An introduction to spectral geometry

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Universidad Nacional de Córdoba, Argentina

VI Workshop on differential geometry — EGEO
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- $\Delta$ is self-adjoint (with respect to $\langle f, g \rangle = \int_M f(x)g(x)dx$.)
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- It commutes with isometries (i.e. $\Delta(\varphi \circ f) = \varphi \circ (\Delta f)$ for every $\varphi$ isometry of $M$).
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- It commutes with isometries (i.e. \(\Delta(\varphi \circ f) = \varphi \circ (\Delta f)\) for every \(\varphi\) isometry of \(M\)).
- It is positive definite (i.e. \(\langle \Delta f, f \rangle \geq 0\).)
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The spectrum of \( \Delta \), denoted by \( \text{Spec}(M, g) \), is the multiset of eigenvalues \( \lambda \) repeated according its multiplicity

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It is discrete, and each eigenvalue $\lambda$ has finite multiplicity.
Inverse spectral geometry studies to what extent does the spectrum encode the geometry of $(M, g)$.
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Known spectral invariants: dimension, volume, heat invariants (Prof. Gilkey is an expert on this matter).
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§2 Flat tori.
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§3  Lens spaces.
§2 Flat tori

In $\mathbb{R}^n$, $\Delta = -\left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)$. 

But $\mathbb{R}^n$ is not compact!
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$v \in \mathbb{R}^n \leadsto f_v : \mathbb{R}^n \rightarrow \mathbb{C},$

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\[ M_{\Lambda} := \mathbb{R}^n / \Lambda. \]

\( v + \Lambda = w + \Lambda \iff v - w \in \Lambda. \)

\( M_{\Lambda} \) is a flat torus. It is homeomorphic to ..., but it is flat.
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$M_\Lambda$ is a flat torus.

It is homeomorphic to $\mathbb{S}^1$, but it is flat.
Given a lattice $\Lambda$ of $\mathbb{R}^n$, we associate the dual lattice

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Let $\nu \in \Lambda^*$ map to $f_\nu : M_\Lambda \to \mathbb{C}$ since, for $x \in \mathbb{R}^n$ and $w \in \Lambda$,

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Hence $f_\nu \in C^\infty(M_\Lambda)$.
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Hence $f_v \in C^\infty(M_\Lambda) \subseteq L^2(M_\Lambda)$, a Hilbert space with

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For \( v, w \in \Lambda^* \),

\[
\langle f_v, f_w \rangle = \frac{1}{\text{vol}(M_\Lambda)} \int_{M_\Lambda} e^{2\pi i \langle x, v - w \rangle} \, dx = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}
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For $\nu \in \Lambda$ and $f \in C^\infty(M_\Lambda)$, the Fourier transform:

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Hence, $\{f_\nu\}_{\nu \in \Lambda^*}$ is an orthonormal basis of $L^2(M_\Lambda)$ (since $C^\infty(M_\Lambda)$ is dense in $L^2(M_\Lambda)$).
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Furthermore, $\{f_v\}_{v \in \Lambda^*}$ are eigenfunctions of $\Delta$, 

\[
\text{Spec}(\mathcal{M}_{\Lambda}) = \{\frac{4\pi^2}{\|v\|^2} : v \in \Lambda^*\}.
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In other words, if $\mu \in \mathbb{R} \geq 0$, then $\text{mult}(\frac{4\pi^2}{\mu}) = \# \{v \in \Lambda^* : \|v\|^2 = \mu\}$. 
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Theorem (Milnor, 1962) The flat tori \(\mathbb{R}^n / \Lambda_1\) and \(\mathbb{R}^n / \Lambda_2\) are isospectral if and only if the quadratic forms \((\Lambda^*_1, \| \cdot \|_2^2)\) and \((\Lambda^*_2, \| \cdot \|_2^2)\) represent the same numbers (with multiplicities), that is, for each \(\mu \in \mathbb{R} \geq 0\),

