \sum_p F_p \\ &= \frac{1}{2} r(r-1) \dim H_*(\mathcal{L}(r)) + 2 g_r \qquad (5.1)\end{aligned}$$

In [3] an explicit formula for $\dim H_*(\mathcal{L}(r))$ is given, so we only need to compute g_r. This will be done following closely the computation of $\dim H_*(\mathcal{L}(r))$ in [3].

Using the 'second Giambelli formula' [2, (24,11)] we have

$$g_r = \sum_{\substack{\mu = \lambda + \epsilon_{d_\lambda} \\ \lambda \in \mathcal{S}_1}} \det G_1(\mu, r)\,, \text{ where } G_1(\mu, r)_{i,j} = \binom{r}{\mu_i' + j - i}.$$

We observe that $\mu = \lambda + \epsilon_{d_\lambda}$ with $\lambda \in \mathcal{S}_1$ if and only if $\mu = (I \cup \{2\}; I \cup \{1\})$ for some $I \subseteq \{3, \ldots, r\}$.

We may also describe the Giambelli matrix $G_1(\mu, r)$ in terms of the Frobenius notation as follows. For $I \subseteq \{3, \ldots, r\}$, let

$$G_2(I,r)_{ij} = \begin{cases} \binom{r}{j+i-1}, & \text{if } i \in I \text{ or } i = 1; \\ \binom{r}{j-1}, & \text{if } i = 2; \\ \binom{r}{j-i}, & \text{if } i \notin I \text{ and } i \neq 1, 2. \end{cases}$$

Then $G_1(\lambda, r)$ is the matrix obtained from $G_2(I, r)$ by performing a row permutation of sign $(-1)^{\# I_0}$, where I_0 is the set of even numbers in I. Since the determinant is linear in rows we can express g_r as the determinant of a single matrix, namely

$$g_r = \det G_3(r),$$

where

$$G_3(r)_{ij} = \begin{cases} \binom{r}{j}, & \text{if } i = 1; \\ \binom{r}{j-1}, & \text{if } i = 2; \\ \binom{r}{j-i} - (-1)^i \binom{r}{j+i-1}, & \text{if } i \geq 3. \end{cases}$$

We recall that (see [3] §2)

$$\dim H_*(\mathcal{L}(r)) = \det \overline{G}(r), \text{ where } \overline{G}(r)_{ij} = \binom{r}{j-i} - (-1)^i \binom{r}{j+i-1}.$$

Notice that $\overline{G}(r)$ and $G_3(r)$ only differ in the first two rows. For convenience we define

$$G(r)_{ij} = \begin{cases} \binom{r}{j}, & \text{if } i = 2; \\ \binom{r}{j-i} - (-1)^i \binom{r}{j+i-1}, & \text{if } i \neq 2. \end{cases}$$

Now

$$g_r = -\det G(r)$$

and $\overline{G}(r)$ and $G(r)$ only differ in the second row. The goal is to prove that

$$\det G(r) = -\frac{1}{4} \frac{\left[\frac{r+1}{2}\right]}{\left[\frac{r+1}{2}\right] - \frac{1}{2}} \det \overline{G}(r). \tag{5.2}$$

Let

$$A_1(r) = \begin{bmatrix} 1 & 0 & & & \\ 1/2 & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \quad \text{and} \quad A_2(r) = \frac{1}{2} \begin{bmatrix} 1 & & & & & -1 \\ & 1 & & & -1 & \\ & & \ddots & \cdot^{\cdot^{\cdot}} & & \\ & & \cdot^{\cdot^{\cdot}} & \ddots & & \\ & 1 & & & 1 & \\ 1 & & & & & 1 \end{bmatrix},$$

with $A_2(r)_{(n+1)(n+1)} = 1$ if $r = 2n+1$. The products $A_1(r)G(r)A_2(r)$ and $\overline{G}(r)A_2(r)$ differ only in the second row. Moreover, collecting together the odd rows on the top and the even rows on the bottom the matrices become two-block matrices. Their first blocks are the same and the second ones are respectively $C(r)$ and $\overline{C}(r)$, with $C(r)_{ij} = \overline{C}(r)_{ij}$ for $i \neq 1$ and

$$\overline{C}(r)_{ij} = \binom{r}{\left[\frac{r+1}{2}\right] + j - 2i} - \binom{r}{\left[\frac{r+1}{2}\right] + j + 2i - 1}, \quad 1 \leq i, j \leq \left[\frac{r}{2}\right];$$
$$C(r)_{1j} = \frac{1}{2}\binom{r}{\left[\frac{r+1}{2}\right] + j - 1} - \frac{1}{2}\binom{r}{\left[\frac{r+1}{2}\right] + j}, \quad 1 \leq j \leq \left[\frac{r}{2}\right].$$

