Moduli spaces of Type $\mathcal{A}$ geometries
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Outline

1. Moduli space Type $\mathcal{A}$ surfaces with torsion and Rank(ρ_s)=2.
2. Moduli space Type $\mathcal{A}$ manifolds with torsion and Rank(ρ_s)=m
3. Moduli space Type $\mathcal{A}$ torsion free surfaces and Rank(ρ)=1.
4. Moduli space Type $\mathcal{A}$ torsion free surfaces with Rank(ρ)=2.
5. Module spaces of Type $\mathcal{B}$ geometries.

Definition

Let $\mathcal{M} := (M, \nabla)$ where ∇ is a connection on the tangent bundle of a smooth manifold M of dimension m. We say ∇ is *torsion free* if $\nabla_X Y - \nabla_Y X = [X, Y]$. Let $\vec{x} = (x^1, \dots, x^m)$ be a system of local coordinates on M. Adopt the *Einstein convention* and sum over repeated indices to expand $\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}{}^k \partial_{x^k}$ in terms of the *Christoffel symbols* $\Gamma = (\Gamma_{ij}{}^k)$; the condition that ∇ is torsion free is then equivalent to the symmetry $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$.

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Theorem

$\mathcal{M}$ is torsion free if and only if for every point P of M, there exists coordinates centered at P so $\Gamma_{ij}{}^k(P) = 0$.

Local homogeneity

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Some homogeneous examples

Type $\mathcal{A}$. Let $M := \mathbb{R}^m$ and let $\Gamma \in (\mathbb{R}^{2m})^* \otimes \mathbb{R}^m$ be constant. The translation group $\mathbb{R}^m$ acts transitively on M and preserves ∇.

Type $\mathcal{B}$. Let $M = \mathbb{R}^+ \times \mathbb{R}^{m-1}$ and let $\Gamma_{ij}{}^k = \frac{C_{ij}{}^k}{x^1}$ for $C \in (\mathbb{R}^{2m})^* \otimes \mathbb{R}^m$ constant. The group $(x^1, \dots) \to (ax^1, ax^2 + b^2, \dots, ax^m + b^m)$ for $a > 0$ acts transitively on M and preserves ∇.

Type $\mathcal{C}$. Let ∇ be the Levi-Civita connection on a complete simply connected pseudo-Riemannian manifold M of constant sectional curvature.

Theorem (Opozda; Arias-Marco and Kowalski)

Let $\mathcal{M} = (M, \nabla)$ be a locally homogeneous surface, possibly with torsion. Then at least one of the following three possibilities holds, which are not exclusive, and which describe the local geometry:

Type $\mathcal{A}$: There exist local coordinates (x^1, x^2) so that $\Gamma_{ij}{}^k$ is constant.

Type $\mathcal{B}$: There exist local coordinates (x^1, x^2) so that $\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$ where $C_{ij}{}^k$ is constant.

Type $\mathcal{C}$: ∇ is the Levi-Civita connection of a metric of constant sectional curvature.

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[a] B. Opozda, "A classification of locally homogeneous connections on 2-dimensional manifolds", J. Diff. Geo. Appl. **21** (2004), 173–198.

[b] T. Arias-Marco and O. Kowalski, "Classification of locally homogeneous affine connections with arbitrary torsion on 2-manifolds", *Monatsh. Math.* **153** (2008), 1–18

Classification of locally homogeneous surfaces II.

Observation

1. There are no surfaces which are both Type $\mathcal{A}$ and Type $\mathcal{C}$.
2. There are surfaces which are both Type $\mathcal{A}$ and Type $\mathcal{B}$; we will characterize these geometries presently from both the Type $\mathcal{A}$ and from the Type $\mathcal{B}$ perspectives.
3. Any surface which is both Type $\mathcal{B}$ and Type $\mathcal{C}$ is modeled either on the hyperbolic plane or on a Lorentzian analogue.

