Positive Ricci curvature and cohomogeneity-two torus actions

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Fernando Galaz-García | August 2, 2016
Motivation

Problem
Construct Riemannian manifolds satisfying given geometric properties.

Possible approach:
- $M$ a compact smooth manifold (without boundary),
- $G$ a compact connected Lie group acting effectively on $M$,
- $g$ a $G$-invariant Riemannian metric.

Question: When does a closed $G$-manifold admit an invariant Riemannian metric with positive Ricci curvature?
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Question: When does a closed $G$-manifold admit an invariant Riemannian metric with positive Ricci curvature?
The **cohomogeneity** of an action is the dimension of its orbit space.

**Positive Ricci curvature on homogeneous spaces and cohomogeneity one manifolds**

- Berestovskii (1995): $M = G/H$ admits an invariant metric of positive Ricci curvature if and only if $|\pi_1(M)| < \infty$.

- Grove, Ziller (2002): $M$ of cohomogeneity one admits an invariant metric of positive Ricci curvature if and only if $|\pi_1(M)| < \infty$. 

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**Motivation and background**

**Cohomogeneity two torus actions**

**Outline of the proof**
Background

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Positive Ricci curvature and cohomogeneity two

- Searle, Wilhelm (2015): \( M \) of cohomogeneity two. If the fundamental group of a principal orbit is finite and the orbit space has positive Ricci curvature, then \( M \) admits an invariant metric of positive Ricci curvature.

- Bazaikin, Matvienko (2007): Every compact, simply connected 4-manifold with an effective action of \( T^2 \) admits an invariant metric of positive Ricci curvature.

Remark: Every compact, simply connected 4-manifold with an effective action of \( T^2 \) is equivariantly diffeomorphic to a connected sum of copies of \( S^4 \), \( \pm \mathbb{C}P^2 \) or \( S^2 \times S^2 \) (Orlik, Raymond 1970).
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Cohomogeneity two torus actions and positive Ricci curvature

Theorem (–, Corro) 2016

Every compact, smooth, simply connected \((n + 2)\)-manifold with a smooth, effective action of a torus \(T^n\) admits an invariant Riemannian metric of positive Ricci curvature.

- There exist compact, simply connected manifolds with a cohomogeneity two torus action in every dimension \(n \geq 2\).
- The topological classification is only known up to dimension \(n \leq 6\).
Cohomogeneity two torus actions and positive Ricci curvature

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*Every compact, smooth, simply connected* $(n + 2)$*-manifold with a smooth, effective action of a torus $T^n$ admits an invariant Riemannian metric of positive Ricci curvature.*

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Torus actions and positive Ricci curvature

Corollary

For every integer $k \geq 4$, every connected sum of the form

\[
\#(k - 3)(S^2 \times S^3),
\]

\[
(S^2 \tilde{\times} S^3)\#(k - 4)(S^2 \times S^3),
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#(k - 4)(S^2 \times S^4)#(k - 3)(S^3 \times S^3),
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(S^2 \tilde{\times} S^4)#(k - 5)(S^2 \times S^4)#(k - 3)(S^3 \times S^3),
\]

has a metric with positive Ricci curvature invariant under a cohomogeneity-two torus action.

- Follows from the topological classification of compact, simply connected 5- and 6-manifolds with cohomogeneity two torus actions (Oh, 1983–1982).
- The manifolds in (1) are not new examples (Sha, Yang, 1991).
Corollary

For every integer $k \geq 4$, every connected sum of the form

\begin{align*}
#(k - 3)(S^2 \times S^3), \\
(S^2 \times S^3)#(k - 4)(S^2 \times S^3), \\
#(k - 4)(S^2 \times S^4)#(k - 3)(S^3 \times S^3), \\
(S^2 \times S^4)#(k - 5)(S^2 \times S^4)#(k - 3)(S^3 \times S^3),
\end{align*}

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Cohomogeneity two torus actions on simply connected manifolds

$M$ a compact simply connected $(n + 2)$-manifold, $n \geq 2$, with a cohomogeneity two action of $T^n$.

