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Which manifolds admit expanding maps?

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KU Leuven Kulak

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Overview

What are expanding maps?

What are nilmanifolds?

Which nilmanifolds admit an expanding map?

What is an Anosov diffeomorphism?

What is the rational Mal’cev completion?

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Notations and definition

M	closed Riemannian manifold
$v \in TM$	tangent vector
$\|v\|$	norm of v
$f : M \to M$	endomorphism of M
$Df : TM \to TM$	derivative of f

Definition

We call f ***expanding*** *if there exist real constants* $c > 0, \lambda > 1$ *such that*

$$\forall v \in TM : \|Df^n(v)\| \geq c\lambda^n \|v\|.$$

Remark: This definition is independent of the choice of Riemannian metric.

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Basic example

Fix $m \in \mathbb{Z}$ with $|m| > 1$ and consider the manifold $S^1 \subseteq \mathbb{C}$ with endomorphism

$$\begin{aligned} f_m : S^1 &\to S^1 \\ z &\mapsto z^m. \end{aligned}$$

For every tangent vector $v \in TM$, we have

$$\|Df_m^n(v)\| = |m|^n \|v\|$$

and thus f_m is expanding.

For $m = 2$:

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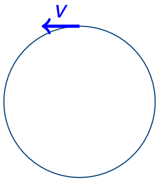

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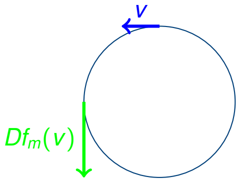

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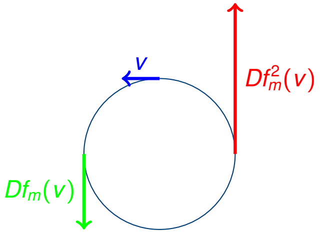

Questions about expanding maps

Observation

Expanding maps are dynamical systems with interesting properties like structural stability, chaos, etc.

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Which manifolds admit an expanding map?

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Is it possible to classify all expanding maps in some way?

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Are there more examples of expanding maps?

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Expanding maps on tori

Recall the main example on $S^1 = \mathbb{R}/\mathbb{Z}$ above:

$$f_m : S^1 \to S^1 : z \mapsto z^m.$$

By lifting these maps to the universal cover, we get

$$\bar{f}_m : \mathbb{R} \to \mathbb{R} : x \mapsto mx,$$

with $\bar{f}_m$ a linear map with only eigenvalues > 1.

Conversely, take any matrix $A \in \mathrm{GL}(n, \mathbb{R})$ such that

$$A(\mathbb{Z}^n) \leq \mathbb{Z}^n \quad \text{and}$$
$$|\lambda| > 1 \quad \forall \lambda \in \mathrm{Spec}(A).$$

Then A induces an expanding map on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Observation

Every n-torus $\mathbb{T}^n$ admits an expanding map.

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More general Lie groups

Can we generalize the previous example?

1. $\mathbb{Z}^n \leq \mathbb{R}^n$ lattice in the simply connected and connected abelian Lie group $\mathbb{R}^n$.
2. $A \in \mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{R})$.
3. $A(\mathbb{Z}^n) \leq \mathbb{Z}^n$.
4. $\forall \lambda \in \mathrm{Spec}(A) : |\lambda| > 1$.

1. $\Gamma \leq G$ lattice in a simply connected and connected Lie group G.
2. $\alpha \in \mathrm{Aut}(G)$.
3. $\alpha(\Gamma) \leq \Gamma$.
4. What is $\mathrm{Spec}(\alpha)$?

Definition

*The **eigenvalues** of $\alpha \in \mathrm{Aut}(G)$ are defined as the eigenvalues of the derivative $D\alpha \in \mathrm{Aut}(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra corresponding to G.*

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Which Lie groups G?

We look for automorphisms with only eigenvalues > 1 in absolute value.

Theorem (Jacobson - 1955)

Let $\mathfrak{g}$ be a Lie algebra and $\varphi \in \operatorname{Aut}(\mathfrak{g})$ an automorphism with $|\lambda| \neq 1$ for all $\lambda \in \operatorname{Spec}(\varphi)$, then $\mathfrak{g}$ is nilpotent.

Definition

Assume that G is a connected and simply connected nilpotent Lie group, $\Gamma \leq G$ a lattice and $\alpha \in \operatorname{Aut}(G)$ with $\alpha(\Gamma) \leq \Gamma$.

1. *The manifold ${}^{G}\!/\!_{\Gamma}$ is called a **nilmanifold**.*
2. *The induced map $\bar{\alpha} : {}^{G}\!/\!_{\Gamma} \to {}^{G}\!/\!_{\Gamma}$ is called a **nilmanifold endomorphism**.*

Remark: The map $\bar{\alpha}$ is expanding if and only if $|\lambda| > 1$ for all $\lambda \in \operatorname{Spec}(\alpha)$.

