Which manifolds admit expanding maps?

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Overview

What are expanding maps?

What are nilmanifolds?

Which nilmanifolds admit an expanding map?

What is an Anosov diffeomorphism?

What is the rational Mal’cev completion?
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Notations and definition

\[ M \]
\[ \nu \in TM \]
\[ \|\nu\| \]
\[ f : M \to M \]
\[ Df : TM \to TM \]
\[
\text{closed Riemannian manifold} \\
\text{tangent vector} \\
\text{norm of } \nu \\
\text{endomorphism of } M \\
\text{derivative of } f
\]

**Definition**

We call \( f \) **expanding** if there exist real constants \( c > 0, \lambda > 1 \) such that

\[ \forall \nu \in TM : \|Df^n(\nu)\| \geq c\lambda^n\|\nu\|. \]

**Remark:** This definition is independent of the choice of Riemannian metric.
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Basic example

Fix $m \in \mathbb{Z}$ with $|m| > 1$ and consider the manifold $S^1 \subseteq \mathbb{C}$ with endomorphism

$$f_m : S^1 \to S^1$$

$$z \mapsto z^m.$$

For every tangent vector $v \in TM$, we have

$$\|Df_m^n(v)\| = |m|^n \|v\|$$

and thus $f_m$ is expanding.

For $m = 2$:
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Questions about expanding maps

Observation

*Expanding maps are dynamical systems with interesting properties like structural stability, chaos, etc.*

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Which manifolds admit an expanding map?

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Is it possible to classify all expanding maps in some way?

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Are there more examples of expanding maps?
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Expanding maps on tori

Recall the main example on $S^1 = \mathbb{R}/\mathbb{Z}$ above:

$$f_m : S^1 \to S^1 : z \mapsto z^m.$$  

By lifting these maps to the universal cover, we get

$$\tilde{f}_m : \mathbb{R} \to \mathbb{R} : x \mapsto mx,$$

with $\tilde{f}_m$ a linear map with only eigenvalues $> 1$.

Conversely, take any matrix $A \in \text{GL}(n, \mathbb{R})$ such that

$$A(\mathbb{Z}^n) \leq \mathbb{Z}^n \quad \text{and} \quad |\lambda| > 1 \quad \forall \lambda \in \text{Spec}(A).$$

Then $A$ induces an expanding map on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

**Observation**

Every $n$-torus $\mathbb{T}^n$ admits an expanding map.
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Every $n$-torus $T^n$ admits an expanding map.
More general Lie groups

Can we generalize the previous example?

1. \( \mathbb{Z}^n \leq \mathbb{R}^n \) lattice in the simply connected and connected abelian Lie group \( \mathbb{R}^n \).

2. \( A \in \text{Aut}(\mathbb{R}^n) = \text{GL}(n, \mathbb{R}) \).

3. \( A(\mathbb{Z}^n) \leq \mathbb{Z}^n \).

4. \( \forall \lambda \in \text{Spec}(A) : |\lambda| > 1 \).

1. \( \Gamma \leq G \) lattice in a simply connected and connected Lie group \( G \).

2. \( \alpha \in \text{Aut}(G) \).

3. \( \alpha(\Gamma) \leq \Gamma \).

4. What is \( \text{Spec}(\alpha) \)?

Definition

The eigenvalues of \( \alpha \in \text{Aut}(G) \) are defined as the eigenvalues of the derivative \( D\alpha \in \text{Aut}(\mathfrak{g}) \) where \( \mathfrak{g} \) is the Lie algebra corresponding to \( G \).
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Which Lie groups $G$?

We look for automorphisms with only eigenvalues $> 1$ in absolute value.

Theorem (Jacobson - 1955)

Let $\mathfrak{g}$ be a Lie algebra and $\varphi \in \text{Aut}(\mathfrak{g})$ an automorphism with $|\lambda| \neq 1$ for all $\lambda \in \text{Spec}(\varphi)$, then $\mathfrak{g}$ is nilpotent.

Definition

Assume that $G$ is a connected and simply connected nilpotent Lie group, $\Gamma \leq G$ a lattice and $\alpha \in \text{Aut}(G)$ with $\alpha(\Gamma) \leq \Gamma$.

1. The manifold $G/\Gamma$ is called a nilmanifold.
2. The induced map $\bar{\alpha} : G/\Gamma \to G/\Gamma$ is called a nilmanifold endomorphism.

Remark: The map $\bar{\alpha}$ is expanding if and only if $|\lambda| > 1$ for all $\lambda \in \text{Spec}(\alpha)$. 

