The Prescribed Ricci Curvature Problem On Three-Dimensional Unimodular Lie Groups

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Outline

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- History of the prescribed Ricci curvature problem

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- Results on manifolds with symmetry

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- On three-dimensional unimodular Lie groups

The Prescribed Ricci Curvature Problem

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Contributors include Adriano, J. Cao, Delanoe, Delay, DeTurck, Eberlein, Goldschmidt, Hamilton, Herzlich, Koiso, Pieterzack, Pina, Pulemotov, Rubinstein, Tenenblat, Wallach, Warner and Xu.

Prescribing Ricci Curvature Locally

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Theorem (DeTurck 1981)

If T is non-degenerate at $p \in M$, then there exists a Riemannian metric g such that $Ric(g) = T$ in a neighbourhood of p.

Prescribing Ricci Curvature Globally

Given a symmetric $(0, 2)$ tensor field T, can we find a Riemannian metric g such that $Ric(g) = T$ globally?

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Given a symmetric $(0,2)$ tensor field T, can we find a Riemannian metric g such that $Ric(g) = T$ globally?

Theorem (DeTurck-Koiso 1984)

If T is positive-definite on a closed manifold M, then there exists a constant $c_0 > 0$ such that for any constant $c > c_0$ and any Riemannian metric g, $Ric(g) \neq cT$ on all of M.

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Theorem (DeTurck 1985)

Let T be positive-definite on a closed manifold M. If there is an Einstein metric g_0 such that the kernel of the Lichnerowicz Laplacian is 1-dimensional, then there exists a function $\lambda : M \to \mathbb{R}$ such that $Ric(g) = \lambda T$.

Left-Invariant Riemannian Metrics

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- Inner products on $\mathfrak{g} = T_e G$ generate left-invariant Riemannian metrics via the group action.
- Ricci curvature is also left-invariant.

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Let G be a Lie group. Given a left-invariant candidate T, can we find a left-invariant g such that $Ric(g) = T$ on G?

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- Hamilton (1984): On $SO(3)$, if T is positive-definite, there exists a constant c and a left-invariant Riemannian metric g such that $Ric(g) = cT$. The constant c is unique and g is unique up to scaling.

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- Kremlev and Nikonorov (2008): Only certain signatures of T are allowed in four dimensions.

Prescribing Invariant Ricci Curvature

Let $M = G/H$ be a homogeneous space. Given a G-invariant candidate T, can we find a G-invariant g such that $Ric(g) = T$ on G/H?

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Theorem (Pulemotov 2016)

If T is positive-semidefinite on a compact homogeneous space $M = G/H$, with H maximal connected, we can solve $Ric(g) = cT$ for c and g.

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If the isotropy representation of M has two inequivalent irreducible summands, then c is unique, and g is unique up to scaling.

On Three-Dimensional Unimodular Lie Groups

Let G be a three-dimensional unimodular Lie group, so

$G \in \{SO(3), SL(2), E(2), E(1,1), \text{The Heisenberg Group}, \mathbb{R}^3\}$.

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- For every left-invariant g on G, there is a basis $\{V_1, V_2, V_3\}$ of $\mathfrak{g}$ in which g is diagonal and

$$[V_2, V_3] = \lambda_1 V_1, \qquad [V_3, V_1] = \lambda_2 V_2, \qquad [V_1, V_2] = \lambda_3 V_3.$$

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- We can impose $\lambda_k \in \{-2, 0, 2\}$, and the possible signatures characterise all three-dimensional unimodular Lie groups.

On Three-Dimensional Unimodular Lie Groups

Theorem (TB'16)

Given a left-invariant T on G, there exists a left-invariant g and a constant $c > 0$ such that $Ric(g) = cT$ if and only if T is diagonalisable in a basis $\{V_1, V_2, V_3\}$ satisfying

$$[V_2, V_3] = \lambda_1 V_1, \qquad [V_3, V_1] = \lambda_2 V_2, \qquad [V_1, V_2] = \lambda_3 V_3,$$

and

On Three-Dimensional Unimodular Lie Groups

Lie Group $(\lambda_1, \lambda_2, \lambda_3)$	Signature of (T_1, T_2, T_3)	Necessary and sufficient conditions on (T_1, T_2, T_3) for existence of a pair (g, c) solving $Ric(g) = cT$	Is c unique?	Is g unique up to scaling?
$SO(3)$ $(2, 2, 2)$	$(+, +, +)$	-	Yes	Yes
$\mathbb{R}^3$ $(0, 0, 0)$	$(0, 0, 0)$	-	No	No

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$SO(3)$	$(+,+,+)$	-	Yes	Yes
$(2,2,2)$	$(+,0,0)$	-	Yes	No
	$(+,-,-)$	Technical	No	Yes
$SL(2)$	$(+,-,-)$	$T_3 + T_1 > 0$	Yes	Yes
$(2,2,-2)$	$(-,-,+)$	$\max\{-T_1, -T_2\} < T_3$	Yes	Yes
		$\min\{-T_1, -T_2\} > T_3$	Yes	Yes
		$T_3 = -T_1 = -T_2$	Yes	No
	$(-,0,0)$	-	Yes	No
$E(2)$	$(0,0,0)$	-	No	No
$(2,2,0)$	$(+,-,-)$	$T_1 + T_2 > 0$	Yes	Yes
$E(1,1)$	$(0,0,-)$	-	Yes	No
$(2,-2,0)$	$(+,-,-)$	$T_1 + T_2 > 0$	Yes	Yes
Heisenberg Group $(2,0,0)$	$(+,-,-)$	-	Yes	Yes
$\mathbb{R}^3$ $(0,0,0)$	$(0,0,0)$	-	No	No

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$SO(3)$ $(2,2,2)$	Yes	Yes
	Yes	No
	No	Yes
$SL(2)$ $(2,2,-2)$	Yes	Yes
	Yes	Yes
	Yes	Yes
	Yes	No
	Yes	No
$E(2)$ $(2,2,0)$	No	No
	Yes	Yes
$E(1,1)$ $(2,-2,0)$	Yes	No
	Yes	Yes
Heisenberg Group $(2,0,0)$	Yes	Yes
$\mathbb{R}^3$ $(0,0,0)$	No	No

In almost all cases, there exists at most one c such that $Ric(g) = cT$ for some g