Evidence of exactness of the mean-field theory in the nonextensive regime of long-range classical spin models

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The q-state Potts model with long-range interactions that decay as $1/r^{\alpha}$ subjected to a uniform magnetic field on d-dimensional lattices is analyzed for different values of q in the nonextensive regime $0 \leqslant \alpha \leqslant d$. We calculate the mean-field solution of the model for all q and performed, for some values of q, Monte Carlo simulations for the spontaneous magnetization in the one-dimensional case. We show that, using a derived scaling which properly describes the nonextensive thermodynamic behavior, both types of calculations present an excellent agreement for $0 \leqslant \alpha < d$. We also consider the two-dimensional antiferromagnetic Ising model with competing antiferromagnetic long-range interactions and ferromagnetic first neighbor ones in the presence of a uniform magnetic field. We calculate the mean-field magnetization for this case and compare it with Monte Carlo numerical data from Sampaio *et al.* They also show a very good agreement for $\alpha < d$. These results, together with some previous ones, led us to conjecture that the mean-field theory is exact for nonextensive classical spin models with $0 \leqslant \alpha < d$.

I. INTRODUCTION

Microscopic pair interactions that decay slowly with the distance r between particles appear in different physical systems. Typical examples are gravitational and Coulomb interactions, where the potential decays as $1/r$. Several other important examples can be found in condensed matter, such as dipolar (both electric and magnetic) and Ruderman-Kittel-Kasuya-Yosida (RKKY) interactions, both proportional to $1/r^3$. Effective interactions with a power-law decay $1/r^{\alpha}$, with some exponent $\alpha \geqslant 0$, appear also in other related problems such as critical phenomena in highly ionic systems,[1] Casimir forces between inert uncharged particles immersed in a fluid near the critical point,[2] and phase segregation in model alloys.[3]

It is known that some of these systems can exhibit nonextensive thermodynamic behavior (see Refs. 4 and 5 and references therein). In other words, for small enough values of the ratio α/d the free energy $F = -\ln Z/\beta$, with $Z \equiv \mathrm{Tr}\exp(-\beta H)$ (H being the Hamiltonian of the system, d the dimensionality, and $\beta \equiv 1/k_B T$), grows faster than the number N of microscopic elements when $N \to \infty$, and the thermodynamic limit is not well defined.

In a recent paper[5] two of us analyzed the thermodynamics associated with the long-range (LR) ferromagnetic Ising Hamiltonian

$$H = -\sum_{(i,j)} J(r_{ij}) S_i S_j \quad (S_i = \pm 1 \forall i), \tag{1}$$

with

$$J(r_{ij}) = \frac{J}{r_{ij}^{\alpha}} \quad (J>0; \alpha \geqslant 0), \tag{2}$$

(where r_{ij} is the distance in crystal units) between sites i and j, and where the sum $\Sigma_{(i,j)}$ runs over all distinct pairs of sites on a d-dimensional hypercubic lattice. It was shown[5] that the quantities per particle (free energy f, internal energy u, entropy s, and magnetization m of a finite model of N spins) behave according to Tsallis conjecture[6] for $N \gg 1$. These quantities present, in the presence of an external magnetic field h, the following asymptotic scaling behaviors:

$$u(N,T,h) \sim N^* u'(T/N^*, h/N^*), \tag{3}$$

$$f(N,T,h) \sim N^* f'(T/N^*, h/N^*), \tag{4}$$

$$s(N,T,h) \sim s'(T/N^*, h/N^*), \tag{5}$$

$$m(N,T,h) \sim m'(T/N^*, h/N^*) \tag{6}$$

for all $\alpha \geqslant 0$, where the functions u', f', s', and m' are the corresponding quantities associated with the same model but with rescaled coupling $J'(r_{ij}) = J(r_{ij})/N^*$ (these functions are independent of the system size N) and the function $N^*(\alpha)$ is defined as

$$N^*(\alpha) = \frac{1}{1-\alpha/d}(N^{1-\alpha/d} - 1) \tag{7}$$

which behaves, for $N \to \infty$, as

$$N^*(\alpha)\sim\begin{cases}\dfrac{1}{\alpha/d-1} & \text{for } \alpha/d>1\\ \ln N & \text{for } \alpha/d=1\\ \dfrac{1}{1-\alpha/d}N^{1-\alpha/d} & \text{for } 0\leqslant\alpha/d<1.\end{cases} \quad (8)$$

For $\alpha>d$ the thermodynamic functions per site do not depend on N and the system is extensive (i.e., the thermodynamic limit exists). When $0\leqslant\alpha\leqslant d$, $N^*(\alpha)$ diverges for $N\to\infty$ and the system is nonextensive. Also presented was numerical evidence[5,7] that for $d=1$ the mean-field (MF) theory becomes *exact* when $0\leqslant\alpha<d$. This led two of us[5] to conjecture that the mean-field theory might be exact for the nonextensive Ising model.

