Nodal solutions of Yamabe equation

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Let (M, g) be a closed Riemannian manifold of $\dim(M) = n \geq 3$. $u \in C^\infty(M)$ is a solution of the Yamabe equation if satisfies (for some $c \in \mathbb{R}$)

$$(YEq) \qquad a_n \Delta_g u + s_g u = c|u|^{p_n - 2} u$$

where $a_n = 4(n-1)/(n-2)$, $p_n = 2n/(n-2)$, and s_g is the scalar curvature of (M, g).

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If u is a positive solution, then $u^{p_n - 2} g$ is a metric of constant scalar curvature c.

(Yamabe problem) There exist positive solutions of (YEq) iff $sign(c) = sign(Y(M, [g]))$

The Yamabe functional is defined by

$$h \in [g] \longmapsto Y(h) := \frac{\int_M s_h dv_h}{vol(M,h)^{\frac{n-2}{n}}}.$$

$[g] := \{fg : \ f \in C^\infty_{>0}(M)\}$.
The Yamabe constant is defined by

$$Y(M,[g]) = \inf_{h \in [g]} Y(h).$$

The infimum is attained by a metric of constant scalar curvature.

The conformal Laplacian of (M, g) is the second order elliptic operator

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The spectrum of L_g is a non-decreasing sequence of eigenvalues

$$\lambda_1(L_g) \leq \lambda_2(L_g) \leq \cdots \leq \lambda_k(L_g) \longrightarrow +\infty$$

If $Y(M, [g]) \geq 0$,

$$Y(M, [g]) = \inf_{h \in [g]} \lambda_1(L_h) vol(M, h)^{\frac{2}{n}}.$$

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- If $\lambda_2(L_h) < 0$ for some $h \in [g]$, then $Y^2(M,[g]) = -\infty$.
- In general, $Y^2(M,g)$ is not attained by a Riemannian metric: (for instance, if M is connected and $Y^2(M) > 0$)
- Second Yamabe constant is related with nodal solutions of the Yamabe equation.

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From now on let (M,g) such that $Y(M,[g]) \geq 0$.

The generalized conformal class of g is the set

$$[g]_{gen} := \{u^{p_n-2}g : u \in L^{p_n}_{\geq 0}(M) - \{0\}\}$$

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$u^{p_n-2}g$ is a generalized metric.
By extending naturally the definition of $\lambda_2(L_h)$ and $vol(M,h)$ we get

$$Y^2(M,[g]) := \inf_{h\in[g]_{gen}} \lambda_2(L_h) vol(M,h)^{\frac{2}{n}}.$$

Theorem [Ammann-Humbert (2006)]
If $Y^2(M,[g]) > 0$ and is attained by a generalized metric $h = u^{p_n-2}g$, then $u = |w|$, where $w \in C^{3,\alpha}(M)$ $(w \in C^\infty(M - \{w=0\}))$ is a nodal solution of Yamabe equation

$$L_g(w) = \lambda_2(h)|w|^{p_n-2}w.$$

Theorem [Ammann-Humbert (2006)]
$Y^2(M,[g])$ is attained if

$$Y^2(M,g) < \left[Y(M,g)^{\frac{n}{2}} + Y(S^n)^{\frac{n}{2}} \right]^{\frac{2}{n}}.$$

(with the strict inequality we avoid concentration phenomena of minimizing sequences of metrics)

Let G be a compact subgroup of the isometry group of (M, g).
The $G-$ *equivariant* Yamabe constant is

$$Y_G(M, [g]_G) = \inf_{h \in [g]_G} Y(h)$$

where $[g]_G$ is the subset of metrics of $[g]$ that are $G-$invariant.
If $G = \{Id\}$, then $Y_G(M, [g]_G) = Y(M, [g])$.

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If $G = \{Id\}$, then $Y_G(M, [g]_G) = Y(M, [g])$.
and the G − *equivariant* second Yamabe constant is

$$Y^2_G(M, [g]_g) = \inf_{h \in [g]_G} \lambda_2^G(L_h) vol(M, h)^{\frac{2}{n}}$$

where $\lambda_2^G(L_h)$ is the second eigenvalue of L_h restricted to $G-$invariant functions.
If $G = \{Id\}$, then $Y^2_G(M, [g]_G) = Y^2(M, [g])$.