It turns out that for $r = 2n+2$, $C(r)$ has it last row equal to $(0, \ldots, 0, 1)$ and the upper-left block has its j-th column equal to the sum of the j-th and $(j-1)$-th columns of $C(r-1)$ for $j > 1$. The same happens for $\overline{C}(r)$. Hence we only need to show that

$$\det C(2n+1) = -\frac{n+1}{2(2n+1)} \det \overline{C}(2n+1).$$

Let I be the identity matrix and let J be the $n \times n$ matrix whose only non-zero coefficients are 1's right above the main diagonal. Set $D(n) = (I-J)^{-1}C(2n+1)(I+J)(I-J)$ and $\overline{D}(n) = (I-J)^{-1}\overline{C}(2n+1)(I+J)(I-J)$. Now redefine $D(n)$ and $\overline{D}(n)$ dividing by 2 their first columns. Thus $D(n)_{ij} = \overline{D}(n)_{ij}$ for $i \neq 1$ and

$$\overline{D}(n)_{ij} = \binom{2n+1}{n+j-2i+1} + \binom{2n+1}{n+j+2i-2}, 1 \leq i, j \leq n,$$

$$D(n)_{1j} = d_j^1 + d_j^2 + d_j^3, 1 \leq j \leq n,$$

with $d_j^1 = \frac{1}{2}\binom{2n+1}{n+j} + \frac{1}{2}\binom{2n+1}{n+j-1}$, $d_j^2 = -\frac{1}{2}\binom{2n+1}{n+j+1} - \frac{1}{2}\binom{2n+1}{n+j-2}$ and $d_j^3 = \binom{2n+1}{n+j+2} + \binom{2n+1}{n+j-3}$.

We still have to prove that $\det D(n) = -\frac{n+1}{2(2n+1)} \det \overline{D}(n)$. From the Giambelli formulas in [2, Corollary 24.35] we know that $\det \overline{D}(n)$ is twice the dimension of the irreducible representation W_1 of $SO(2n+1)$ with highest weight $n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n$. On the other hand, using the linearity of the determinant in the first row we can express $\det D(n)$ as a sum of three determinants d^1, d^2 and d^3, corresponding to the matrices with first rows d_j^1, d_j^2 and d_j^3. It is clear that $d^3 = 0$, $d^1 = \frac{1}{2}\det \overline{D}(n)$ and $-d^2$ is the dimension of the irreducible $SO(2n+1)$ representation W_2 with highest weight $n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1}$. In other words

$$\det D(n) = d^1 + d^2 + d^3 = \dim W_1 - \dim W_2.$$

From the Weyl character formula it follows that $\dim W_2 = \frac{n}{2n+1}\dim W_1$ which implies that $\det D(n) = -\frac{n+1}{2(2n+1)}\det \overline{D}(n)$ and therefore (5.2) is proved.

Theorem 5.2. *Let $\mathcal{L}(r) = V \oplus \Lambda^2 V$ be the free 2-step nilpotent complex Lie algebra of rank r. Then*

$$\begin{aligned}\dim H_*(\mathcal{L}(r), \mathcal{L}(r)) &= \frac{1}{2}\Big(r(r-1) + \frac{\left[\frac{r+1}{2}\right]}{\left[\frac{r+1}{2}\right] - \frac{1}{2}}\Big)\dim H_*(\mathcal{L}(r)) \\ &= \begin{cases} 2^n(c_n - 2(2n+1))\beta(n)^2, & \text{if } r = 2n+1; \\ 2^n c_n \beta(n)\beta(n+1) & \text{if } r = 2n+2 \end{cases}\end{aligned}$$

where $\beta(n) = \prod_{1\le i\le j\le n} \frac{2(i+j)-1}{2i-1}$ for $n > 0$ and $\beta(0) = 1$, and $c_n = (2n+1)(2n+2) + \frac{2n+2}{2n+1}$.

Proof. It is a straightforward consequence of Theorem 1.1 of [3] and equations 5.1 and 5.2. □

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