Spinorial equations for special geometries

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based on *I.Agricola, SGC, T.Friedrich, J.Höll*, JGP 2015
& work with *S.Salamon*

(para Sergio)

traditional framework

(M^n, g) Riemannian manifold

- $\phi \in \Lambda^* M$ with open orbit and stabiliser $G \subseteq SO(n)$
- $\nabla\phi \in T^*M \otimes \dfrac{\mathfrak{so}(n)}{\mathfrak{g}} = W_1 \oplus \ldots \oplus W_N$ is often determined

  by the deRham complex ($d\phi$ and the like)

Example:

n	ϕ	G	N
2m	$\omega \in \Lambda^2, \xi \in \Lambda^{3,0}$	$SU(m)$	7

Gives rise to classes of almost Hermitian geometry, eg

$$
\begin{aligned}
\nabla\phi \in W_3 \oplus W_4 \oplus W_5 &\iff \text{Hermitian}\\
\nabla\phi \in W_2 \oplus W_5 &\iff \text{almost Kähler}\\
\nabla\phi \in W_1 \oplus W_2 \oplus W_3 &\iff \text{1/2 flat}\\
\nabla\phi \in W_1 &\iff \text{nearly Kähler}
\end{aligned}
$$

This also applies to $G = U(m), Sp(k), G_2, Spin(7), Sp(k)Sp(1)$ etc...

taking sides

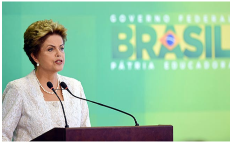
As ‘spin geometry’ is usually relegated to doctoral courses, if offered at all, I ask:

Should spinors
REMAIN members of Riem Geometry or LEAVE Riem Geometry?

remain camp
- Weyl, Atiyah
- Milnor, Connes
- Dirac, Schrödinger, Witten

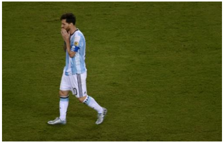
leave camp
- Cartan
- . . .
- . . .

The outcome is not straightforward, given recent in/out decisions:



(Rouxit)

(Brexit)



(Mexit)

I advocate decisively for ‘remain’ (and conquer)

a tasting

- [Atiyah-Singer] index theorem & al.
- [Witten] positive mass thm (cf. Yamabe solution)
- in low dimensions strong relationship to special metrics, for

$Spin(3)/\mathbb{Z}_2 = SO(3)$	chirality
$Spin(4) = SU(2)^2$	self-duality
$Spin(5) = Sp(2)$	
$Spin(6) = SU(4)$	(*the focus of this talk*)
$Spin(7)$	much related to $G_2, Sp(2)Sp(1)$
$Spin(8)$	triality

- [Seiberg-Witten, Donaldson] invariants in dimension 4
- if there is a parallel spinor, the metric is Ricci-flat (holonomy principle)
- [Friedrich, Grunewald, Hijazi, Kath] $\nabla_X \phi = \lambda X \cdot \phi \iff$

n	$3, 4$	5	6	7	8
M^n	S^n	Sasaki-Einstein	nearly Kähler	weak G_2	S^8

dim 6

- (Half-)spin representation $Spin(6) \longrightarrow SU(\Sigma), \quad \Sigma = \mathbb{C}^4$
- $Spin(6)$-invariant complex structure $\overline{\phi} = e_1 \cdot \ldots \cdot e_6 \cdot \phi$
- Spin bundle $\mathbb{S} = P \times_{Spin(6)} (\Sigma \oplus \overline{\Sigma}), \qquad P/\mathbb{Z}_2 = \text{ON frames}$

Spinors $\phi \in \Sigma$ rise from *quadratic relations*:

$$\Sigma = span\{\phi^0, \phi^1, \phi^2, \phi^3\} \qquad \dim_{\mathbb{C}} \Lambda^2\Sigma = 6$$

so

$$e^1 + ie^2 = \phi^0 \wedge \phi^1 \qquad e^3 + ie^4 = \phi^0 \wedge \phi^2 \qquad e^5 + ie^6 = \phi^0 \wedge \phi^3$$
$$e^1 - ie^2 = \phi^2 \wedge \phi^3 \qquad e^3 - ie^4 = \phi^3 \wedge \phi^1 \qquad e^5 - ie^6 = \phi^1 \wedge \phi^2$$

is a basis of 1-forms: $(T^*M)^{\mathbb{C}} \cong \Lambda^2\Sigma$

- $(\Sigma \oplus \overline{\Sigma})^{\otimes 2} = \underbrace{\mathbb{C}}_{g(e_i, e_j) = e^i \wedge e^j \phi^{0123}} \oplus T^*_{\mathbb{C}} \oplus \Lambda^2 T^*_{\mathbb{C}} \oplus \underbrace{\Lambda^3 T^*_{\mathbb{C}}}_{S^2\Sigma \oplus S^2\overline{\Sigma}} \oplus \ldots$

parameter spaces

New definition: an $SU(3)$ manifold is a spin (M^6, g, ϕ) with $||\phi|| = 1$

- *real* spinors $\mathbb{R}\phi \quad \longleftrightarrow \quad SU(3)$ structures (J, ξ)

$$\mathbb{RP}^7 \cong \frac{SO(6)}{SU(3)}$$

given by

$$\begin{cases} J(X)\cdot\phi \ := \ \overline{X\cdot\phi} \\ \xi(X,Y,Z) \ := \ -\langle X\cdot Y\cdot Z\cdot\phi, \phi\rangle \end{cases}$$