Definition

The curvature operator R, the Ricci tensor ρ, and symmetric Ricci tensor ρ_s are given by

$$R(\xi_1,\xi_2) := \nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]},$$
$$\rho(\xi_1,\xi_2) := \operatorname{Tr}\{\xi_3 \to R(\xi_3,\xi_1)\xi_2\},$$
$$\rho_s(\xi_1,\xi_2) := \tfrac{1}{2}\{\rho(\xi_1,\xi_2)+\rho(\xi_2,\xi_1)\}.$$

If $\mathcal{M}$ is a Type $\mathcal{A}$ manifold of dimension $m \geq 2$, then

$$R_{abc}{}^{d} = \Gamma_{ae}{}^{d}\Gamma_{bc}{}^{e} - \Gamma_{be}{}^{d}\Gamma_{ac}{}^{e},$$
$$\rho_{bc} = \Gamma_{ae}{}^{a}\Gamma_{bc}{}^{e} - \Gamma_{be}{}^{a}\Gamma_{ac}{}^{e},$$
$$\rho_{s,bc} = \tfrac{1}{2}\{\rho_{bc}+\rho_{cb}\}.$$

If $\mathcal{M}$ is also torsion free, then $\rho = \rho_s$. If $m = 2$ and if ρ is zero, then $\mathcal{M}$ is flat. However, if $m = 3$ and if we take $\Gamma_{13}{}^{1} = \Gamma_{31}{}^{1} = 21$, $\Gamma_{23}{}^{2} = \Gamma_{32}{}^{2} = 28$, and $\Gamma_{33}{}^{3} = 25$, then $\mathcal{M}$ is torsion free, $\rho = 0$ and $R \neq 0$. So this geometry is not flat although $\rho = 0$.

Definition: Let G be a Lie group which acts smoothly on a smooth manifold N.

• Let $G_P := \{g \in G : gP = P\}$ be the isotropy group.

• The action is *fixed point free* if $G_P = \{\text{id}\}$ for all P.

• The action is *proper* if given points $P_n \in N$ and $g_n \in G$ with $P_n \to P \in N$ and $g_n P_n \to \tilde{P} \in N$, we can choose a convergent subsequence so $g_{n_k} \to g \in G$.

Theorem. Let G be a fixed point free proper smooth action of a Lie group G on a smooth manifold N. Then the orbit space N/G inherits a smooth structure so that $G \to N \to N/G$ is a principal G bundle.

S. Gallot, D. Hulin, J. Lafontaine, "Riemannian Geometry 3rd ed", Springer Universitext (2014). Theorem 1.95 Page 32.

Parameter spaces for Type $\mathcal{A}$ geometry

Definition

1) Let $\mathcal{W}(m) := (\mathbb{R}^{2m})^* \otimes \mathbb{R}^m$ be the parameter space for Type A geometry with torsion; let $\mathcal{Z}(m) := S^2(\mathbb{R}^m) \otimes \mathbb{R}^m$ be the parameter space for torsion free Type $\mathcal{A}$ geometry.
2) If $\Gamma \in \mathcal{Z}(m)$, let $G_\Gamma^+ := \{g \in \mathrm{GL}^+(2, \mathbb{R}^m) : g\Gamma = \Gamma\}$ and $G_\Gamma := \{g \in \mathrm{GL}(2, \mathbb{R}^m) : g\Gamma = \Gamma\}$ be the isotropy subgroups of the natural action of these groups on $(\mathbb{R}^{2m})^* \otimes \mathbb{R}^m$.
3) Let $\mathcal{W}(p, q)$ (resp. $\mathcal{Z}(p, q)$) be the set of Type $\mathcal{A}$ connections with torsion (resp. without torsion) so that $\rho_{s,\Gamma}$ has signature (p, q) for $p + q = m$. p-timelike and q-spacelike.
4) Let $\mathfrak{W}(p, q)$ (resp. $\mathfrak{Z}(p, q)$) be the associated moduli spaces in the unoriented category. Identify two structures if exists the germ of a diffeomorphism intertwining them.
5) Let $\mathfrak{W}^+(p, q)$ (resp. $\mathfrak{Z}^+(p, q)$) be the associated moduli spaces in the oriented category. Require the diffeomorphism to preserve the orientation.

