Parallel 2-forms and Killing Yano 2-forms in low dimensional Lie groups and some examples in flag manifolds

Cecilia Herrera

Universidad Nacional de Córdoba

2 de agosto de 2016
Let \((M, g, J)\) be an **almost Hermitian manifold**, where \(g\) is the metric and \(J: TM \rightarrow TM\) is the almost complex structure. Recall that

\[
J^2 = -I
\]

Let \(\nabla\) be the Levi-Civita connection.

\(J\) is said **parallel** if \(\nabla J = 0\), that is

\[
(\nabla_X J) Y := \nabla_X JY - J\nabla_X JY = 0, \quad \forall X, Y \in \mathcal{X}(M).
\]

and \((M, g, J)\) is said to be a **Kähler manifold**.

If \(J\) satisfies only the weaker condition

\[
(\nabla_X J) X = 0, \quad \forall X \in \mathcal{X}(M),
\]

then \((M, g, J)\) is said to be a **nearly Kähler manifold**.
Let \((M, g, J)\) be an **almost Hermitian manifold**, where \(g\) is the metric and \(J : TM \to TM\) is the almost complex structure. Recall that \(J^2 = -I\).

Let \(\nabla\) be the Levi-Civita connection.

\(J\) is said \textbf{parallel} if \(\nabla J = 0\), that is

\[
(\nabla_X J) Y := \nabla_X JY - J \nabla_X JY = 0, \quad \forall X, Y \in \mathfrak{X}(M).
\]

and \((M, g, J)\) is said to be a **Kähler manifold**.

If \(J\) satisfies only the weaker condition

\[
(\nabla_X J) X = 0, \quad \forall X \in \mathfrak{X}(M),
\]

then \((M, g, J)\) is said to be a **nearly Kähler manifold**.
Let \((M, g, J)\) be an **almost Hermitian manifold**, where \(g\) is the metric and \(J : TM \to TM\) is the almost complex structure. Recall that

\[ J^2 = -I \]

Let \(\nabla\) be the Levi-Civita connection.

\(J\) is said **parallel** if \(\nabla J = 0\), that is

\[ (\nabla_X J) Y := \nabla_X JY - J\nabla_X JY = 0, \quad \forall X, Y \in \mathcal{X}(M). \]

and \((M, g, J)\) is said to be a **Kähler manifold**.

If \(J\) satisfies only the weaker condition

\[ (\nabla_X J) X = 0, \quad \forall X \in \mathcal{X}(M), \]

then \((M, g, J)\) is said to be a **nearly Kähler manifold**.
Let \((M, g, J)\) be an **almost Hermitian manifold**, where \(g\) is the metric and \(J : TM \to TM\) is the almost complex structure.

Recall that

\[ J^2 = -I \]

Let \(\nabla\) be the Levi-Civita connection.

\(J\) is said **parallel** if \(\nabla J = 0\), that is

\[ (\nabla_X J) Y := \nabla_X JY - J\nabla_X JY = 0, \quad \forall X, Y \in \mathcal{X}(M). \]

and \((M, g, J)\) is said to be a **Kähler manifold**.

If \(J\) satisfies only the weaker condition

\[ (\nabla_X J) X = 0, \quad \forall X \in \mathcal{X}(M), \]

then \((M, g, J)\) is said to be a **nearly Kähler manifold**.
Comments:

- Kähler manifolds are nearly Kähler manifolds.
- The converse is true in dimension 2 and 4.
- There are examples of homogeneous spaces of dimension 6 that are strict nearly Kähler manifolds.

1. The maximal flag manifold $\mathbb{F} = SU(3)/S^1 \times S^1$
2. $S^3 \times S^3$
3. $CP(3)$
4. $S^6$
Comments:

- Kähler manifolds are nearly Kähler manifolds.
- The converse is true in dimension 2 and 4.
- There are examples of homogeneous spaces of dimension 6 that are strict nearly Kähler manifolds.
  1. The maximal flag manifold $\mathbb{F} = SU(3)/S^1 \times S^1$
  2. $S^3 \times S^3$
  3. $CP(3)$
  4. $S^6$
Comments:

- Kähler manifolds are nearly Kähler manifolds.
- The converse is true in dimension 2 and 4.
- There are examples of homogeneous spaces of dimension 6 that are strict nearly Kähler manifolds.
  1. The maximal flag manifold $\mathbb{F} = SU(3)/S^1 \times S^1$
  2. $S^3 \times S^3$
  3. $CP(3)$
  4. $S^6$
Comments:

• Kähler manifolds are nearly Kähler manifolds.
• The converse is true in dimension 2 and 4.
• There are examples of homogeneous spaces of dimension 6 that are strict nearly Kähler manifolds.