- *complex* spinors $\mathbb{C}\phi = \ell \quad \longleftrightarrow \quad$ almost complex structures J

$$\mathbb{CP}^3 \cong \frac{SU(4)}{S(U(3)\times U(1))} = \frac{SO(6)}{U(3)}$$

given by

$$\begin{cases} \Lambda^{1,0} \ := \ \ell \wedge \ell^{\perp} & (\alpha\text{-planes in Klein quadric } \mathbb{CP}^5) \\ \Lambda^{0,1} \ := \ \Lambda^2 \ell^{\perp} & (\beta\text{-planes}) \end{cases}$$

square roots

Fibration (determinant) governing everything:

$$\begin{array}{ccc} \mathbb{RP}^7 & \xrightarrow{S^1} & \mathbb{CP}^3 \\ \mathbb{R}\phi & \longmapsto & \mathbb{C}\phi = \mathbb{R}\phi \oplus \mathbb{R}\overline{\phi} \end{array}$$

$\Lambda^{3,0} = \ell^2$ means that a complex spinor $\ell = \mathbb{C}\phi$ is the square root of a holomorphic 3-form, a fact used by Hitchin to show that a Kähler M^6 is spin iff K_M has a square root

As always, physicists knew already about $\sqrt{}$:

$$i\,\hbar\,\frac{\partial}{\partial t} = \sqrt{c^2\,\hbar^2\,\Delta + m^2\,c^4} \quad \rightsquigarrow \quad \mathrm{D} = \sqrt{\Delta + s/4}$$

[Dirac] [Schrödinger-Lichnerowicz]

where $\mathrm{D} : \Sigma \xrightarrow{\nabla} T \otimes \Sigma = \Lambda^2\Sigma \otimes \Sigma \xrightarrow{\wedge} \Lambda^3\Sigma \cong \overline{\Sigma}$ is the Dirac operator

simpler & more uniform

$$\Sigma \cong \mathbb{R}^8 = \mathbb{R}\phi \oplus \mathbb{R}\overline{\phi} \oplus TM^6 \cdot \phi$$

Lemma: $\nabla\phi = \eta \otimes \overline{\phi} + A \otimes \phi$, with $\eta \in \Lambda^1$, $A \in End(TM)$

Theorem: the geometry of the $SU(3)$ mfd (M^6, g, ϕ) is determined by the tensor $A \lrcorner \xi - \frac{2}{3}\eta \otimes \omega$

$\rightsquigarrow$ all almost Hermitian types described by spinorial eqn's (2^7)

Examples:

- half-flatness:

$d(\mathrm{Re}\,\xi) = 0 \qquad d(\omega^2) = 0 \qquad \Longleftrightarrow \qquad \nabla_X\phi = A(X)\cdot\phi \quad \forall X$

2 eqns, 2 unknowns $\qquad$ *1 eqn, 1 unknown*

- nearly Kähler:

$d(\mathrm{Re}\,\xi) = \omega^2 \qquad d\omega = \mathrm{Im}\,\xi \qquad \Longleftrightarrow \qquad \nabla_X\phi = \lambda X\cdot\phi \quad \forall X$

2 eqns, 3 unknowns $\qquad$ *1 eqn, 1 unknown*

spin Hodge

Harmonic spinors $\mathcal{H}(\mathbb{S}) = \mathrm{Ker}(\mathrm{D} : \Sigma \to \overline{\Sigma})$

depend on the choice of metric (yet, how?) and little control on $\dim \mathcal{H}(\mathbb{S})$

Theorem:

- $\mathrm{D}\,\phi = 0 \iff \star d(\star\omega) + 2\eta = 0 \qquad (\eta \neq 0,\ A \neq c\,J)$
- 'complementary' components $W_1, W_{\bar{1}}$ of $\nabla\phi$ determined by

$$\langle \mathrm{D}\,\phi, \overline{\phi}\rangle \;=\; -tr(JA), \qquad \langle \mathrm{D}\,\phi, \phi\rangle \;=\; -tr(A)$$

Example: $M^6 = SL(2,\mathbb{C}) = \dfrac{SL(2,\mathbb{C}) \times SU(2)}{SU(2)_{\mathrm{diag}}} = G/H$ (reductive)

$$\mathfrak{g} = \mathfrak{h} \oplus \{(A,B) \mid A = \overline{A}^t,\ tr\,A = 0,\ B = -\overline{B}^t,\ tr\,B = 0\}$$

The spinor determined by $J(A,B) = (iA, iB)$ and $\eta = 0$ is harmonic.