Isomorphism type of Type $\mathcal{A}$ manifolds with torsion

Theorem. (Brozos-Vázquez, García-Río, and Gilkey[d])

$$\mathfrak{W}^+(p,q)=\mathcal{W}(p,q)/\operatorname{GL}^+(m,\mathbb{R}),\ \mathfrak{W}(p,q)=\mathcal{W}(p,q)/\operatorname{GL}(m,\mathbb{R}),$$
$$\mathfrak{Z}^+(p,q)=\mathcal{Z}(p,q)/\operatorname{GL}^+(m,\mathbb{R}),\quad \mathfrak{Z}(p,q)=\mathcal{Z}(p,q)/\operatorname{GL}(m,\mathbb{R}).$$

[d] "Homogeneous affine surfaces: Moduli spaces" arXiv 1604.06610.

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Proof

Let $\mathcal{M}$ be a Type $\mathcal{A}$ manifold. Choose a Type $\mathcal{A}$ coordinate atlas so that $\Gamma\in(\mathbb{R}^{2m})^*\otimes\mathbb{R}^m$ is constant on each chart. The symmetric Ricci tensor is an invariantly defined pseudo-Riemannian metric on $\mathcal{M}$ which is preserved by the coordinate transformations. Since Γ is constant, the components of ρ_s are constant. Thus ρ_s is flat and the coordinate transformations are affine; they take the form $\vec{x}\to A\vec{x}+\vec{b}$ where $A\in\operatorname{GL}(m,\mathbb{R})$ and $\vec{b}\in\mathbb{R}^m$; if we are dealing with oriented structures, then $A\in\operatorname{GL}^+(m,\mathbb{R})$. The translations do not affect Γ. The desired result now follows. □

The oriented moduli spaces in dimension 2

Theorem. (Gilkey)

1) Let $(p,q)=(1,1)$ or $(p,q)=(0,2)$. The action of $\mathrm{GL}^+(2,\mathbb{R})$ on $\mathcal{W}(p,q)$ is proper and fixed point free; $\mathfrak{W}^+(p,q)$ and $\mathfrak{Z}^+(p,q)$ admit smooth structures so that $\mathcal{W}^+(p,q)\to\mathfrak{W}^+(p,q)$ and $\mathcal{Z}^+(p,q)\to\mathfrak{Z}^+(p,q)$ are principal $\mathrm{GL}^+(2,\mathbb{R})$ bundles.
2) Let $(p,q)=(2,0)$ (neg. defn.). Let $\{{\Gamma_{0,11}}^1=-1$, ${\Gamma_{12}}^2=1$, ${\Gamma_{0,21}}^2=1$, ${\Gamma_{0,22}}^1=1$, and ${\Gamma_{0,ij}}^k=0$ otherwise$\}$ define $\Gamma_0\in\mathcal{Z}(2,0)$. Let $C_0:=\mathrm{GL}^+(2,\mathbb{R})\cdot\Gamma_0$; C_0 is a closed orbit in $\mathcal{Z}(2,0)$ and $\mathcal{W}(2,0)$. If $\Gamma\in\mathcal{W}(2,0)$ satisfies $G_\Gamma^+\neq\{\mathrm{id}\}$, then $\Gamma\in C_0\subset\mathcal{Z}(2,0)$ and $G_\Gamma^+=\mathbb{Z}_3$. The punctured oriented moduli spaces admit smooth structures so the projections $\mathcal{W}^+(2,0)-C_0\to\mathfrak{W}^+(2,0)-[C_0]$ and $\mathcal{Z}^+(2,0)-C_0\to\mathfrak{Z}^+(2,0)-[C_0]$ are principal $\mathrm{GL}^+(2,\mathbb{R})$ bundles. The unpunctured oriented moduli spaces $\mathfrak{W}^+(2,0)$ and $\mathfrak{Z}^+(2,0)$ are $\mathbb{Z}_3$ orbifolds.

P. Gilkey, "The moduli space of Type $\mathcal{A}$ surfaces with torsion and non-singular symmetric Ricci tensor", arXiv:1605.06698.