Mean-field theory accurately describes the thermodynamic behavior of sufficiently high dimensional spin systems. In fact, it becomes exact in the limit $d\to\infty$. However, for low dimensional systems, where fluctuations play an important role, significant departures from MF theory are expected, at least for short-range models. Hence the accurate agreement between numerical and MF solutions for the whole magnetic equation of state[5] and for the correlation function[7] in the above $d=1$ model is remarkable, suggesting that MF theory may be ''universal'' for nonextensive classical spin models, in the sense that it describes exactly its thermodynamical behavior independently of the dimension. Therefore it is interesting to check this hypothesis in different low dimensional systems.

In this work we extend the previous analysis[5] and present other evidence of the exactness of the MF theory for $0\leqslant\alpha<d$ for the one-dimensional ferromagnetic q-state LR Potts model for different values of q (including the first-order phase transition predicted by MF theory for $q>2$) and also for the two-dimensional antiferromagnetic LR Ising model in an external field. Our mean-field calculation for the q-state Potts model presented here also agrees in the $q\to1$ limit with Monte Carlo calculations for the long-range percolation problem in $d=1$ obtained by other authors.[8] Our results, together with some previous ones, led us to conjecture that the MF theory might be exact for any nonextensive classical spin model excluding the borderline case $\alpha=d$, where there are probably corrections to the MF results (some previous experimental results[9] on dipolar ferromagnetic materials do not exclude this possibility).

The outline of this paper is the following. In Sec. II we analyze the ferromagnetic q-state LR Potts model subjected to a uniform magnetic field h. First, we show that the previous analysis[5] for the Ising model ($q=2$) in the nonextensive region is straightforwardly extended to the generic $q>2$ case. Then, in Sec. II A we derive the mean-field solution of this model for arbitrary values of q, α, and h, in particular the MF predictions for the LR bond percolation which correspond to the $q\to1$ and $h\to0^+$ limit. In Sec. II B we compare the MF solution with our Monte Carlo simulation of the one-dimensional model for $h=0$, $q=2$, 3, and 5 and different values of α. In Sec. III we calculate the mean-field solution of the two-dimensional Ising model with competing LR antiferromagnetic and short-range ferromagnetic interactions in an external field and compare them with the Monte Carlo results of Sampaio, de Albuquerque, and Menezes.[10] Some comments and conclusions are presented in Sec. IV.

II. POTTS MODEL WITH LONG-RANGE FERROMAGNETIC INTERACTIONS

In this section we address the LR ferromagnetic q state Potts model, i.e., we consider the Hamiltonian

$$H=-\frac{1}{2}\sum_{i,j} J(r_{ij})\delta(\sigma_i,\sigma_j)-h\sum_i \delta(\sigma_i,1)$$
$$(\sigma_i=1,2,\ldots,q,\forall i), \quad (9)$$

where to each site i we associate a spin variable σ_i, which can assume q integer values; the sum $\Sigma_{i,j}$ runs over all pairs of sites of a d-dimensional lattice of N sites ($i\neq j$); δ is the Kronecker delta function, $J(r_{ij})$ is given by Eq. (2), and h is an external magnetic field in the $\sigma=1$ direction. The $\alpha\to\infty$ limit corresponds to the first neighbor model. For $q=2$ the $\alpha=0$ limit corresponds, after a rescaling $J\to J/N$, to the Curie-Weiss model.

This model, in its plain formulation [$\alpha\to\infty$ of Eq. (9)] or in a more general one with many-body interactions, is at the heart of a complex network of relations between geometrical and/or thermal statistical models, like, for example, various types of percolation, vertex models, generalized resistor and diode network problems, classical spin models, etc. (see Ref. 11 and references therein).

On the other hand, the Potts model with LR interactions has been much less studied. In the extensive regime $\alpha>d$ it presents a very rich thermodynamic behavior, even in the one-dimensional case.[12–15] To the best of our knowledge, no study has been carried out for the nonextensive regime $0\leqslant\alpha\leqslant d$.

Let us introduce the sums $\phi_i(\alpha)=\Sigma_{j\neq i}1/r_{ij}^{\alpha}$. A sufficient condition (and believed to be necessary[16,17]) for the existence of the thermodynamic limit of this system is that

$$\phi(\alpha)=\lim_{N\to\infty}\frac{1}{N}\sum_i \phi_i(\alpha)<\infty. \quad (10)$$