These manifolds were studied in the 1970s-1980s.


- The orbit space $M^*$ is homeomorphic to $D^2$.
- The only isotropy groups are $T^2$, $T^1$ and trivial.
- The boundary of $M^*$ consists of $m \geq n$ edges $\Gamma_i$ with circle isotropy $G(a_i)$ and $m$ vertices $F_i$ between the edges $\Gamma_i$ and $\Gamma_{i+1}$, with isotropy $G(a_i) \times G(a_{i+1})$. 
Cohomogeneity two torus actions on simply connected manifolds

Orbit space structure of a cohomogeneity-two torus action on a compact, simply connected manifold $M$. 

$M^*$ trivial isotropy

$G(a_i)$ isotropy

$G(a_i) \times G(a_{i+1})$ isotropy

$G(a_i)$ isotropy

$F_{i+1}$

$\Gamma_i$

$F_i$

$F_2$

$\Gamma_2$

$\Gamma_1$

$F_1$
Cohomogeneity two torus actions on simply connected manifolds

The orbit space is decorated with isotropy information, the so-called weights.

**Definition**

Let $M$ and $N$ be two compact, simply connected smooth $(n + 2)$-manifolds with effective $T^n$ actions. The orbit spaces $M^*$ and $N^*$ are isomorphic if there exists a weight-preserving diffeomorphism between them.

**Theorem (Kim, McGavran, Pak 1974, Oh 1983)**

Two closed, simply connected smooth $(n + 2)$-manifolds with an effective $T^n$-action are equivariantly diffeomorphic if and only if their orbit spaces are isomorphic.
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**Definition**

Let $M$ and $N$ be two compact, simply connected smooth $(n + 2)$-manifolds with effective $T^n$ actions. The orbit spaces $M^*$ and $N^*$ are **isomorphic** if there exists a weight-preserving diffeomorphism between them.

**Theorem (Kim, McGavran, Pak 1974, Oh 1983)**

Two closed, simply connected smooth $(n + 2)$-manifolds with an effective $T^n$-action are equivariantly diffeomorphic if and only if their orbit spaces are isomorphic.
Outline of the proof

Let $M$ be a compact, simply connected $(n + 2)$-manifold with a cohomogeneity two action of $T^n$. Assume $n \geq 2$.

- Let $m$ be the number of vertices in the orbit space (i.e. the number of orbits with isotropy $T^2$).
- Construct an $(m + 2)$-manifold $N_m$ with an effective $T^m$-action and a free action of a $T^{m-n}$ subgroup of $T^m$ so that $N_m/T^{m-n}$ has an induced cohomogeneity two action of $T^n$ with the same weights as the $T^n$ action on $M$.

\[ N_m = (D^2 \times T^m)/\sim \]

- By the equivariant classification theorem, $M$ and $N_m/T^{m-n}$ are equivariantly diffeomorphic.
- To construct the metric, one considers two cases:
  
  (a) the orbit space has at least 5 vertices.
  
  (b) the orbit space has at most 4 vertices.
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Outline of the proof

Construction of the metric

Case (a): The orbit space has at least 5 vertices.

- Construct a piecewise-smooth $C^1$ metric on $N_m = (D^2 \times T^m)/\sim$ that is invariant under the $T^{m-n}$ action.
- This induces a piecewise-smooth $C^1$ Riemannian metric $g$ on $N_m/T^{m-n}$.
- The metric $g$ has positive Ricci curvature (O’Neill formulas).
- Smooth out the metric $g$ while preserving positive Ricci curvature.

Case (b): The orbit space has at most 4 vertices.

- The manifold $M$ is equivariantly diffeomorphic to $S^4$, $S^5$, $S^3 \times S^3$ or to a quotient of $S^3 \times S^3$ by a free linear torus action.
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Thank you