The unoriented moduli spaces in dimension 2

Theorem. (Gilkey)

1) Let $(p,q)=(1,1)$ or let $(p,q)=(0,2)$. The unoriented moduli space $\mathfrak{W}(p,q)$ (resp. $\mathfrak{Z}(p,q)$) admits a smooth structure as a 4-dimensional manifold without boundary (resp. 2-dimensional manifold with boundary) so $\mathfrak{W}^+(p,q)\to\mathfrak{W}(p,q)$ (resp. $\mathfrak{Z}^+(p,q)\to\mathfrak{Z}(p,q)$) is a ramified double cover where the ramification occurs over a smooth 2-dimensional surface (resp. curve which creates the boundary).
2) Let $(p,q)=(2,0)$. Assertion 1 holds for the punctured unoriented moduli space. Let s_3 be the symmetric group; it is generated by permutations T_i of order 2 and 3. Let s_3 act on $\mathbb{C}\oplus\mathbb{C}$ or $\mathbb{C}$ so that T_3 is multiplication by a third root of unity and T_2 is complex conjugation. Then $\mathfrak{W}(p,q)$ (resp. $\mathfrak{Z}(2,0)$) has a s_3 orbifold structure where s_3 acts on $\mathbb{C}\oplus\mathbb{C}$ (resp. $\mathbb{C}$) at the singular orbit.

Definition

Let $\mathfrak{P}_m = \mathfrak{P}_m(\Gamma)$ be a polynomial defined on $(\mathbb{R}^{2m})^* \otimes \mathbb{R}^m$ which is divisible by $\det(\rho_{s,\Gamma})$ and which doesn't vanish identically on $S^2(\mathbb{R}^m) \otimes \mathbb{R}^m$. Let

$$\mathcal{W}(p,q;\mathfrak{P}_m) := \{\Gamma \in \mathcal{W}(p,q) : \mathfrak{P}_m(\Gamma) \neq 0\},$$
$$\mathcal{Z}(p,q;\mathfrak{P}_m) := \{\Gamma \in \mathcal{Z}(p,q) : \mathfrak{P}_m(\Gamma) \neq 0\}.$$

These are open dense subsets of $\mathcal{W}(p,q)$ and $\mathcal{Z}(p,q)$, respectively; the Christoffel symbols so $\mathfrak{P}_m(\Gamma) \neq 0$ are *generic*.

Higher dimensional Type $\mathcal{A}$ manifolds II.

Theorem (Gilkey-Park[b])

There exists $\mathfrak{P}_m$ so that $\operatorname{GL}(m,\mathbb{R})$ preserves $\mathcal{W}(p,q;\mathfrak{P}_m)$ and $\mathcal{Z}(p,q;\mathfrak{P}_m)$ and so that the action is proper and fixed point free. Thus there are smooth structures on the associated moduli spaces

$$\mathcal{W}(p,q;\mathfrak{P}_m)/\operatorname{GL}^+(m,\mathbb{R}),\qquad \mathcal{W}(p,q;\mathfrak{P}_m)/\operatorname{GL}(m,\mathbb{R}),$$
$$\mathcal{Z}(p,q;\mathfrak{P}_m)/\operatorname{GL}^+(m,\mathbb{R}),\qquad \mathcal{Z}(p,q;\mathfrak{P}_m)/\operatorname{GL}(m,\mathbb{R}),$$

and the projections from $\mathcal{W}(p,q;\mathfrak{P}_m)$ and $\mathcal{Z}(p,q;\mathfrak{P}_m)$ to these moduli spaces are smooth principal bundles. Furthermore, the projections from the oriented to the unoriented moduli spaces are $\mathbb{Z}_2$ covering projections.

[b] Moduli spaces of oriented Type $\mathcal{A}$ manifolds of dimension at least 3 arXiv:1607.01563

Theorem (Gilkey-Park)

Suppose that $m\geq 3$.

1) There exists $c(m)$ so that if G_Γ^+ contains no elements of infinite order, then every element of G_Γ^+ has order at most $c(m)$.