Let us now consider a d-dimensional hypercube of side $L+1$ and $N=(L+1)^d$, and let $i=0$ be the central site of the hypercube. We have that

$$\phi(\alpha)=\lim_{N\to\infty}\phi_0(\alpha). \quad (11)$$

When $L\gg1$ ($N\gg1$) $\phi_0(\alpha)$ shows the following asymptotic behavior:[5]

$$\phi_0(\alpha)\sim C_d(\alpha)2^{\alpha}N^*(\alpha), \quad (12)$$

where $N^*(\alpha)$ is given by Eq. (7) and $C_d(\alpha)$ is a continuous function[5] of α independent of N, with $C_d(0)=1$ $\forall d$. It can be proved that[5]

$$C_1(\alpha)=\begin{cases}1 & \text{for } 0\leqslant\alpha\leqslant1\\ \dfrac{\alpha-1}{2^{\alpha-1}}\zeta(\alpha) & \text{for } \alpha>1\end{cases}, \quad (13)$$

where $\zeta(x)$ is the Riemann zeta function. From Eqs. (7) and (12) we see that the thermodynamic limit is well defined for $\alpha > d$ (where the system presents extensive behavior), while for $\alpha \leqslant d$ the system becomes nonextensive. Following the same procedure as in Ref. 5 it can be shown that the scaling behaviors (3)-(6) of the thermodynamic functions hold $\forall q \geqslant 2$ and $\forall \alpha \geqslant 0$. For $q=2$ the system undergoes a second-order phase transition at finite temperature for all $\alpha > d$ when $d \geqslant 2$[18], and for $1 \leqslant \alpha \leqslant 2$ when $d=1$.[19] For $\alpha \to d^+$, the critical temperature of the Ising model shows the following asymptotic behavior:[18]

$$k_B T_c \sim J \phi(\alpha). \tag{14}$$

For $d \geqslant 2$ and $\alpha \gg d$ (short-range case[11]) there exists a critical value q_c such that the phase transition is a second-order one when $q \leqslant q_c(d)$ ($q_c=4$ for $d=2$) and a first-order one for $q > q_c(d)$. For $d=1$ and $1 < \alpha \leqslant 2$ Monte Carlo simulations[14,15] show that, for $q>2$, there is an α-dependent threshold value $q_c(\alpha)$ such that the transition is of first order when $q > q_c(\alpha)$ and of second order for $q \leqslant q_c(\alpha)$.

A. Mean-field theory

In order to develop a mean-field version of Hamiltonian (9) we use Mittag and Stephen's[20] spin representation for the Potts model, i.e., we associate to each site j a spin variable λ_j which can take the values $\lambda_j = 1, \omega, \omega^2, \ldots, \omega^{q-1}$, where $\omega = e^{2\pi i/q}$ is a qth root of unity. In other words, if the site j is in the state σ then $\lambda_j = \omega^{\sigma-1}$. Then, using the property

$$q^{-1} \sum_{k=1}^{q} \lambda^k \lambda'^{\,q-k} = \delta(\lambda, \lambda') \tag{15}$$

we can rewrite the Hamiltonian (9) as

$$H = -\frac{1}{2q} \sum_{i,j} J(r_{ij}) \sum_{l=1}^{q-1} \lambda_i^l \lambda_j^{q-l} - \frac{h}{q} \sum_i \sum_{l=1}^{q-1} \lambda_i^l - C(J,h), \tag{16}$$

where the constant term $C(J,h)$ is

$$C(J,h) = \frac{1}{2q} \sum_{i,j} J(r_{ij}) + \frac{hN}{q}. \tag{17}$$

The fraction of sites in the state σ, $n_\sigma = (1/N) \Sigma_i \langle \delta(\sigma_i - \sigma) \rangle$, in this representation is given by

$$n_\sigma = \frac{1}{q} \left[1 + \sum_{l=1}^{q-1} \omega^{q-l(\sigma-1)} \langle \lambda^l \rangle \right], \tag{18}$$

and the order parameter for a symmetry breaking in the $\sigma = 1$ direction is

$$m = \frac{q n_1 - 1}{q-1}, \tag{19}$$

which can be written, using Eq. (18), as

$$m = \frac{1}{q-1} \sum_{l=1}^{q-1} \langle \lambda^l \rangle. \tag{20}$$

The mean-field solution for this model can be easily found from the variational method in Ref. 21 by using a noninteracting trial Hamiltonian H_0 given by

$$H_0 = -\eta \sum_{i=1}^{N} \sum_{l=1}^{q-1} \lambda_i^l, \tag{21}$$

where η is the variational parameter to be found as a function of temperature. The variational free energy $\overline{F}$ is given by

$$\overline{F} = F_0 + \langle H - H_0 \rangle_0 \tag{22}$$

$$= F_0 + \left(\eta - \frac{h}{q} \right) \sum_i \sum_{l=1}^{q-1} \langle \lambda_i^l \rangle_0 - \frac{1}{2q} \sum_{i,j} J(r_{ij}) \sum_{l=1}^{q-1} \langle \lambda_i^l \lambda_j^{q-l} \rangle_0 - C(J,h), \tag{23}$$

where the free energy F_0 associated with H_0 is

$$F_0 = -\frac{N}{\beta} \ln\{\exp[\beta\eta(q-1)] + (q-1)\exp(-\beta\eta)\} \tag{24}$$

and $\langle \cdots \rangle_0$ denotes the canonical average using the Boltzmann measure proportional to $\exp(-\beta H_0)$.

