GROUPS AND DYNAMICS

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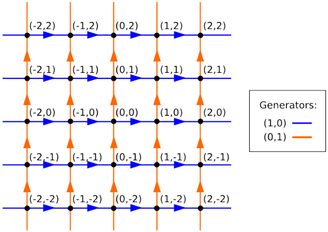
Edges: connect any two elements that differ by (right) multiplication by a generator.



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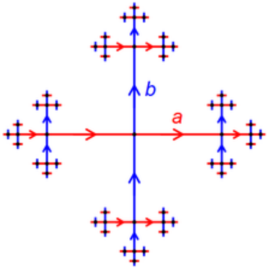

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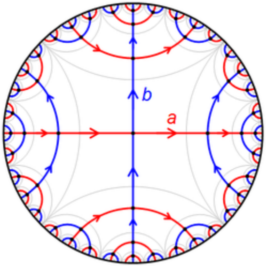

Full symmetry groups

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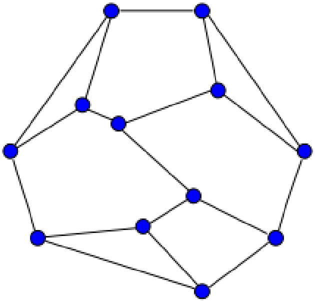

A General Principle

If a group (class of groups) acts nicely on a nice space, then the action should reveal some algebraic structure ($\rightsquigarrow$ nice theorem).

Groups acting on manifolds

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Conjecture (Zimmer)

The group $\mathrm{SL}(n+2, \mathbb{R})$ has no action by homeomorphisms on a compact n-dimensional manifold with infinite image.

The Burnside groups

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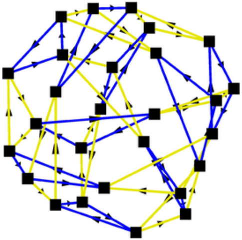

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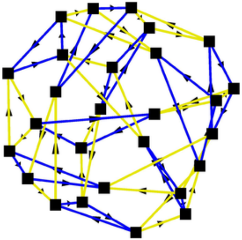

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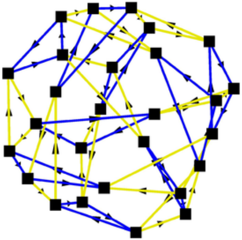

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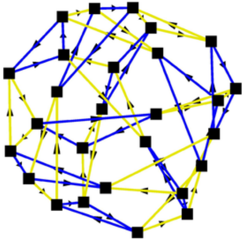

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- For $n > 666$ odd, $B(n)$ is infinite (non-amenable; Adian-Novikov).

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• According to a theorem of Kerékjártó (based on the work of Brouwer), every finite-order homeomorphism of the sphere is conjugate to a rotation.

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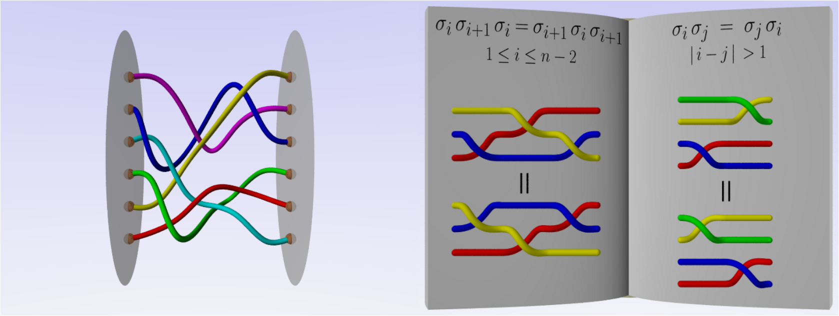
Question

Does there exist an algebraic characterization of groups that do act faithfully by homeomorphisms of a certain 2-manifold ?

Ordering braids



$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \colon \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \rangle$

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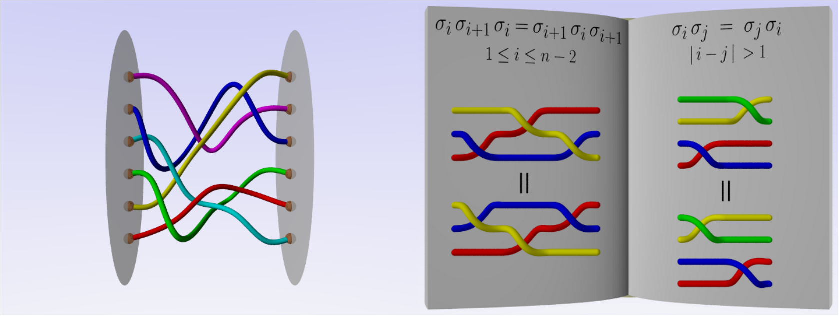

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The braid group B_n is left-orderable.