(here, as happens often, $\phi \in \mathrm{Ker}\,\mathrm{D} \iff$ the $SU(3)$ class is W_3)

examples – twistor spaces

The twistor spaces of self-dual Einstein 4-manifolds

$$\frac{SO(5)}{U(2)} \longrightarrow S^4, \qquad \frac{U(3)}{U(1)\times U(1)\times U(1)} \longrightarrow \mathbb{CP}^2$$

carry a family g_t of metrics with $scal(g_t) = 2c(6 - t + 1/t)$

[Hitchin, Friedrich-Kurke] g_1 is Kähler

[Eells-Salamon, Friedrich] $g_{1/2}$ is nearly Kähler, induced by ϕ_ϵ ($\epsilon = \pm 1$)

Theorem:

- For $t \neq 0$ let $A_\epsilon \;=\; \epsilon\,\mathrm{diag}\left(\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{1-t}{2\sqrt{t}}, \frac{1-t}{2\sqrt{t}}\right)$. Then

$$\nabla_X \phi_\epsilon \;=\; A_\epsilon(X)\cdot\phi_\epsilon$$

(except when $t = 1/2$: type $W_{\bar{1}\bar{2}}$ and $\mathrm{D}\,\phi_\epsilon = \epsilon\sqrt{c}\frac{t+1}{\sqrt{t}}\phi_\epsilon$)

- For $t = 1$: ϕ_ϵ are Kählerian KS, don't define a compatible $SU(3)$ structure

G_2 story

Similar picture, same recipe:

• (M^7, Φ) G_2 manifold $\longleftrightarrow$ (M^7, g, ϕ) spin with $||\phi|| = 1$

• $\Phi(X, Y, Z) = \langle X \cdot Y \cdot Z \cdot \phi, \phi \rangle$ expressing $SO(7)/G_2 \cong \mathbb{RP}^7$

• $\Sigma = \mathbb{R}^8 = \mathbb{R}\phi \oplus TM^7 \cdot \phi$ real

(here no conjugation, but still the same dim as the Σ on p.8)

Prop$^{\mathbf{n}}$: $\nabla_X \phi = A(X) \cdot \phi$, and the fundamental tensor is $-\dfrac{2}{3} A \lrcorner \Phi$

Theorem: ϕ is harmonic iff the G_2 structure is of class W_{23}.

Manifest power of spin approach

$(\overline{V} = \mathbb{R}^7, \Phi)$ induces $SU(3)$ structure on any hypersurface $V = \vec{n}^{\perp}$:

– (usually) restrict $\Phi|_V$, so that $\vec{n} \lrcorner \Phi$ defines a complex str on V

– (much simpler) both structures, on $\overline{V}$ and V, correspond to the same choice of real spinor $\phi \in \Sigma$.

hypersurface theory

[Friedrich] $M^2 \hookrightarrow \mathbb{R}^3$ isometric: $\mathrm{D}\,\phi = H\phi \iff \nabla_X\phi \;=\; \frac{1}{2}\alpha(X)\cdot\phi$

$\rightsquigarrow$ example of Killing spinor: $\nabla_X\phi \;=\; \lambda X\cdot\phi$

For $M^6 \hookrightarrow Y^7$ the best 'app' are *generalised* Killing spinors:

$$\overline{\nabla}_X\phi \;=\; A(X)\cdot\phi$$

We are at freedom to choose $A \in \mathrm{Sym}^2 TM$ (Weingarten map) and the metric connection with skew torsion $\overline{\nabla} = \nabla + 2s\mathcal{T}$, where $\mathcal{T} \in \Lambda^3$ (torsion), $s \in \mathbb{R}$ (parameter). Thus we can control the reduction process (from 7 to 6 dims) and the oxidation (from 6 to 7). As an application

Theorem (*spin cones*): (M^6, g, ϕ), $h : I \to S^1$, $f : I \to \mathbb{R}_+$ smooth. Then $(\,M^6 \times I\,,\; f(t)^2 g + dt^2\,)$ has a family of G_2 structures

$$\widetilde{\phi}_t = \;(Re\,h)\phi + (Im\,h)\overline{\phi}.$$

This subsumes

- holonomy cones [Bär]
- $f(t) = t$ straight spin cone [Agricola-Höll]
- $f = \sin,\; h(t) = e^{it/2},\; A = \frac{1}{2}\mathrm{Id}$ 'sine' cone [Acharya & al.]

memento: 2 appointments

Rio, 1–2 Sept 2016

1ST JOINT MEETING BRAZIL-ITALY IN MATHEMATICS

“Geometric structures, Lie theory and applications”

`http://www.sbm.org.br/jointmeeting-italy/special-sessions/`

Campinas, 5–7 Dec 2016

“Trends in geometry and topology”

`http://jovens.ime.unicamp.br`