Using equality (15) one gets that

$$\langle \lambda_i \rangle_0 = \langle \lambda_i^2 \rangle_0 = \cdots = \langle \lambda_i^{q-1} \rangle_0 = m_0 \quad (\forall i), \tag{25}$$

where the variational order parameter m_0 [defined by an equation similar to Eq. (20)] is related to η through

$$m_0 = \frac{\exp(\beta\eta q) - 1}{\exp(\beta\eta q) + (q-1)} \tag{26}$$

and from the property $\lambda^{q-l} = (\lambda^l)^*$ it follows that

$$\langle \lambda_i^l \lambda_j^{q-l} \rangle_0 = m_0^2. \tag{27}$$

The minimization condition leads to

$$\eta = \frac{1}{q} [J\phi(\alpha) m_0 + h] \tag{28}$$

which, combined with Eq. (26), gives the following mean-field equation for the order parameter m_0:

$$m_0 = \frac{\exp\{\beta[J\phi(\alpha)m_0 + h]\} - 1}{\exp\beta[J\phi(\alpha)m_0 + h] + (q-1)}. \tag{29}$$

In the $\alpha \to \infty$ limit (short-range interactions) we have $\phi(\alpha) \to z$, z being the coordination number of the lattice, and we recover, for $h=0$, Mittag and Stephen's[20] result. For $q=2$ the Hamiltonian (9) is equivalent to the Ising one with long-range interactions, providing that $J^{Potts} = 2J^{Ising}$ and $h^{Potts} = 2h^{Ising}$. In this case Eq. (29) reduces to

$$m_0 = \tanh\{\beta/2[J\phi(\alpha)m_0 + h]\} \tag{30}$$

and we recover the result from Ref. 5, i.e., the long-range version of the Curie-Weiss equation which describes a second-order phase transition for $h=0$ at $k_B T_C / J^{Ising} \phi(\alpha) = 1$.

The variational free energy $\overline{F}$ calculated at the minimum [condition (28)] gives the following mean-field free energy per site f:

$$-\beta f=\ln q+\frac{\beta J\phi(\alpha)}{2q}+\frac{1}{q}\left\{\beta h+\frac{1}{2}(q-1)\beta J\phi(\alpha)m_0^2+(q-1)\beta h m_0-[1+(q-1)m_0]\ln[1+(q-1)m_0]-(q-1)(1-m_0)\ln(1-m_0)\right\}, \tag{31}$$

where we have used Eqs. (23)–(29).

For $q>2$ and $h=0$ the transition is of first order (in agreement with Ref. 15) and it is easy to verify that at the transition temperature the order parameter m jumps to the value $(q-2)/(q-1)$. The first-order transition temperature for $q>2$ is given by

$$kT_c/J=\left(\frac{q-2}{q-1}\right)\frac{\phi(\alpha)}{2\ln(q-1)}. \tag{32}$$

Equation (31) for $h=0$ and Eq. (32) recover, in the $\alpha\to\infty$ limit, Mittag and Stephen's results.[20]

In the $q\to 1$ limit, which corresponds to a bond percolation where the bond probability occupancy between any two sites i and j is p/r_{ij}^{α}, the order parameter *probability percolation* P_∞ (defined as the probability that a randomly chosen bond of an infinite lattice belongs to a cluster of infinite size) can be derived from[22]

$$P_\infty(p)=1+\lim_{\overline{h}\to 0^+}\frac{\partial}{\partial\overline{h}}\left\{\lim_{q\to 1}\frac{\partial}{\partial q}\lim_{N\to\infty}\frac{1}{N}\ln(Z)\right\}, \tag{33}$$

where $\overline{h}=\beta h$, Z is the partition function of the Potts model with coupling constants J/r_{ij}^{α}, and p is the first neighbor bond probability given by $p=1-\exp(-J/k_BT)$.