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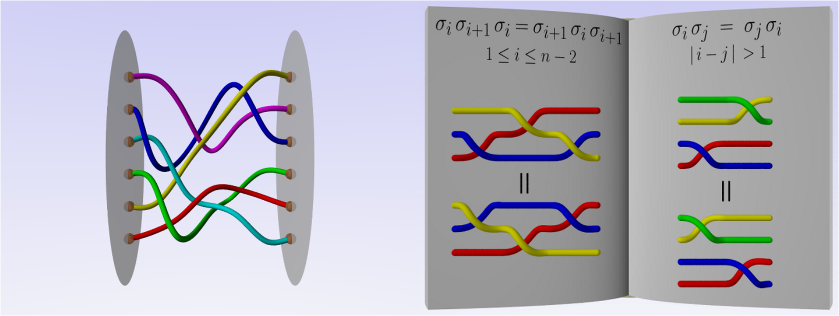

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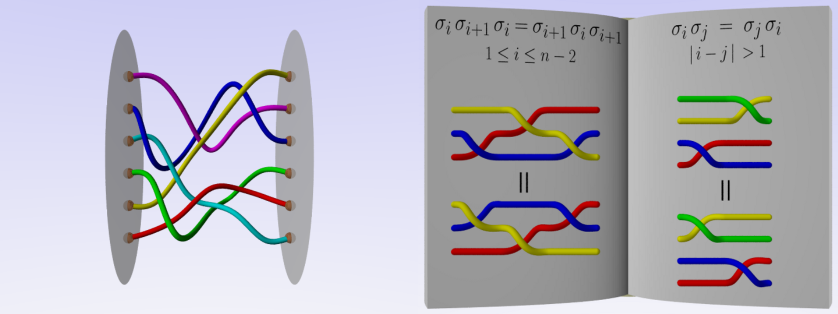

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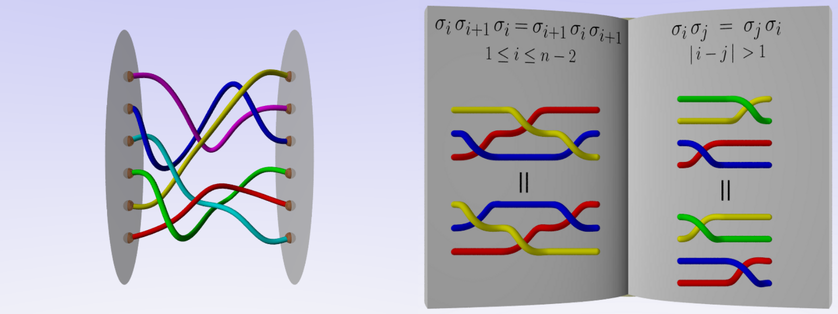

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The regularity enters into the game

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Every nontrivial finitely-generated subgroup of $\mathrm{Diff}^1_+([0,1])$ admits a nontrivial homomorphism into $\mathbb{R}$ (*i.e.* $\mathrm{Diff}^1_+([0,1])$ is *locally indicable*).

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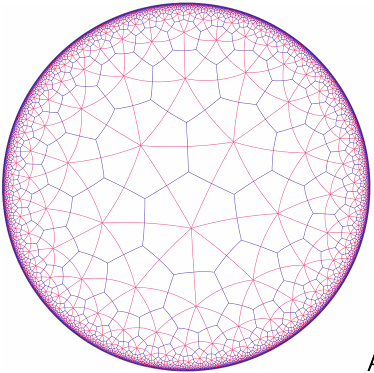
Local indicability does not hold for $\mathrm{Homeo}_+([0,1])$ (even for the group of Lipschitz homeomorphisms). An example (also due to Thurston) is the lifting to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the (2,3,7)-triangle subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Local indicability



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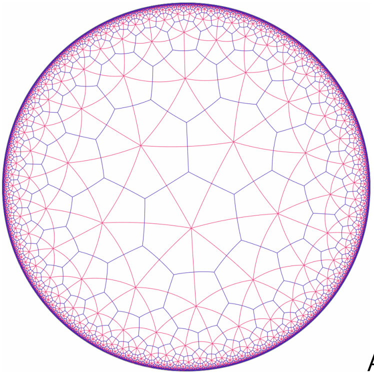

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Theorem (N)

Local indicability is not the only obstruction for embeddings into $\mathrm{Diff}^1_+([0,1])$.

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The group of piecewise dyadic homeomorphisms of the binary Cantor set is finitely presented and simple (Thompson's group V). It contains a copy of every finite group.

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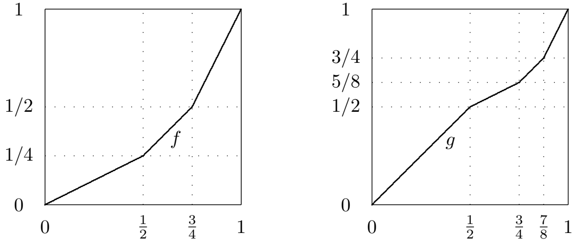

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– For Lipschitz homeomorphisms, rigidity comes from the associated cohomological equation.

MUCHAS GRACIAS