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Example of a nilmanifold

Let R be a commutative ring with 1 and consider the Heisenberg group:

$$H_3(R) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in R \right\}$$

- For $R = \mathbb{R}$ we get the nilpotent, simply connected and connected Lie group $G = H_3(\mathbb{R})$.
- If $R = \mathbb{Z}$, we get the discrete Heisenberg group $\Gamma = H_3(\mathbb{Z})$, which is a lattice of G and the quotient is a 3-dimensional nilmanifold.
- The map $\alpha : G \to G$ given by

$$\alpha \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax & abz \\ 0 & 1 & by \\ 0 & 0 & 1 \end{pmatrix}.$$

for $a, b \in \mathbb{Z}$ is an automorphism with $\alpha(\Gamma) \leq \Gamma$. If both $|a| > 1$ and $|b| > 1$, then $\bar{\alpha}$ is an expanding endomorphism on ${}^{G}\!/_{H_3(\mathbb{Z})}$.

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- For $k > 0$ define $N_k \leq G$ as the lattice

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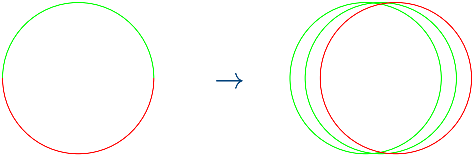

Theorem (Gromov - 1981)

Up to finite cover, every expanding map is topologically conjugate to an (expanding) nilmanifold endomorphism.

Actually, this theorem is a consequence of his celebrated theorem about polynomial growth:

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A group of polynomial growth is virtually nilpotent.

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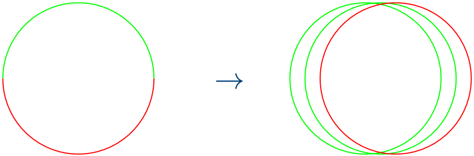

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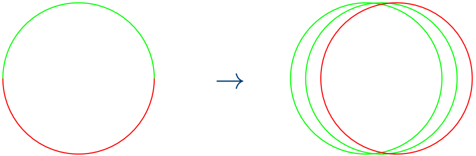

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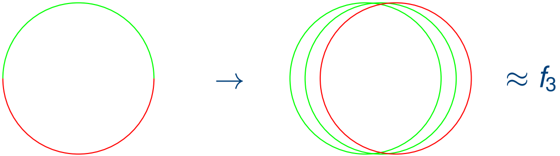

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- We can assume that φ is semisimple (Aut($\mathfrak{g}$) is linear algebraic group).
- $$\mathfrak{g} = \bigoplus_{r>0} \mathfrak{g}_r \qquad \text{with } [\mathfrak{g}_r, \mathfrak{g}_s] \subseteq \mathfrak{g}_{r+s};$$
  where $\mathfrak{g}_r$ is the sum of eigenspaces for eigenvalues λ with $|\lambda| = \exp(r)$.
- After renaming we get
  $$\mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}_i \qquad \text{with } [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

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  $$\mathfrak{g} = \bigoplus_{i\in\mathbb{N}} \mathfrak{g}_i \qquad \text{with } [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

Definition

*We say that the Lie algebra $\mathfrak{g}$ has a **positive grading**.*

Algebraic characterization for expanding maps

Theorem ()

Let G/Γ be a nilmanifold, then it holds that:

The nilmanifold G/Γ admits an expanding map.

$$\Downarrow$$

The Lie group G has an expanding automorphism.

$$\Downarrow$$

The Lie algebra $\mathfrak{g}$ has a positive grading.

Remarks:

1. The result also holds for the bigger class of *infra-nilmanifolds*.
2. There is a similar statement for the existence of non-trivial self-covers.
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Overview

What are expanding maps?

What are nilmanifolds?

Which nilmanifolds admit an expanding map?

What is an Anosov diffeomorphism?

What is the rational Mal'cev completion?

What is an Anosov diffeomorphism?

Definition

Let M be a closed Riemannian manifold with diffeomorphism $f : M \to M$, then we call f an ***Anosov diffeomorphism*** *if:*