1. The maximal flag manifold $\mathbb{F} = SU(3)/S^1 \times S^1$
2. $S^3 \times S^3$
3. $CP(3)$
4. $S^6$
Let \((M, g)\) be a Riemannian manifold with a skew-symmetric isomorphism \(E : TM \to TM\) (i.e, \(g(EX, Y) = -g(X, EY)\)). We are not requiring the condition \(E^2 = -I\).

Consider the associated 2-forms
\[
\omega(X, Y) := g(EX, Y)
\]
\[
\mu(X, Y) := g(E^{-1}X, Y)
\]
and the Nijenhuis tensor
\[
N_E(X, Y) := [EX, EY] - E([X, EY] + [EX, Y]) + E^2[X, Y].
\]

\(E\) is parallel if one of the following equivalent conditions holds.

1) \(\nabla E = 0\), 2) \(d\omega = d\mu = N_E = 0\), 3) \(\nabla\omega = 0\).

This generalizes the notion of Kähler manifolds.
Let \((M, g)\) be a Riemannian manifold with a skew-symmetric isomorphism \(E : TM \to TM\) (i.e, \(g(EX, Y) = -g(X, EY)\)). **We are not requiring the condition** \(E^2 = -I\).

Consider the associated 2-forms

\[
\omega(X, Y) := g(EX, Y) \\
\mu(X, Y) := g(E^{-1}X, Y).
\]

and the Nijenhuis tensor

\[
N_E(X, Y) := [EX, EY] - E([X, EY] + [EX, Y]) + E^2[X, Y].
\]

\(E\) is **parallel** if one of the following equivalent conditions holds.

1) \(\nabla E = 0\),
2) \(d\omega = d\mu = N_E = 0\),
3) \(\nabla\omega = 0\).

This generalizes the notion of Kähler manifolds.
Let \((M, g)\) be a Riemannian manifold with a skew-symmetric isomorphism \(E : TM \to TM\) (i.e., \(g(EX, Y) = -g(X, EY)\)). **We are not requiring the condition** \(E^2 = -I\).

Consider the associated 2-forms

\[
\omega(X, Y) := g(EX, Y),
\]
\[
\mu(X, Y) := g(E^{-1}X, Y).
\]

and the Nijenhuis tensor

\[
N_E(X, Y) := [EX, EY] - E([X, EY] + [EX, Y]) + E^2[X, Y].
\]

\(E\) is **parallel** if one of the following **equivalent conditions** holds.

1) \(\nabla E = 0\), 2) \(d\omega = d\mu = N_E = 0\), 3) \(\nabla\omega = 0\).

This generalizes the notion of Kähler manifolds.
Let \((M, g)\) be a Riemannian manifold with a skew-symmetric isomorphism \(E : TM \to TM\) (i.e, \(g(EX, Y) = -g(X, EY)\)). **We are not requiring the condition** \(E^2 = -I\).

Consider the associated 2-forms

\[
\omega(X, Y) := g(EX, Y)
\]
\[
\mu(X, Y) := g(E^{-1}X, Y).
\]

and the Nijenhuis tensor

\[
N_E(X, Y) := [EX, EY] - E([X, EY] + [EX, Y]) + E^2[X, Y].
\]

\(E\) is **parallel** if one of the following **equivalent conditions** holds.

1) \(\nabla E = 0\), 2) \(d\omega = d\mu = N_E = 0\), 3) \(\nabla \omega = 0\).

This generalizes the notion of Kähler manifolds.
Requiring only the condition

$$(\nabla_X E) X = 0, \ \forall X \in \mathfrak{X}(M),$$

we generalize the notion of nearly Kähler manifold. This condition is also equivalent to

$$d\omega (X, Y, Z) = 3\nabla_X \omega (Y, Z), \ \forall X, Y, Z \in \mathfrak{X}(M).$$

A 2-form satisfying this condition is called a **Killing-Yano form** and $E$ is called a **Killing-Yano tensor**.