Theorem [H-Madani (2016)]
Let $\Lambda_G = \inf_{x \in M} \sharp\{O_G(x)\}$. Then

$$Y_G^2(M,g) < \left[Y_G(M,g)^{\frac{n}{2}} + Y(S^n)^{\frac{n}{2}} \Lambda_G \right]^{\frac{2}{n}}$$

if one of the following items is satisfied:

a) $\dim O_G(x) \geq 1$ for every $x \in M^n$.

b) There exists P that belongs to a minimal finite orbit such that $\omega(P) \leq \frac{n-2}{6}$ and

- $n \geq 11$ if $Y_G(M,g) > 0$.
- $n \geq 9$ if $Y_G(M,g) = 0$.

$$\omega(x) := \inf\{ l \in \mathbb{N}_0 : \ \nabla^l W(x) \neq 0 \}$$

$$\omega(x) = +\infty \text{ if } \nabla^l W(x) = 0 \ \forall \ l$$

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[Ammann-Humbert] Proved this theorem when $G = \{Id\}$ and M is non locally conformally flat.

$$Y_G^2(M,g) \leq \sup_{v \in V - \{0\}} \frac{\int_M a_n |\nabla v|^2_{g+th} + s_g v^2 dv_g}{\int_M u^{p_{m+n}-2} v^2 dv_g} \left(\int_M u^{p_n} dv_g\right)^{\frac{2}{n}}.$$

for any $u \in L^{p_n}_{G,\geq 0}(M)$ and V any 2-dimensional subspace of $H^2_{1,G}(M)$ and $H^2_{1,G}(M - \{u = 0\})$.

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When $Y_G(M,g) > 0$, $\Lambda_G < \infty$

$$u_\varepsilon := Y(\phi_\varepsilon)^{\frac{n-2}{4}} \phi_\varepsilon + Y_G(M,g)^{\frac{n-2}{4}} \psi$$

and

$$V_\varepsilon = span\{\phi_\varepsilon, \psi\}$$

Where $Y(\varphi) = Y_G(M,g)$ and

$$u_\varepsilon := Y(\phi_\varepsilon)^{\frac{n-2}{4}}\phi_\varepsilon + Y_G(M,g)^{\frac{n-2}{4}}\psi$$

Let P such that $O_G(P)$ is minimal, then

$$\phi_{P,\varepsilon}(Q) = \eta_{P,\delta}\left(\frac{\varepsilon}{\varepsilon^2 + d^2(P,Q)}\right)^{\frac{n-2}{2}}$$

and

$$\psi_{P,\varepsilon}(Q) = (1 + r^{\omega+2}\sum_{k=1}^{[\frac{\omega}{2}]} c_k\varphi_k(\xi))\phi_{P,\varepsilon}(Q)$$

Finally,

$$\phi_\varepsilon = \sum_{\sigma\in G/H} \psi_{P,\varepsilon}\circ\sigma^{-1}$$

$H \subset G$ the stabilizer of P.

$$u_\varepsilon := Y(\phi_\varepsilon)^{\frac{n-2}{4}}\phi_\varepsilon + Y_G(M,g)^{\frac{n-2}{4}}\varphi$$

$$V_\varepsilon = span\{\phi_\varepsilon, \varphi\}$$

for ε small enough.

$$Y_G^2(M,g) \leq \sup_{v \in V_\varepsilon - \{0\}} \frac{\int_M a_n |\nabla v|^2_{g+th} + s_g v^2 dv_g}{\int_M u_\varepsilon^{p_{m+n}-2} v^2 dv_g}$$

$$\times \left(\int_M u_\varepsilon^{p_n} dv_g\right)^{\frac{2}{n}} < \left[Y_G(M,g)^{\frac{n}{2}} + Y(S^n)^{\frac{n}{2}}\Lambda_G\right]^{\frac{2}{n}}.$$

We always have

$$2^{\frac{2}{n}} Y_G(M,g) \leq Y_G^2(M,g) \leq \left[Y_G(M,g)^{\frac{n}{2}} + Y(S^n)^{\frac{n}{2}} \Lambda_G \right]^{\frac{2}{n}}$$

In Particular if $G = \{Id\}$

$$2^{\frac{2}{n}} Y(M,[g]) \leq Y^2(M,[g]) \leq \left[Y(M,g)^{\frac{n}{2}} + Y(S^n)^{\frac{n}{2}} \right]^{\frac{2}{n}}$$

$Y^2(S^n, g_0^n) = 2^{\frac{2}{n}} Y(S^n)$.