(i) *there exists a continuous splitting of the tangent bundle*

$$TM = E^s \oplus E^u;$$

(ii) *the subbundles E^s and E^u are preserved under the map $Df : TM \to TM$, i.e.*

$$Df(E^s) = E^s \text{ and } Df(E^u) = E^u;$$

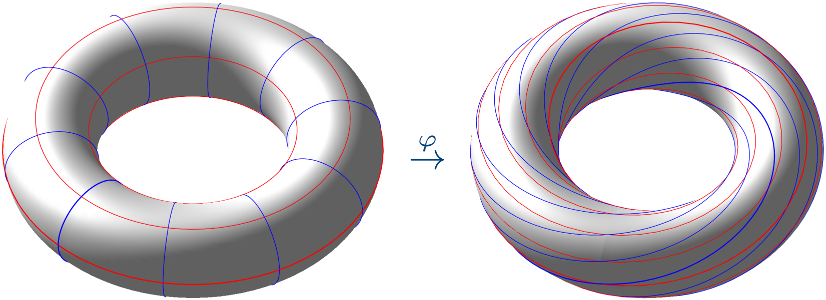
(iii) *there exist real constants $0 < \lambda < 1$ and $c > 0$ such that*

$$\forall v \in E^s,\ \forall k > 0 : \|Df^k(v)\| \leq c\lambda^k \|v\|,$$

$$\forall v \in E^u,\ \forall k > 0 : \|Df^k(v)\| \geq \frac{1}{c\lambda^k} \|v\|.$$

Arnold's cat map

Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then A induces an Anosov diffeomorphism φ on $\mathbb{T}^2$:



The map φ is called **Arnold's cat map**.

Arnold's cat map

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Observation

Arnold’s cat map is chaotic.

Arnold's cat map

Conjecture

Anosov diffeomorphisms are chaotic.

Examples of Anosov diffeomorphisms

1. By combining the matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

   we can construct Anosov diffeomorphisms on all $\mathbb{T}^n$ with $n > 1$.
2. On the contrary, $\mathbb{T}^1 = S^1$ doesn't admit an Anosov diffeomorphism, since it has no expanding diffeomorphisms.
3. Every hyperbolic affine infra-nilmanifold automorphism is an Anosov diffeomorphism.

Conjecture

Every Anosov diffeomorphism is topologically conjugate to a hyperbolic affine infra-nilmanifold automorphism.

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More examples

Question

Which nilmanifolds admit an Anosov diffeomorphism?

More examples

(1) The nilmanifolds ${}^{H_3(\mathbb{R})}/_{N_k}$ do not admit an Anosov diffeomorphism. Indeed, $Z(N_k) \cong \mathbb{Z}$ and thus every automorphism has eigenvalue ± 1.

(2)
- If $R = \mathbb{Z}[\sqrt{2}]$ (or $\mathbb{Z}[\sqrt{d}]$ with d a square-free natural number), then we can embed $\Gamma = H_3(R)$ as a discrete subgroup of $H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$ via

$$H_3(R) \hookrightarrow G \oplus G : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \sigma(x) & \sigma(z) \\ 0 & 1 & \sigma(y) \\ 0 & 0 & 1 \end{pmatrix} \right)$$

  where $\sigma : R \to R$ is the unique ring morphism with $\sigma(\sqrt{2}) = -\sqrt{2}$. Then $H_3(R)$ is a lattice in $H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$.

- Take α the automorphism of Γ as above with $a = b = 1 + \sqrt{2}$. Then α extends to an automorphism of $H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$ which induces an Anosov diffeomorphism on the quotient.

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What is the rational Mal'cev completion?

Since G is a simply connected and connected Lie group, the exponential map is a diffeomorphism:

$$G \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathfrak{g}.$$

The group multiplication and the Lie bracket are related by the **Baker-Campbell-Hausdorff formula**.

The group Γ is a lattice of the Lie group G. The rational span $\mathfrak{n}^{\mathbb{Q}} = \mathbb{Q}\log(\Gamma)$ is a rational Lie algebra and $\Gamma^{\mathbb{Q}} = \exp(\mathfrak{n}^{\mathbb{Q}})$ is a subgroup of G. The group $\Gamma^{\mathbb{Q}}$ is called the rational Mal'cev completion of Γ. Note that every injective group morphism of Γ extends to an automorphism of $\Gamma^{\mathbb{Q}}$, so $\mathrm{Endo}(\Gamma) \subseteq \mathrm{Aut}(\Gamma^{\mathbb{Q}})$.

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$$(\mathbb{Z}^n)^{\mathbb{Q}} = \mathbb{Q}^n.$$

2. For the Heisenberg group, it holds that

$$(H_3(\mathbb{Z}))^{\mathbb{Q}} = H_3(\mathbb{Q}) = (N_k)^{\mathbb{Q}}.$$

Indeed, the diffeomorphism log is given by

$$\log \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & z - \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

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Application of the characterization

The easiest examples of nilmanifolds admit an expanding map, for example:

1. Assume that $\dim(G) \leq 6$, then every nilmanifold G/Γ admits an expanding map.
2. Assume that the nilpotency class of G is ≤ 2, then every nilmanifold G/Γ admits an expanding map.

Theorem (D. - 2015)

There exists a nilmanifold admitting an Anosov diffeomorphism but no expanding map.

Question

What is the smallest possible dimension of such an example?

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