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Example of a nilmanifold

Let $R$ be a commutative ring with 1 and consider the Heisenberg group:

$$H_3(R) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in R \right\}$$

- For $R = \mathbb{R}$ we get the nilpotent, simply connected and connected Lie group $G = H_3(\mathbb{R})$.
- If $R = \mathbb{Z}$, we get the discrete Heisenberg group $\Gamma = H_3(\mathbb{Z})$, which is a lattice of $G$ and the quotient is a 3-dimensional nilmanifold.
- The map $\alpha : G \to G$ given by

$$\alpha \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax & abz \\ 0 & 1 & by \\ 0 & 0 & 1 \end{pmatrix}.$$

for $a, b \in \mathbb{Z}$ is an automorphism with $\alpha(\Gamma) \leq \Gamma$. If both $|a| > 1$ and $|b| > 1$, then $\bar{\alpha}$ is an expanding endomorphism on $G/H_3(\mathbb{Z})$. 
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- For $k > 0$ define $N_k \leq G$ as the lattice

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Are there more examples?

Theorem (Gromov - 1981)

*Up to finite cover, every expanding map is topologically conjugate to an (expanding) nilmanifold endomorphism.*

Actually, this theorem is a consequence of his celebrated theorem about polynomial growth:

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*A group of polynomial growth is virtually nilpotent.*
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Nilmanifolds with an expanding map

Start with an expanding automorphism $\varphi : g \to g$.

- We can assume that $\varphi$ is semisimple ($\text{Aut}(g)$ is linear algebraic group).

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g = \bigoplus_{r > 0} g_r \quad \text{with} \quad [g_r, g_s] \subseteq g_{r+s};
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where $g_r$ is the sum of eigenspaces for eigenvalues $\lambda$ with $|\lambda| = \exp(r)$.

- After renaming we get

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g = \bigoplus_{i \in \mathbb{N}} g_i \quad \text{with} \quad [g_i, g_j] \subseteq g_{i+j}.
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**Definition**

We say that the Lie algebra $g$ has a **positive grading**.
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Algebraic characterization for expanding maps

Theorem ()

Let $G/\Gamma$ be a nilmanifold, then it holds that:

The nilmanifold $G/\Gamma$ admits an expanding map.

$\Downarrow$

The Lie group $G$ has an expanding automorphism.

$\Downarrow$

The Lie algebra $\mathfrak{g}$ has a positive grading.

Remarks:

1. The result also holds for the bigger class of infra-nilmanifolds.
2. There is a similar statement for the existence of non-trivial self-covers.
3. Independent proof by Y. Cornulier for the situation of nilmanifolds.
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- The nilmanifold $G/\Gamma$ admits an expanding map.
- The Lie group $G$ has an expanding automorphism.
- The Lie algebra $\mathfrak{g}$ has a positive grading.

Remarks:

1. The result also holds for the bigger class of infra-nilmanifolds.
2. There is a similar statement for the existence of non-trivial self-covers.
3. Independent proof by Y. Cornulier for the situation of nilmanifolds.
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Every injective group morphism of $\Gamma$ induces an automorphism of the Lie group $G$, so

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(1) (2)
Overview

What are expanding maps?

What are nilmanifolds?

Which nilmanifolds admit an expanding map?

What is an Anosov diffeomorphism?

What is the rational Mal’cev completion?
What is an Anosov diffeomorphism?

**Definition**

Let $M$ be a closed Riemannian manifold with diffeomorphism $f : M \to M$, then we call $f$ an **Anosov diffeomorphism** if:

(i) there exists a continuous splitting of the tangent bundle

$$TM = E^s \oplus E^u;$$

(ii) the subbundles $E^s$ and $E^u$ are preserved under the map $Df : TM \to TM$, i.e.

$$Df(E^s) = E^s \text{ and } Df(E^u) = E^u;$$

(iii) there exist real constants $0 < \lambda < 1$ and $c > 0$ such that

$$\forall v \in E^s, \forall k > 0 : \| Df^k(v) \| \leq c \lambda^k \| v \|,$$

$$\forall v \in E^u, \forall k > 0 : \| Df^k(v) \| \geq \frac{1}{c \lambda^k} \| v \|.$$
Arnold’s cat map

Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then $A$ induces an Anosov diffeomorphism $\varphi$ on $\mathbb{T}^2$.