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Witt in 1942 proved that the quadratic forms associated to \(\Lambda_1 = E_8 \oplus E_8\), \(\Lambda_2 = D_{16} + 16\) satisfy the above condition. Witt used modular forms. For a simple proof, see [Conway, The sensual quadratic form]. There were more examples, going down the dimension until 4. It is also known that such example does not exist in dimension 3.
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§3 Lens spaces

3.1 Spectrum of $S^n$

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Theorem
If $f$ is a harmonic ($\Delta f = 0$) homogeneous polynomial of degree $k$, then

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*If \( f \) is a harmonic \((\Delta f = 0)\) homogeneous polynomial of degree \( k \), then*

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- (r^2)^{-\frac{k}{2}} \left( \sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i^2}(x) \right)
\]
Proof.
Let $f$ be a harmonic homogeneous polynomial of degree $k$. Let $r = |x|$, thus $r^2 = \sum_i x_i^2$.

\[
(\Delta \hat{f})(x) = -k(k+2)(r^2)^{-\left(\frac{k}{2}+2\right)} r^2 f(x) \\
+ (n+1)k(r^2)^{-\left(\frac{k}{2}+1\right)} f(x) \\
+ 2k^2(r^2)^{-\left(\frac{k}{2}+1\right)} f(x) \\
- (r^2)^{-\frac{k}{2}} (\Delta f)(x) \\
\]

= 0
Proof.
Let $f$ be a harmonic homogeneous polynomial of degree $k$.
Let $r = |x|$, thus $r^2 = \sum_i x_i^2$. If $x \in S^n$, then $r = 1$.

$$(\Delta S f)(x) = (\Delta \hat{f})(x) = -k(k + 2)(r^2)^{-(\frac{k}{2}+2)} r^2 f(x) + (n + 1)k(r^2)^{-(\frac{k}{2}+1)} f(x) + 2k^2(r^2)^{-(\frac{k}{2}+1)} f(x)$$
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(\Delta_S f)(x) = (\hat{\Delta} f)(x) = -k(k + 2)f(x) + (n + 1)kf(x) + 2k^2 f(x)
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Let $f$ be a harmonic homogeneous polynomial of degree $k$. Let $r = |x|$, thus $r^2 = \sum_i x_i^2$.

$$(\Delta_S f)(x) = k(k + n - 1)f(x)$$
\[ \mathcal{P}_k := \mathbb{C}[x_0, \ldots, x_n]^{(k)}, \]
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**Theorem**
\( \Delta : \mathcal{P}_k \to \mathcal{P}_{k-2} \) is surjective.
\( \mathcal{P}_k := \mathbb{C}[x_0, \ldots, x_n]^{(k)} \), \( H_k := \{ f \in \mathcal{P}_k : \Delta f = 0 \} \),

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\[ \mathcal{P}_k = H_k \oplus r^2 \mathcal{P}_{k-2} = H_k \oplus r^2 (H_{k-2} \oplus r^2 \mathcal{P}_{k-4}) \]
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   r^k H_0 & \text{if } n \text{ is even}, \\
   r^{k-1} H_1 & \text{if } n \text{ is odd}.
\end{cases}
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Thus, every polynomial restricted to \( S^n \) (\( r = 1 \)) is sum of harmonic polynomials.
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Thus, every polynomial restricted to \( S^n (r = 1) \) is sum of harmonic polynomials. By Weierstrass approximation theorem, one shows that

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L^2(S^n) = \bigoplus_{k \geq 0} H_k
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**Theorem**

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We conclude that the spectrum of $S^n$ is:
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since $\mathcal{P}_k = H_k \oplus r^2\mathcal{P}_{k-2} \simeq H_k \oplus \mathcal{P}_{k-2}$. 
We conclude that the spectrum of $S^n$ is:

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Hence

$$\text{mult}(\lambda_k) = \binom{k+n}{n} - \binom{k-2+n}{n}.$$