One can easily show, from Eqs. (29), (31), and (33), that the probability percolation $P_\infty(p)$ is, as expected,[17] exactly the $q\to 1$ and $\overline{h}\to 0^+$ limit of the order parameter m_0, namely,

$$P_\infty=m_0(q\to 1,\overline{h}\to 0^+)=1-\exp\left[-\frac{J\phi(\alpha)}{k_BT}m_0(q\to 1,\overline{h}\to 0^+)\right] \tag{34}$$

or, in terms of p,

$$P_\infty(p;\alpha)=1-(1-p)^{\phi(\alpha)P_\infty(p;\alpha)} \tag{35}$$

from which it follows that, due to the divergence of $\phi(\alpha)$ in the nonextensive regime, $P_\infty(0<p\leq 1;0\leq\alpha\leq d)=1$ and hence the critical probability $p_c(0\leq\alpha\leq d)=0$ in agreement with the exact result.[17]

One can easily prove, from Eq. (35), that the percolation probability P_∞ of a finite system with N bonds presents an asymptotic scaling behavior similar to Eq. (6), namely,

$$P_\infty(N,p)\sim P'_\infty(p^*)\quad (N\gg 1), \tag{36}$$

where p^* is the variable p calculated at $T^*=T/N^*$, namely,

$$p^*=1-\exp\left(-\frac{JN^*}{k_BT}\right) \tag{37}$$

and P'_∞ is the probability percolation associated with the long range bond percolation whose bond probability occupancies are p^*/r_{ij}^{α}. Using this rescaled variable p^*, the MF order parameter equation becomes, for $N\gg 1$ [see Eq. (12)],

$$P_\infty(p^*;\alpha)=1-(1-p^*)^{C_d(\alpha)2^{\alpha}P_\infty(p^*;\alpha)} \tag{38}$$

which leads, for different values of $\alpha\in[0,d]$, to monotonously increasing distinct order parameters as p^* varies from the critical probabilites $p_c^*(\alpha,d)=1-\exp\{-1/[C_d(\alpha)2^{\alpha}]\}$ to $p^*=1$. However, if one introduces a more convenient variable, namely,

$$r^*\equiv 1-\exp\left(-\frac{J\phi(\alpha)}{k_BT}\right)=1-(1-p^*)^{\phi(\alpha)/N^*(\alpha)} \tag{39}$$

then all these MF probability percolation curves for different values of α and d coalesce into a single curve described by the equation

$$P_\infty(r^*)=1-(1-r^*)^{P_\infty(r^*)}. \tag{40}$$

The critical value r_c^* where $P_\infty(r^*)$ vanishes is $r_c^*=1-\exp(-1)=0.632\,12\ldots$, which leads to the MF critical probability

$$p_c(\alpha)=1-\exp\left(\frac{-1}{\phi(\alpha)}\right). \tag{41}$$

Combining Eqs. (41), (12), and (8) one verifies that $p_c(\alpha\to d^+)$ vanishes asymptotically as

$$p_c(\alpha\to d^+)\sim\frac{1}{C_d(d)2^d}\left(\frac{\alpha}{d}-1\right) \tag{42}$$

which, in the particular case of $d=1$, gives

$$p_c(\alpha\to 1^+,d=1)\sim\frac{1}{2}(\alpha-1). \tag{43}$$

Notice that the asymptotic behavior [Eq. (43)] coincides with the lower bound for $p_c(1<\alpha\leq 2,d=1)$[23]. Very recent Monte Carlo calculations in the one-dimensional long-range percolation problem[8] have also shown a very good agreement with Eq. (40).

B. Monte Carlo results

We performed a Monte Carlo simulation using the heat bath algorithm on the one-dimensional Hamiltonian (9) with $h=0$ and periodic boundary conditions for $N=300$, 600, and 1200, for $q=2$, 3, and 5, and different values of $0\leq\alpha<1$. We calculated the magnetization per spin (19) as a function of $T^*=T/N^*$ for different system sizes and performed a numerical extrapolation for $1/N\to 0$.

In Fig. 1 we compare the numerical results of the $q=2$ case for $m(T^*)$ vs $2k_BT^*/2^{\alpha}J$ for different values of α with the mean-field solution [Eq. (30)]. We see that all the nu-

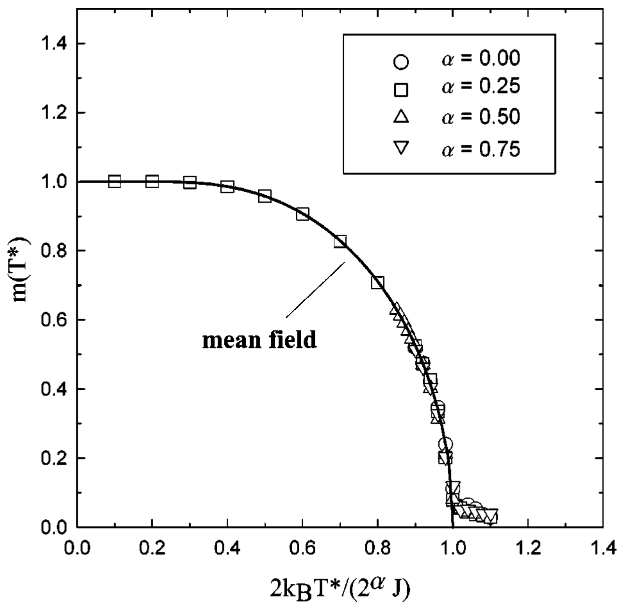

FIG. 1. Monte Carlo extrapolated results (symbols) of $m(T^*)$ vs $2k_BT^*/(2^\alpha J)$ compared with the mean-field solution (solid line) for $q=2$. The error bars are smaller than or equal to the symbol sizes.

merical curves (which recover those obtained in Ref. 5) fall into a single one in excellent agreement with the MF prediction.