**Comments:**

- If $E$ is parallel, it is a Killing-Yano tensor. The converse is also true in dimension 2 and 4 if $\omega$ has constant norm.

- The Iwasawa Manifold $\Gamma \backslash H_3(\mathbb{C})$, where $H_3(\mathbb{C})$ is the complex Heisenberg group of dimension 3 and $\Gamma = H_3(\mathbb{Z} + i\mathbb{Z})$ admit Killing Yano tensor and it hasn’t nearly Kähler structure.
Requiring only the condition

\[(\nabla_X E) X = 0, \ \forall X \in \mathcal{X}(M),\]

we generalize the notion of nearly Kähler manifold. This condition is also equivalent to

\[d\omega (X, Y, Z) = 3\nabla_X \omega (Y, Z), \ \forall X, Y, Z \in \mathcal{X}(M).\]

A 2-form satisfying this condition is called a **Killing-Yano form** and \(E\) is called a **Killing-Yano tensor**.

**Comments:**

- If \(E\) is parallel, it is a Killing-Yano tensor. The converse is also true in dimension 2 and 4 if \(\omega\) has constant norm.
- The Iwasawa Manifold \(\Gamma \backslash H_3(\mathbb{C})\), where \(H_3(\mathbb{C})\) is the complex Heisenberg group of dimension 3 and \(\Gamma = H_3(\mathbb{Z} + i\mathbb{Z})\) admit Killing Yano tensor and it hasn’t nearly Kähler structure.
Requiring only the condition

\[(\nabla_X E) X = 0, \quad \forall X \in \mathcal{X}(M),\]

we generalize the notion of nearly Kähler manifold. This condition is also equivalent to

\[d\omega (X, Y, Z) = 3\nabla_X \omega (Y, Z), \quad \forall X, Y, Z \in \mathcal{X}(M).\]

A 2-form satisfying this condition is called a **Killing-Yano form** and \(E\) is called a **Killing-Yano tensor**.

Comments:

- If \(E\) is parallel, it is a Killing-Yano tensor. The converse is also true in dimension 2 and 4 if \(\omega\) has constant norm.

- The Iwasawa Manifold \(\Gamma \backslash H_3(\mathbb{C})\), where \(H_3(\mathbb{C})\) is the complex Heisenberg group of dimension 3 and \(\Gamma = H_3(\mathbb{Z} + i\mathbb{Z})\) admit Killing Yano tensor and it hasn’t nearly Kähler structure.
Requiring only the condition

\[(\nabla_X E)X = 0, \ \forall X \in \mathcal{X}(M),\]

we generalize the notion of nearly Kähler manifold. This condition is also equivalent to

\[d\omega (X, Y, Z) = 3\nabla_X \omega (Y, Z), \ \forall X, Y, Z \in \mathcal{X}(M).\]

A 2-form satisfying this condition is called a **Killing-Yano form** and \(E\) is called a **Killing-Yano tensor**.

**Comments:**

- If \(E\) is parallel, it is a Killing-Yano tensor. The converse is also true in dimension 2 and 4 if \(\omega\) has constant norm.
- The Iwasawa Manifold \(\Gamma \backslash H_3(\mathbb{C})\), where \(H_3(\mathbb{C})\) is the complex Heisenberg group of dimension 3 and \(\Gamma = H_3(\mathbb{Z} + i\mathbb{Z})\) admit Killing Yano tensor and it hasn’t nearly Kähler structure.
Requiring only the condition

\[(\nabla_X E) X = 0, \ \forall X \in \mathcal{X}(M),\]

we generalize the notion of nearly Kähler manifold. This condition is also equivalent to

\[d\omega (X, Y, Z) = 3\nabla_X \omega (Y, Z), \ \forall X, Y, Z \in \mathcal{X}(M).\]

A 2-form satisfying this condition is called a **Killing-Yano form** and \(E\) is called a **Killing-Yano tensor**.