The map $\varphi$ is called **Arnold’s cat map**.
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Observation

Arnold’s cat map is chaotic.
Arnold’s cat map

Conjecture

Anosov diffeomorphisms are chaotic.
Examples of Anosov diffeomorphisms

1. By combining the matrices

\[
\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},
\]

we can construct Anosov diffeomorphisms on all $\mathbb{T}^n$ with $n > 1$.

2. On the contrary, $\mathbb{T}^1 = S^1$ doesn’t admit an Anosov diffeomorphism, since it has no expanding diffeomorphisms.

3. Every hyperbolic affine infra-nilmanifold automorphism is an Anosov diffeomorphism.

Conjecture

Every Anosov diffeomorphism is topologically conjugate to a hyperbolic affine infra-nilmanifold automorphism.
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Every Anosov diffeomorphism is topologically conjugate to a hyperbolic affine infra-nilmanifold automorphism.
More examples

Question

Which nilmanifolds admit an Anosov diffeomorphism?
More examples

(1) The nilmanifolds $H_3(\mathbb{R})/N_k$ do not admit an Anosov diffeomorphism. Indeed, $Z(N_k) \cong \mathbb{Z}$ and thus every automorphism has eigenvalue $\pm 1$.

(2) If $R = \mathbb{Z}[\sqrt{2}]$ (or $\mathbb{Z}[\sqrt{d}]$ with $d$ a square-free natural number), then we can embed $\Gamma = H_3(R)$ as a discrete subgroup of $H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$ via

$$H_3(R) \hookrightarrow G \oplus G : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \sigma(x) & \sigma(z) \\ 0 & 1 & \sigma(y) \\ 0 & 0 & 1 \end{pmatrix}$$

where $\sigma : R \rightarrow R$ is the unique ring morphism with $\sigma(\sqrt{2}) = -\sqrt{2}$. Then $H_3(R)$ is a lattice in $H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$.

Take $\alpha$ the automorphism of $\Gamma$ as above with $a = b = 1 + \sqrt{2}$. Then $\alpha$ extends to an automorphism of $H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$ which induces an Anosov diffeomorphism on the quotient.

On the other hand, the lattice $H_3(\mathbb{Z}) \oplus H_3(\mathbb{Z})$ doesn’t admit Anosov diffeomorphisms.
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What is the rational Mal’cev completion?

Since $G$ is a simply connected and connected Lie group, the exponential map is a diffeomorphism:

$$G \xrightarrow{\exp} \mathfrak{g}.$$  

The group multiplication and the Lie bracket are related by the **Baker-Campbell-Hausdorff formula**.

The group $\Gamma$ is a lattice of the Lie group $G$. The rational span $\mathfrak{n}^\mathbb{Q} = \mathbb{Q} \log(\Gamma)$ is a rational Lie algebra and $\Gamma^\mathbb{Q} = \exp(\mathfrak{n}^\mathbb{Q})$ is a subgroup of $G$. The group $\Gamma^\mathbb{Q}$ is called the rational Mal’cev completion of $\Gamma$. Note that every injective group morphism of $\Gamma$ extends to an automorphism of $\Gamma^\mathbb{Q}$, so $\text{Endo}(\Gamma) \subseteq \text{Aut}(\Gamma^\mathbb{Q})$. 
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Examples

1. In the abelian case, we have

\[(\mathbb{Z}^n)^\mathbb{Q} = \mathbb{Q}^n.\]

2. For the Heisenberg group, it holds that

\[(H_3(\mathbb{Z}))^\mathbb{Q} = H_3(\mathbb{Q}) = (N_k)^\mathbb{Q}.\]

Indeed, the diffeomorphism log is given by

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\log \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & z - \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.
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Theorem

Let \(\Gamma_1\) and \(\Gamma_2\) be two lattices of \(G\), then \(\Gamma_1^\mathbb{Q} = \Gamma_2^\mathbb{Q}\) if and only if \(\Gamma_1\) and \(\Gamma_2\) are commensurable.
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(1) Study group morphisms of commensurable nilpotent groups.

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Application of the characterization

The easiest examples of nilmanifolds admit an expanding map, for example:

1. Assume that $\dim(G) \leq 6$, then every nilmanifold $G/\Gamma$ admits an expanding map.

2. Assume that the nilpotency class of $G$ is $\leq 2$, then every nilmanifold $G/\Gamma$ admits an expanding map.

Theorem (D. - 2015)

There exists a nilmanifold admitting an Anosov diffeomorphism but no expanding map.

Question

What is the smallest possible dimension of such an example?
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Thank you!