In Fig. 2 we make the same comparison for the $q=3$ and $q=5$ cases. The solid lines represent the MF solution given by Eq. (29) for $h=0$. The dotted lines in this figure indicate the mean-field prediction for the first-order transition temperature jumps (32). Again we observe, in both cases, a very good agreement between our simulations and MF results for $0\leqslant\alpha<1$, including the first order phase transition for $q>2$.

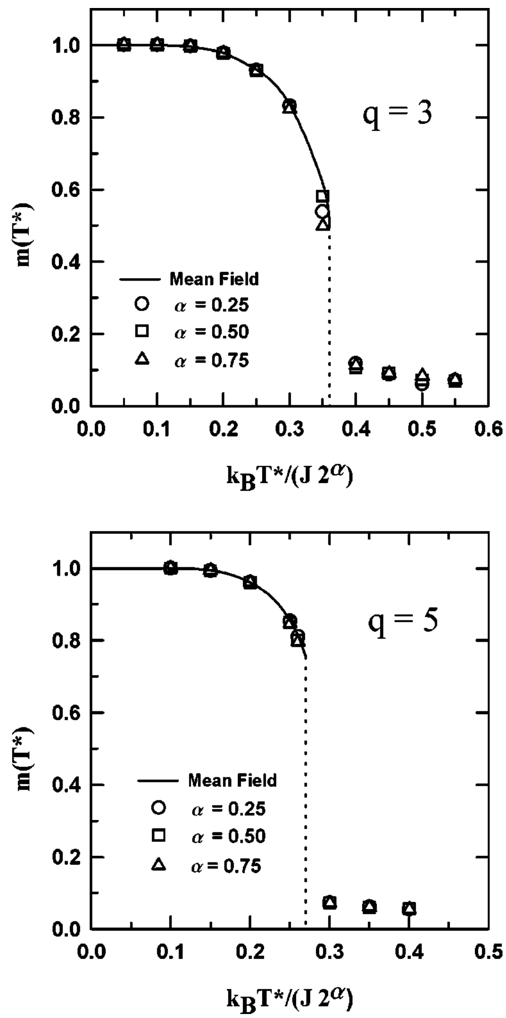

FIG. 2. Monte Carlo extrapolated results (symbols) of $m(T^*)$ vs. $k_BT^*/(2^\alpha J)$ compared with the mean-field solution (solid line) for $q=3$ and $q=5$. The error bars are smaller than or equal to the symbol sizes.

III. ANTIFERROMAGNETIC ISING MODEL WITH LONG-RANGE INTERACTIONS

We now consider the square lattice Ising model with competing LR antiferromagnetic and short-range ferromagnetic interactions in an external field, which is described by the Hamiltonian

$$H=-J_F\sum_{\langle i,j\rangle} S_iS_j+J\sum_{(i,j)}\frac{1}{r_{ij}^\alpha}S_iS_j-h\sum_i S_i \times(S_i=\pm 1\ \forall i), \qquad (44)$$

where $J>0$, $J_F>0$, and the sum $\Sigma_{\langle i,j\rangle}$ runs over nearest-neighbor sites of the square lattice. The above Hamiltonian reduces, for $J_F=1$ and $J=0.5$, to the model studied by Sampaio, de Albuquerque, and Menezes[10] through Monte Carlo simulations.

A mean-field version of this model can be obtained by considering the Hamiltonian

$$H_{MF}=-\sum_i h_{eff}^iS_i \qquad (45)$$

with

$$h_{eff}^i=-J\sum_{j\neq i}\frac{1}{r_{ij}^\alpha}m_j+J_F\sum_{jnni}m_j+h, \qquad (46)$$

where the sum Σ_{jnni} runs over all nearest-neighbor sites of i and

$$m_j\equiv\frac{1}{Z_{MF}}\mathrm{Tr}_{\{S_i\}}\{S_je^{-\beta H_{MF}(\{S_i\})}\}, \qquad (47)$$

with

$$Z_{MF}=\mathrm{Tr}_{\{S_i\}}e^{-\beta H_{MF}(\{S_i\})}. \qquad (48)$$

Dividing the square lattice into two square interpenetrated sublattices A and B we can propose a solution of the form

$$m_i=\begin{cases} m^A & \text{if } i\in A\\ m^B & \text{if } i\in B\end{cases}. \qquad (49)$$