2) $\left\{\Gamma\in\mathcal{W}(p,q):\dim\{G_\Gamma\}\geq 1\right\}$ is closed.
$\left\{\Gamma\in\mathcal{Z}(p,q):\dim\{G_\Gamma\}\geq 1\right\}$ is closed.

3) If $p=0$ or $q=0$, then

- **a.** The action of $\mathrm{GL}(m,\mathbb{R})$ on $\mathcal{Z}(p,q)$ and $\mathcal{W}(p,q)$ is proper.
- **b.** If $\Gamma\in\mathcal{W}(p,q)$, then either G_Γ^+ is finite or $\dim\{G_\Gamma^+\}\geq 1$.
  If $\Gamma\in\mathcal{Z}(p,q)$, then either G_Γ^+ is finite or $\dim\{G_\Gamma^+\}\geq 1$.
- **c.** $\left\{\Gamma\in\mathcal{W}(p,q):G_\Gamma\neq\{\mathrm{id}\}\right\}$ is closed in $\mathcal{W}(p,q)$.
  $\left\{\Gamma\in\mathcal{Z}(p,q):G_\Gamma\neq\{\mathrm{id}\}\right\}$ is closed in $\mathcal{Z}(p,q)$.

4) If $p\geq 1$ and $q\geq 1$, then the action of $\mathrm{GL}(m,\mathbb{R})$ on $\mathcal{Z}(p,q)$ and on $\mathcal{W}(p,q)$ is not proper.

Choose $X\in T_PM$ so $\rho(X,X)\neq 0$. Set

$$\alpha_X(\mathcal{M}):=\nabla\rho(X,X;X)^2\cdot\rho(X,X)^{-3}.$$

$$\epsilon_X(\mathcal{M}):=\text{Sign}\{\rho(X,X)\}=\pm 1.$$

Theorem. (Brozos-Vázquez, García-Río, and Gilkey[c])

Let $\mathcal{M}=(M,\nabla)$ be a torsion free Type $\mathcal{A}$ surface. Let $\rho:=\rho_\nabla$.
1) If $\text{Rank}\{\rho\}=1$, then $\alpha(\mathcal{M}):=\alpha_X(\mathcal{M})$ and $\varepsilon(\mathcal{M}):=\varepsilon_X(\mathcal{M})$ are invariants of the underlying structure and are independent of the particular X chosen, and the moduli space of torsion free Type $\mathcal{A}$ surfaces with $\text{Rank}\{\rho\}=1$ is $(-\infty,0]\dot\cup[0,\infty)$.
2) $\mathcal{M}$ is also of Type $\mathcal{B}$ if and only if $\text{Rank}(\rho)=1$ and either $\alpha(M)\notin(0,16)$ or $\alpha(M)=0$ and $\epsilon(M)<0$.

[c] "Homogeneous affine surfaces: Killing vector fields and gradient Ricci solitons", http://arxiv.org/abs/1512.05515.

Theorem. (Brozos-Vázquez, García-Río, and Gilkey[d])

Let $\mathcal{M} = (M, \nabla)$ be a locally homogeneous oriented torsion free surface of Type $\mathcal{A}$ where $\text{Rank}\{\rho\} = 2$. Set

$$\check{\rho}_{ij} := \Gamma_{ik}{}^{l}\Gamma_{jl}{}^{k},\ \psi := \text{Tr}_{\rho}\{\check{\rho}\} = \rho^{ij}\check{\rho}_{ij},\ \Psi := \det(\check{\rho})/\det(\rho),$$
$$\chi(\Gamma) := \rho(\Gamma_{ab}{}^{b}\Gamma_{ij}{}^{k}\check{\rho}_{kl}\rho^{ij}dx^{a} \wedge dx^{l}, \text{dvol}).$$

1. ψ, Ψ, and χ are invariantly defined on $\mathcal{M}$ and do not depend on the particular choice of Type $\mathcal{A}$ coordinates.