Theorem [H (2015)] Let (M^m, g) and (N^n, h) be closed manifolds, $(m \geq 2)$ and $s_g > 0$. Then,

$$\lim_{t \to +\infty} Y^2(M \times N, [g + th]) = 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e^n]).$$

Theorem [H (2015)] Let (M^m, g) and (N^n, h) be closed manifolds, $(m \geq 2)$ and $s_g > 0$. Then,

$$\lim_{t \to +\infty} Y^2(M \times N, [g + th]) = 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e^n]).$$

This is the second Yamabe constant's version of
Theorem [Akutagawa, Florit, and Petean (2007)]

$$\lim_{t \to +\infty} Y(M \times N, [g + th]) = Y(M \times \mathbb{R}^n, [g + g_e^n]).$$

Corollary Let (M^m, g) and (N^n, h) be closed manifolds, $(m, n \geq 2)$ and $s_g > 0$.For t large enough, there exists a nodal solution of the Yamabe equation on $(M \times N, g + th)$

$$L_{g+h}(v) = c|v|^{p_{m+n}-2}v$$

Moreover, $v \in C^{3,\alpha}(M \times N)$ and is smooth in $M \times N - \{v^{-1}(0)\}$.

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Proof: If

$$Y^2(M \times N, g + th) < \left[Y(M \times N, [g + th])^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}} \right]^{\frac{2}{m+n}}$$

$\Rightarrow Y^2(M \times N, g + th) > 0$ is attained.

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$$Y^2(M \times N, [g + th]) \quad \leq \left[Y(M \times N, g + th)^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}\right]^{\frac{2}{m+n}}$$

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$$\begin{array}{ccc} Y^2(M \times N, [g + th]) & \leq & \left[Y(M \times N, g + th)^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}\right]^{\frac{2}{m+n}} \\ \Big\downarrow t \to +\infty & & \Big\downarrow t \to +\infty \\ 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, g + g_e^n) & & \left[Y(M \times \mathbb{R}^n, g + g_e^n)^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}\right]^{\frac{2}{m+n}} \end{array}$$

Corollary Let (M^m, g) and (N^n, h) be closed manifolds, $(m, n \geq 2)$ and $s_g > 0$.For t large enough, there exists a nodal solution of the Yamabe equation on $(M \times N, g + th)$

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$$\Big\downarrow t \to +\infty \qquad\qquad \Big\downarrow t \to +\infty$$

$$2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, g + g_e^n) \qquad \left[Y(M \times \mathbb{R}^n, g + g_e^n)^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}} \right]^{\frac{2}{m+n}}$$

But $Y(M \times \mathbb{R}^n, g + g_e^n) < Y(S^{m+n})$ if $n \geq 2$. (Akutagawa-Florit-Petean)

Corollary Let (M^m, g) and (N^n, h) be closed manifolds, $(m, n \geq 2)$ and $s_g > 0$.For t large enough, there exists a nodal solution of the Yamabe equation on $(M \times N, g + th)$

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$$\Big\downarrow t \to +\infty \qquad\qquad \Big\downarrow t \to +\infty$$

$$2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, g + g_e^n) < \left[Y(M \times \mathbb{R}^n, g + g_e^n)^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}} \right]^{\frac{2}{m+n}}$$

Corollary Let (M^m, g) and (N^n, h) be closed manifolds, $(m, n \geq 2)$ and $s_g > 0$.For t large enough, there exists a nodal solution of the Yamabe equation on $(M \times N, g + th)$

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Proof: If

$$Y^2(M \times N, g + th) < \left[Y(M \times N, [g + th])^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}\right]^{\frac{2}{m+n}}$$

$\Rightarrow Y^2(M \times N, g + th) > 0$ is attained.

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Therefore,

$$Y^2(M \times N, g + th) < \left[Y(M \times N, [g + th])^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}\right]^{\frac{2}{m+n}}$$

Remarks $(M^m \times N^n, g + th)$ is not locally conformally flat ($s_g > 0 \Longrightarrow Y(M \times N, [g + h]) > 0$ if t large enough). So if $m + n \geq 11$ we can apply Ammann and Humbert's Theorem. Actually our result improve the previous results when $4 \leq m + n < 11$.