Let us introduce the functions

$$\phi^{(1)}(\alpha) \equiv \sum_{j\in A} \frac{1}{r_{ij}^{\alpha}}\bigg|_{i\in A}, \tag{50}$$

$$\phi^{(2)}(\alpha) \equiv \sum_{j\in B} \frac{1}{r_{ij}^{\alpha}}\bigg|_{i\in A}$$

with

$$\phi^{(1)}(\alpha)+\phi^{(2)}(\alpha)=\phi(\alpha).$$

It can be easily seen that

$$\phi^{(1)}(\alpha)=2^{-\alpha/2}\phi(\alpha), \tag{51}$$

$$\phi^{(2)}(\alpha)=(1-2^{-\alpha/2})\phi(\alpha).$$

Then, substituting Eqs. (49) and (50) into Eq. (46) we obtain

$$h_{eff}^{i}=\begin{cases} -J\phi^{(1)}(\alpha)m^A-J\phi^{(2)}(\alpha)m^B+4J_Fm^B+h & \text{if } i\in A \\ -J\phi^{(2)}(\alpha)m^A-J\phi^{(1)}(\alpha)m^B+4J_Fm^A+h & \text{if } i\in B \end{cases}. \tag{52}$$

Combining Eqs. (52), (45), and (47) we arrive, after some algebra, to the following set of MF equations for the magnetization $m=m^A+m^B$ and for the staggered magnetization $m_s=m^A-m^B$:

$$m=\frac{\sinh(2\beta\{h-[J\phi(\alpha)-4J_F]m\})}{\cosh(2\beta\{h-[J\phi(\alpha)-4J_F]m\})+\cosh(2\beta\{[J\phi(\alpha)(2^{1-\alpha/2}-1)+4J_F]m_s\})}, \tag{53}$$

$$m_s=\frac{-\sinh(2\beta\{[J\phi(\alpha)(2^{1-\alpha/2}-1]+4J_F)m_s\})}{\cosh(2\beta\{h-[J\phi(\alpha)-4J_F]m\})+\cosh(2\beta\{[J\phi(\alpha)(2^{1-\alpha/2}-1)+4J_F]m_s\})}. \tag{54}$$

For $h\neq 0$ it is easy to verify that the only solution of Eqs. (54) and (53) is $m_s=0$ and

$$m=\tanh(\beta\{h-[J\phi(\alpha)-4J_F]m\}). \tag{55}$$

In Fig. 3 we compare a numerical solution of Eq. (55) with the Monte Carlo data of Sampaio, de Albuquerque, and Menezes[10] for $\alpha=1$, $J_F=1$, $J=1/2$, $\beta N^*(\alpha)=(0.3)^{-1}$, and different values of N. The function $C_d(\alpha)$ for $d=2$ was evaluated numerically by performing calculations for different values of N and then extrapolating to $1/N\to 0$. We obtained, for $\alpha=1$, $C_2(1)=0.8813\pm 0.0001$. The comparison for other values of $\alpha<2$ gave similar results.

A similar comparison for $\alpha=2$ is made in Fig. 4 $[C_2(2)=0.746\pm 0.001]$. The Monte Carlo data of Sampaio, de Albuquerque, and Menezes [10] do not agree very well with the MF magnetization, suggesting that corrections to the mean-field result should be taken into consideration in this

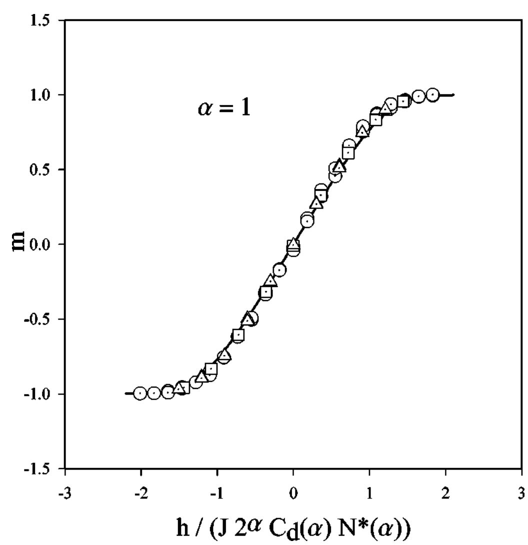

FIG. 3. Monte Carlo simulations of Sampaio, de Albuquerque, and Menezes (Ref. 10) for the magnetization m vs. a rescaled magnetic field for square lattice sizes of 32×32 (circles), 48×48 (squares), and 64×64 (triangles) for $\alpha=1$, $J_F=1$, $J=1/2$, and $T^*=T/N^*=0.3$. The solid line represents the MF solution given by Eq. (55) for $N=64\times 64$. The error bars are smaller than or equal to the symbol sizes.