2) $\Xi(p,q) := (\psi, \Psi, \chi)$ is a 1-1 map from each $\mathfrak{Z}^{+}(p,q)$ to a closed surface in $\mathbb{R}^3$ and provides a complete set of invariants in the oriented context.

3. $\Theta(p,q) := (\psi, \Psi)$ defines a 1-1 map from $\mathfrak{Z}(p,q)$ to a simply connected closed subset $\mathfrak{V}(p,q)$ of $\mathbb{R}^2$ and provides a complete set of invariants in the unoriented context.

[d] "Homogeneous affine surfaces: Moduli spaces", to appear JMAA.

Proof of Invariance

Since the rank of the symmetric Ricci tensor is 2, the structure group is $\mathrm{GL}^+(m,\mathbb{R})$ or $\mathrm{GL}(m,\mathbb{R})$. Contracting an upper index against a lower index is invariant under the action of the general linear group. It now follows that ψ, Ψ, and χ are invariants and descend to invariants on the moduli space which are smooth away from the singular orbit.

Description of $\mathfrak{V}(p,q)$

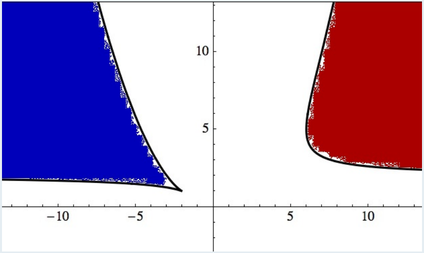
Consider the two curves

$$\sigma_+(t) := (4t^2 + \tfrac{1}{t^2} + 2, 4t^4 + 4t^2 + 2),$$
$$\sigma_-(t) := (-4t^2 - \tfrac{1}{t^2} + 2, 4t^4 - 4t^2 + 2).$$

Note that $\sigma_\pm(t) = \sigma_\mp(\sqrt{-1}t)$. The curve σ_+ is smooth; the curve σ_- has a cusp at $(-2,1)$ when $t = \frac{1}{\sqrt{2}}$. These two curves divide the plane into 3 open regions. The set $\mathfrak{V}(2,0)$ (in blue below) lies in the second quadrant and is bounded on the right by σ_-, the set $\mathfrak{V}(0,2)$ (in red below) lies in the first quadrant, and is bounded on the left by σ_+ and the set $\mathfrak{V}(1,1)$ (in white) lies in between and is bounded on the left by σ_- and on the right by σ_+.

Remark on the Domains

$\Theta(p,q)$ is 1-1 on $\mathfrak{Z}(p,q)$ and $\Theta(1,1)(\mathfrak{Z}(1,1))$ intersects $\Theta(0,2)(\mathfrak{Z}(0,2))$ (resp. $\Theta(2,0)(\mathfrak{Z}(2,0))$) along their common boundary σ_+ (resp. σ_-). This does not mean that $\mathfrak{Z}(1,1)$ intersects $\mathfrak{Z}(0,2)$ (resp. $\mathfrak{Z}(2,0)$) nor does it mean that $\Theta(p,q)$ is not 1-1 on the respective domains. There is an apparent cusp in the picture. For $(p,q)=(1,1)$, this is an artifact of the parametrization and for $(p,q)=(2,0)$, we have replaced a corner with an angle of $\frac{2\pi}{3}$ by a cusp; again, this is an artifact.



Theorem (Gilkey-Park)

Let $\Gamma \in \mathcal{Z}(p,q)$ for $p+q=3$. Assume $G_\Gamma^+ \neq \{\text{id}\}$. We can make a linear change of coordinates so that one of the following 4 possibilities holds:

Case 1) There exist $(a,b,c,d) \in \mathbb{R}^4$ with $ad \neq 0$ and $-b^2+bd+c(-c+d) \neq 0$ so $G_\Gamma^+ = \text{SO}(1,1)$, ${\Gamma_{12}}^3 = a$, ${\Gamma_{13}}^1 = b$, ${\Gamma_{23}}^2 = c$, ${\Gamma_{33}}^3 = d$,
$\rho = ad(e^1 \otimes e^2 + e^2 \otimes e^1) + (-b^2+bd+c(-c+d))e^3 \otimes e^3$.

