A Note On Polar Representations

Francisco J. Gozzi

USP
São Paulo, Brasil.

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Polar Representations

An orthogonal representation of a compact Lie group G is called *polar* if it admits an orthogonal cross-section, i.e., a linear subspace intersecting every G-orbit and doing so orthogonally.

They were first considered by J. Szenthe and J. Dadok.
Consider connected G.

- Dadok '85: Polar representations are orbit equivalent to s-representations.
- Eschenburg-Heintze '99: Complete linear classification of irreducible polar representations.

Riemannian Polar G-manifolds

A complete Riemannian manifold M together with a proper isometric action of a Lie group G is said to be *polar* if it admits a *section*, i.e., an immersed complete submanifold Σ of M intersecting every G-orbit and doing so orthogonally.

Examples:

- The standard linear action of T^n on $\mathbb{R}^{2n}$.
- The action of a compact Lie group G on itself by conjungation.
- For a symmetric pair (G, K) the left action of K on G/K.
- Any action of cohomogeneity one.
- Slice representations of a polar action are polar representations.

Orbit space structure

$N(\Sigma) = \{g \in G | g \cdot \Sigma = \Sigma\}$
$Z(\Sigma)$ point-wise stabilizer of the section.
$W = N(\Sigma)/Z(\Sigma)$ acts isometrically on Σ so that

$$\Sigma/W \cong M/G.$$

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Construct a (candidate) fundamental domain $C \subset \Sigma$ as a connected component of the complement of codimension one strata in Σ, later closed.

A *Coxeter Polar* action is a polar action without exceptional strata and such that C gives a strict fundamental domain of the action.

Coxeter polar actions

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Boundary of $C \subseteq \Sigma$ is stratified by totally geodesic faces.

Faces of C have constant W-isotropy in Σ and constant G-isotropy in M.

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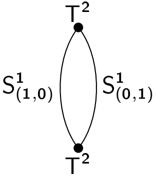
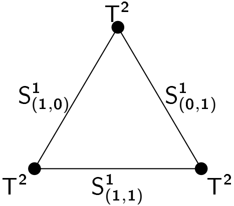
- Grove-Ziller, 2012: Coxeter polar data $(C, G(C))$ determines a Coxeter polar manifold $M(C, G(C))$ up to equivariant diffeomorphism.
  A polar action of a connected Lie group on a simply-connected manifold is Coxeter polar.

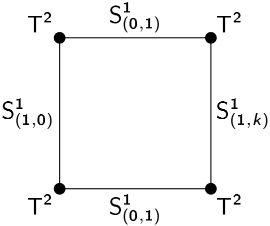
Some examples:



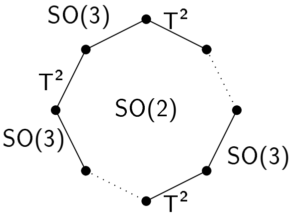
$M = \mathbb{S}^4$



$M = \mathbb{CP}^2$



$$M = \begin{cases} \mathbb{S}^2 \times \mathbb{S}^2 & k \text{ even} \\ \mathbb{CP}^2 \# -\mathbb{CP}^2 & k \text{ odd} \end{cases}$$



$$G = \mathrm{SO}(3) \times \mathrm{T}(1),\ M = \#_n \mathbb{S}^3 \times \mathbb{S}^2.$$

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This is false for reducible polar representations.

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Proposition (.)

A Coxeter polar representation is determined by its history and dimension.

Proof:
Assume G is connected.
We can determine the polar group W from the given history.
Generating reflections are given by the unique involutions in $N_{K_i}(H)_{/}H$, for next-to-minimal subgroups K_i.

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W is a Coxeter group, which decomposes uniquely as a product of irreducible factors,

$$W = W_1 \times \cdots \times W_l.$$

The representation and section decompose accordingly as

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_l.$$

$$\Sigma = V_0 \oplus \Sigma_1 \oplus \cdots \oplus \Sigma_l.$$

Σ_i is point-wise fixed by the action of
$W_{\Sigma_i} := (W_1 \times \cdots \times \hat{W}_i \times \cdots W_l) \subset N(H)/H$.
The isotropy group G_{p_i} of a generic regular point p_i in Σ_i is the unique minimal group in the history such that

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Notice that the restricted action of G on V_i has principal isotropy group G_{p_i}.
Make the action effective and recognize it.
We have determined the representation

$$G \xrightarrow{\oplus_i \rho_i} \mathrm{SO}(V_1) \times \cdots \times \mathrm{SO}(V_l)$$

The dimension n is only required to determine the trivial subspace V_0. □

Thank you!