Let $(M^m \times N^n, g+h)$, the N-Yamabe constant is

$$Y_N(M \times N, g+h) := \inf_{u \in C^\infty(N) - \{0\}} Y(u^{p_{m+n}-2}(g+h)).$$

$$Y(M \times N, g+h) \leq Y_N(M \times N, g+h)$$

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$$Y(M \times N, g+h) \leq Y_N(M \times N, g+h)$$

The second $N-$Yamabe constant

$$Y_N^2(M \times N, g+h) := \inf_{\tilde{h} \in [g+h]_N} \lambda_2^N(L_{\tilde{h}}) vol(M \times N, \tilde{h})^{\frac{2}{m+n}},$$

where $[g+h]_N : \{\tilde{h} = u^{p_{m+n}-2}(g+h),\ u \in L^{p_{m+n}}(N) - \{0\}\}$ and $\lambda_2^N(\tilde{h})$ is the 2-nd eigenvalue of $L_{\tilde{h}}$ restricted to functions that depend only on the variable N.

$$Y^2(M \times N, g+h) \leq Y_N^2(M \times N, g+h)$$

Theorem[H] (2015) Let (M^m, g) $(m \geq 2)$ $s_g \equiv c > 0$ and (N^n, h) be a closed manifold. Then,

$$\lim_{t \to +\infty} Y_N^2(M \times N, g + th) = 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e^n).$$

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$Y^2(M \times N, g + h)$ is always attained, because $i : H_1^2(N) \longrightarrow L^{p_{m+n}}(N)$ is a compact operator $(p_{m+n} < p_n)$, so we can avoid concentration phenomena of minimizing sequences. Therefore, is this situation we always have a nodal solution of

$$L_{g+h}(u) = c|u|^{p_{m+n}-2}u$$

where u is a functions that depends on N.

Let G be compact subgroup of the isometry group of (M, g). Assume that $Y_G(M, g) > 0$, then we have
Theorem[H-Madani] (2016) If $Y_G^2(M, g)$ is attained by $h = u^{p_n - 2} g$ with $u \in L^{p_n}_{\geq 0, G}$, then $u = |w|$, with w a $G-$invariant nodal solution of the Yamabe equation

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$u \in C^{2,\alpha}$ and is smooth outside the nodal set.

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For instance, if the orbits of the group action are not finite $\Longrightarrow$ $Y_G^2(M, g)$ is attained.
(the inclusion of $H^2_{1,G}(M)$ in $L^{p_n}(M)$ is compact if $\inf_{x \in M} \dim(O_G(x)) \geq 1$)

Let $G = G_1 \times G_2 \subseteq I(M_1 \times M_2)$ where $G_1 \subseteq I(M, g)$ and $G_2 \subseteq I(N, h)$.
In the same way as the second $N-$ Yamabe constant, we can define the $G-$equivariant $N-$second Yamabe constant as:

$$Y^2_{N,G}(M \times N, g+h) := \inf_{\bar{g} \in [g+h]_{N,G}} \lambda^N_{2,G}(L_{\bar{g}}) vol(M \times N, \bar{g})^{\frac{2}{m+n}}.$$

where
$[g+h]_{N,G} := \{u^{p_n - 2}(g+h)\ ,\ u \in C^\infty_{G,>0}(N) /\ \sigma^*(u) = u\ \forall\ \sigma \in G\}$

If $G_2 = \{Id\}$, then $Y^2_{N,G}(M \times N, g+h) = Y^2_N(M \times N, g+h)$.

Proposition [H-M] Let (M, g) of constant scalar curvature. Then, the $G-$equivariant $N-$second Yamabe constant is always achieved.

Corollary [H-M] Let (M^m, g) of c. s. c. and (N^n, h) be closed Riemannian manifolds and let $G \subseteq I(N, g)$. There exists a $G-$invariant nodal solution of the Yamabe equation on $(M \times N, g + h)$ that depends only on the N variable.

Theorem [H-Madani] (2016)

Let (M^m, g) with $m \geq 2$ (of positive scalar curvature). Let G_1 be a subgroup of $I(M,g)$. Let G_1 be a subgroup of $I(M,g)$ and $G = G_1 \times \{Id\}$.

a)

$$\lim_{t \to +\infty} Y_G(M \times N, g + th) = Y_G(M \times \mathbb{R}^n, g + g_e^n).$$

b)

$$\lim_{t \to +\infty} Y_G^2(M \times N, g + th) = 2^{\frac{2}{m+n}} Y_G(M \times \mathbb{R}^n, g + g_e^n).$$

c)

$$\lim_{t \to +\infty} Y_{N,G}^2(M \times N, g + th) = 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n, G}(M \times \mathbb{R}^n).$$

Thanks!