A Note On Polar Representations

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Polar Representations

An orthogonal representation of a compact Lie group $G$ is called *polar* if it admits an orthogonal cross-section, i.e., a linear subspace intersecting every $G$-orbit and doing so orthogonally.

They were first considered by J. Szenthe and J. Dadok. Consider connected $G$.

- Dadok ’85: Polar representations are orbit equivalent to s-representations.
Riemannian Polar $G$-manifolds

A complete Riemannian manifold $M$ together with a proper isometric action of a Lie group $G$ is said to be polar if it admits a section, i.e., an immersed complete submanifold $\Sigma$ of $M$ intersecting every $G$-orbit and doing so orthogonally.

Examples:
- The standard linear action of $\mathbb{T}^n$ on $\mathbb{R}^{2n}$.
- The action of a compact Lie group $G$ on itself by conjugation.
- For a symmetric pair $(G, K)$ the left action of $K$ on $G/K$.
- Any action of cohomogeneity one.
- Slice representations of a polar action are polar representations.
Orbit space structure

\[ N(\Sigma) = \{ g \in G | g \cdot \Sigma = \Sigma \} \]

\( Z(\Sigma) \) point-wise stabilizer of the section.

\( W = N(\Sigma)/Z(\Sigma) \) acts isometrically on \( \Sigma \) so that

\[ \Sigma/W \cong M/G. \]
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Construct a (candidate) fundamental domain \( C \subset \Sigma \) as a connected component of the complement of codimension one strata in \( \Sigma \), later closed.

A *Coxeter Polar* action is a polar action without exceptional strata and such that \( C \) gives a strict fundamental domain of the action.
Coxeter polar actions

\[ C \cong \Sigma/W \cong M/G. \]

Boundary of \( C \subseteq \Sigma \) is stratified by totally geodesic faces.

Faces of \( C \) have constant \( W \)-isotropy in \( \Sigma \) and constant \( G \)-isotropy in \( M \).
Coxeter polar actions

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- **Grove-Ziller, 2012**: Coxeter polar data \( (C, G(C)) \) determines a Coxeter polar manifold \( M(C, G(C)) \) up to equivariant diffeomorphism.
  
  A polar action of a connected Lie group on a simply-connected manifold is Coxeter polar.
Some examples:

\[ M = S^4 \]

\[ M = \mathbb{CP}^2 \]

\[ M = \begin{cases} S^2 \times S^2 & k \text{ even} \\ \mathbb{CP}^2 \# - \mathbb{CP}^2 & k \text{ odd} \end{cases} \]
\[ G = \text{SO}(3) \times T(1), \quad M = \#_n S^3 \times S^2. \]
Back to representations:

An irreducible polar representation is identified by the group $G$ and a principal isotropy subgroup. This is false for reducible polar representations.
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**Proposition** (.)

A Coxeter polar representation is determined by its history and dimension.
Proof:
Assume $G$ is connected.
We can determine the polar group $W$ from the given history.
Generating reflections are given by the unique involutions in $N_{K_i}(H)/H$, for next-to-minimal subgroups $K_i$. 

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Proof:
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$W$ is a Coxeter group, which decomposes uniquely as a product of irreducible factors,

$$W = W_1 \times \cdots \times W_l.$$ 

The representation and section decompose accordingly as

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_l.$$ 

$$\Sigma = V_0 \oplus \Sigma_1 \oplus \cdots \oplus \Sigma_l.$$
\( \Sigma_i \) is point-wise fixed by the action of \\
\( \mathcal{W}_{\Sigma_i} := (\mathcal{W}_1 \times \cdot \cdot \cdot \times \hat{\mathcal{W}}_i \times \cdot \cdot \cdot \mathcal{W}_l) \subset N(H)/H. \) \\
The isotropy group \( G_{p_i} \) of a generic regular point \( p_i \) in \( \Sigma_i \) is the unique minimal group in the history such that \\

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G_{p_i} \supset \mathcal{W}_{\Sigma_i} \cdot H.
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Notice that the restricted action of $G$ on $V_i$ has principal isotropy group $G_{p_i}$.

Make the action effective and recognize it.
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Make the action effective and recognize it.

We have determined the representation

$$G \xrightarrow{\oplus i \rho_i} SO(V_1) \times \cdots \times SO(V_l)$$

The dimension $n$ is only required to determine the trivial subspace $V_0$. 

$\Box$
Thank you!