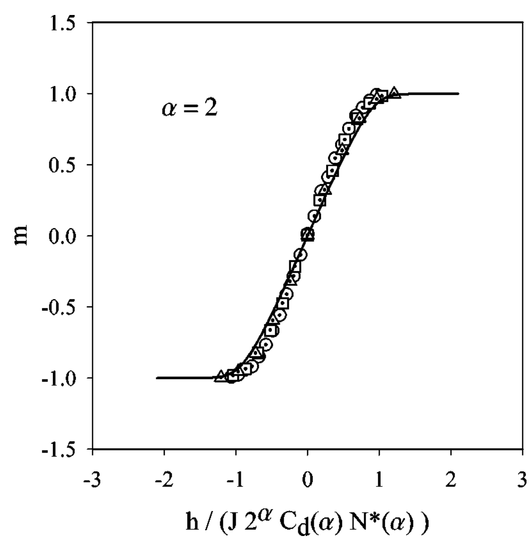

FIG. 4. Monte Carlo simulations of Sampaio, de Albuquerque, and Menezes (Ref. 10) for the magnetization m vs a rescaled magnetic field for square lattice sizes of 32×32 (circles), 48×48 (squares), and 64×64 (triangles) for $\alpha=2$, $J_F=1$, $J=1/2$, and $T^*=T/N^*=0.3$. The solid line represents the MF solution given by Eq. (55) for $N=64\times 64$. The error bars are smaller than or equal to the symbol sizes.

borderline case (where $\alpha=d$). Notice that theoretical[24] predictions and experimental results[9] obtained for the critical behavior of $d=3$ uniaxial ferromagnets with exchange and strong dipolar interactions show that corrections to the MF behavior are needed in this $\alpha=d=3$ case.

Summarizing this section, we verify that the Monte Carlo simulations for the equation of state of the $d=2$ LR Ising antiferromagnet are in very good agreement with the MF prediction when $0\leqslant\alpha<2$.

IV. CONCLUSIONS

We have analyzed, in this paper, two long-range spin models with power-law decaying interactions ($r^{-\alpha}$) under an uniform magnetic field h in low spatial dimensions: (i) the q-state LR Potts ferromagnet on d-dimensional hypercubic lattices (including the $q\to1$ and $h\to0^+$ case of LR bond percolation) and (ii) the LR square Ising antiferromagnet with first neighbor ferromagnetic interactions. Both models present nonextensive thermodynamic behaviors when $0\leqslant\alpha\leqslant d$, but their thermodynamic functions become finite when convenient scaled variables are used.[6] We have derived this scaling for $q\geqslant2$ and have shown that the mean-field probability percolation (i.e., the percolation order parameter) satisfies a similar scaling. The derived MF solution for the free energy of the LR Potts model led to spontaneous magnetization curves which agree very well with our Monte Carlo simulations for $d=1$ and $q=2,\ 3$, and 5 states for different values of $0<\alpha<1$. Concerning the $\alpha=0$ case, the MF theory is exact for any value of q (see, for example, Ref. 25 and references therein). An excellent agreement occurred also between the derived MF equation of state for the above $d=2$ antiferromagnetic LR model and the Monte Carlo simulations of Sampaio, de Albuquerque, and Menezes [10] for distinct values of $0\leqslant\alpha<2$. Since fluctuations play a negligible role for increasingly higher dimensions (in fact, MF theory becomes exact in the limit $d\to\infty$, even for models with short-range interactions), our results strongly suggest that the mean-field theory is exact for these two LR nonextensive models with $0\leqslant\alpha<d,\ \forall d\geqslant1$. Accordingly, this would predict, for the $d=1$ LR Potts ferromagnet, a first-order transition for $q>2$ and $0\leqslant\alpha<1$, which matches nicely with previous results[14,15] exhibiting first-order transition for $q>q_c(\alpha)$. A good agreement between MF and Monte Carlo calculations has also been found in other works concerning $d=1$ nonextensive classical LR spin and related models, namely, the order parameter of the LR bond percolation problem,[8] the magnetization of the classical XY model,[26] the correlation fucntion of the Ising model,[7] and the frozen-active transition line of the Domany-Kinzel LR cellular automaton.[27] Furthermore, the MF theory is exact for $\alpha=0$ in many classical LR spin models,[16] as well as for $\alpha\to d^+$, at least in the Ising[18] and spherical[28] LR models. These facts together with our results lead us to conjecture that *the mean-field theory might be exact for d-dimensional LR nonextensive classical spin models with* $0\geqslant\alpha<d$.

The above results show that mean-field behavior is robust against the range of interactions α within the nonextensive region, for a large class of classical magnetic and related systems. If our conjecture were true, this would have important practical implications: if you are considering systems with slow enough decaying interactions then you do not need sophisticated approximations.

We hope that our conjecture is proved, at least for some particular spin model. It would also be interesting to extend the previous analysis to more general systems of interacting particles with long-range interactions, such as a gas of classical particles interacting through a Newtonian potential.

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