Case 2) There exist $(a,b,c,d) \in \mathbb{R}^4$ with $ad \neq 0$ and $bd - b^2 + c^2 \neq 0$ so $G_\Gamma^+ = \text{SO}(2)$, ${\Gamma_{11}}^3 = a$, ${\Gamma_{13}}^1 = b$, ${\Gamma_{13}}^2 = c$, ${\Gamma_{22}}^3 = a$, ${\Gamma_{23}}^1 = -c$, ${\Gamma_{23}}^2 = b$, ${\Gamma_{33}}^3 = d$,
$\rho_\Gamma = ad(e^1 \otimes e^1 + e^2 \otimes e^2) + 2(bd - b^2 + c^2)e^3 \otimes e^3$.

Case 3) There exists $T \in G_\Gamma^+$ with $\nu(T) = 3$. Furthermore, $\nu(S) \leq 3$ for every $S \in G_\Gamma^+$. Then T acts by a rotation through an angle of $\frac{2\pi}{3}$ on $\text{Span}\{e_1, e_2\}$ and there exists $(a, b, c, d) \in \mathbb{R}^4$ so

$$\Gamma_{11}{}^1 = 1, \quad \Gamma_{11}{}^3 = a, \quad \Gamma_{12}{}^2 = -1, \quad \Gamma_{13}{}^1 = b, \quad \Gamma_{13}{}^2 = c,$$
$$\Gamma_{22}{}^1 = -1, \quad \Gamma_{22}{}^3 = a, \quad \Gamma_{23}{}^1 = -c, \quad \Gamma_{23}{}^2 = b, \quad \Gamma_{33}{}^3 = d.$$

with $2(-b^2 + c^2 + bd) \neq 0$, and $ad - 2 \neq 0$.
$\rho_\Gamma = (ad - 2)(e^1 \otimes e^1 + e^2 \otimes e^2) + 2(bd - b^2 + c^2)e^3 \otimes e^3$. We have $G_\Gamma^+ = \mathbb{Z}_3$ except for the following exceptional structures:

Case 3a) $a = 0$, $b = 0$, $c = 1$, $d = 0$, and
$G_\Gamma^+ = \{\text{id}, T, T^2, S_1, TS_1, T^2S_1\} \approx s_3$.

Case 3b) $c = 0$, $a = b = \pm\frac{1}{\sqrt{2}}$, $d = \pm\sqrt{2}$, and $G_\Gamma^+ = a_4$.

Case 4) All elements of G_Γ^+ have order 2. There are two structures:

Case 4a) $G_\Gamma^+ = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\Gamma_{12}{}^3 = 1$, $\Gamma_{13}{}^2 = 1$, $\Gamma_{23}{}^1 = -1$, and $\rho = -2(e^1 \otimes e^1 + e^2 \otimes e^2) + 2e^3 \otimes e^3$.

Case 4b) $G_\Gamma^+ = \mathbb{Z}_2$, $\Gamma_{ij}{}^k = 0$ unless the index 3 appears an odd number of times.

$$\begin{array}{llll} \Gamma_{11}{}^3 := a, & \Gamma_{12}{}^3 = b, & \Gamma_{13}{}^1 := c, & \Gamma_{13}{}^2 = d, \\ \Gamma_{21}{}^3 := b, & \Gamma_{22}{}^3 = e, & \Gamma_{23}{}^1 := f, & \Gamma_{23}{}^2 = g, \\ \Gamma_{31}{}^1 := c, & \Gamma_{31}{}^2 := d, & \Gamma_{32}{}^1 := f, & \Gamma_{32}{}^2 := g, \\ \Gamma_{33}{}^3 := h. & & & \end{array}$$

$$\rho_{11} = -2bd + a(-c + g + h),$$

$$\rho_{12} = \rho_{21} = -de - af + bh,$$

$$\rho_{33} = -c^2 - 2df + ch + g(-g + h),$$

One requires $\det(\rho) \neq 0$.

The moduli space of Type $\mathcal{B}$ surfaces with torsion

Definition

Let $\mathcal{WB}(p,q)$ be the set of connections $(x^1)^{-1}C_{ij}{}^k$ with torsion of Type $\mathcal{B}$ on $\mathbb{R}^+ \times \mathbb{R}$ where the symmetric Ricci tensor is non-degenerate and has signature (p,q) for $p+q=2$. Let $\mathcal{I} := \{T_{a,b} : (x^1,x^2) \to (x^1, ax^1+bx^2)\}$ for $b \neq 0$ and $\mathcal{I}^+$ the connected component of the identity where $b > 0$.

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Theorem. [Gilkey]

If $\Gamma,\tilde\Gamma\in\mathcal{WB}(p,q)$ define locally isomorphic structures for $p+q=2$, then there exists $T\in\mathcal{I}$ so that $T\Gamma=\tilde\Gamma$; thus only the linear action is relevant. The action of $\mathcal{I}^+$ on $\mathcal{WB}(p,q)$ is proper and without fixed points so the projection from $\mathcal{WB}(p,q)$ to the oriented moduli space space $\mathfrak{WB}(p,q):=\mathcal{WB}(p,q)/\mathcal{I}^+$ is a principal $\mathcal{I}^+$ bundle. No element of $\mathcal{W}(p,q)$ is also of Type $\mathcal{A}$.

Definition

Let $\Gamma_{ij}{}^k := \frac{1}{x^1} C_{ij}{}^k$ be a torsion free Type $\mathcal{B}$ connection on $\mathbb{R}^+ \times \mathbb{R}$.

1. Let $\mathcal{ZB}(2) := \{C \in \mathbb{R}^6 : (C_{12}{}^2, C_{22}{}^1, C_{22}{}^2) \neq (0,0,0)\}$. These are the Type $\mathcal{B}$ geometries which are not of Type $\mathcal{A}$.
2. Let $\mathfrak{ZB}^+(2)$ the oriented moduli space.
3. Let $\mathfrak{ZB}(2)$ the unoriented moduli space.
4. Let $\mathcal{I}^+ := \{T : T(x^1, x^2) = (x^1, bx^1 + cx^2)\}$ for $c > 0\}$.

Theorem (Brozos-Vázquez, García-Río, and Gilkey[d])

Let $\Gamma = (x^1)^{-1}C$ define a torsion-free Type $\mathcal{B}$ surface.

1) If $\rho \neq 0$, then the surface is of Type A if and only if $({C_{12}}^2, {C_{22}}^1, {C_{22}}^2) = (0,0,0)$.

2) $\mathfrak{Z}\mathfrak{B}^+(2) = \mathcal{Z}\mathcal{B}(2)/\mathcal{I}^+$.

3) The action of $\mathcal{I}^+$ on $\mathcal{Z}\mathcal{B}(2)$ is proper and without fixed points.

4) $\mathfrak{Z}\mathfrak{B}^+(2)$ is a 4-dimensional manifold.

5) $\mathcal{Z}\mathcal{B}(2) \to \mathfrak{Z}\mathfrak{B}^+(2)$ is a principal $\mathcal{Z}\mathcal{B}(2)$ bundle.

6) $\mathfrak{Z}\mathfrak{B}(2)$ is a smooth 4-dimensional manifold.

7) $\mathfrak{Z}\mathfrak{B}^+(2) \to \mathfrak{Z}\mathfrak{B}(2)$ is a ramified double cover where the ramification set is surface diffeomorphic to $\mathbb{C} - \{0\} \cup \mathbb{C} - \{0\}$.

8) $\mathfrak{Z}\mathfrak{B}^+(2)$ and $\mathfrak{Z}\mathfrak{B}(2)$ are simply connected.

9) $H^2_{\text{DeRham}}(\mathfrak{Z}\mathfrak{B}^+(2)) = \mathbb{R}$ and $H^2_{\text{DeRham}}(\mathfrak{Z}\mathfrak{B}(2)) = \mathbb{R}$.

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