**Comments:**

- If \(E\) is parallel, it is a Killing-Yano tensor. The converse is also true in dimension 2 and 4 if \(\omega\) has constant norm.
- The Iwasawa Manifold \(\Gamma \backslash H_3(\mathbb{C})\), where \(H_3(\mathbb{C})\) is the complex Heisenberg group of dimension 3 and \(\Gamma = H_3(\mathbb{Z} + i\mathbb{Z})\) admit Killing Yano tensor and it hasn’t nearly Kähler structure.
Requiring only the condition

$$(\nabla_X E) X = 0, \; \forall X \in \mathcal{X}(M),$$

we generalize the notion of nearly Kähler manifold. This condition is also equivalent to

$$d\omega (X, Y, Z) = 3\nabla_X \omega (Y, Z), \; \forall X, Y, Z \in \mathcal{X}(M).$$

A 2-form satisfying this condition is called a **Killing-Yano form** and $E$ is called a **Killing-Yano tensor**.

**Comments:**

- If $E$ is parallel, it is a Killing-Yano tensor. The converse is also true in dimension 2 and 4 if $\omega$ has constant norm.

- The Iwasawa Manifold $\Gamma \backslash H_3(\mathbb{C})$, where $H_3(\mathbb{C})$ is the complex Heisenberg group of dimension 3 and $\Gamma = H_3(\mathbb{Z} + i\mathbb{Z})$ admit Killing Yano tensor and it hasn’t nearly Kähler structure.
Let $G$ be a **Lie group of dimension 4**, and $g$ a left invariant metric.

Any Killing-Yano tensor is also parallel (Andrada, Barberis and Moroianu).

In what follows we’ll search 4-dimensional metric Lie algebras that admit a parallel skew symmetric isomorphism. 

Such a lie algebra must be **solvable** by a result of Chu.

---

**Theorem (Andrada-Dotti)**

The only 4-dimensional solvable Lie algebras with non-trivial center that admits a metric and a parallel invertible tensor are $\mathbb{R}^4$, $\text{aff}(\mathbb{R}) \times \mathbb{R}^2$ and $\mathfrak{e}(2) \times \mathbb{R}$

- $\text{aff}(\mathbb{R})$ is the 2 dimensional non abelian Lie algebra.
- $\mathfrak{e}(2)$ is the Lie algebra of the group of rigid motions in the plane.
Let $G$ be a **Lie group of dimension 4**, and $g$ a left invariant metric. Any Killing-Yano tensor is also parallel (Andrada, Barberis and Moroianu).

In what follows we’ll search 4-dimensional metric Lie algebras that admit a parallel skew symmetric isomorphism. Such a lie algebra must be **solvable** by a result of Chu.

**Theorem (Andrada-Dotti)**

The only 4-dimensional solvable Lie algebras with non-trivial center that admits a metric and a parallel invertible tensor are $\mathbb{R}^4$, $\text{aff}(\mathbb{R}) \times \mathbb{R}^2$ and $\varepsilon(2) \times \mathbb{R}$

-$\text{aff}(\mathbb{R})$ is the 2 dimensional non abelian Lie algebra.
-$\varepsilon(2)$ is the Lie algebra of the group of rigid motions in the plane.
Let $G$ be a **Lie group of dimension 4**, and $g$ a left invariant metric.

Any Killing-Yano tensor is also parallel (Andrada, Barberis and Moroianu).

In what follows we’ll search 4-dimensional metric Lie algebras that admit a parallel skew symmetric isomorphism.

Such a lie algebra must be **solvable** by a result of Chu.

**Theorem (Andrada-Dotti)**

The only 4-dimensional solvable Lie algebras with non-trivial center that admits a metric and a parallel invertible tensor are $\mathbb{R}^4$, $\text{aff}(\mathbb{R}) \times \mathbb{R}^2$ and $\mathfrak{e}(2) \times \mathbb{R}$

- $\text{aff}(\mathbb{R})$ is the 2 dimensional non abelian Lie algebra.
- $\mathfrak{e}(2)$ is the Lie algebra of the group of rigid motions in the plane.
Let $G$ be a **Lie group of dimension 4**, and $g$ a left invariant metric.

Any Killing-Yano tensor is also parallel (Andrada, Barberis and Moroianu).

In what follows we’ll search 4-dimensional metric Lie algebras that admit a parallel skew symmetric isomorphism.

Such a lie algebra must be **solvable** by a result of Chu.

---

**Theorem (Andrada-Dotti)**

*The only 4-dimensional solvable Lie algebras with non-trivial center that admits a metric and a parallel invertible tensor are $\mathbb{R}^4$, $\text{aff}(\mathbb{R}) \times \mathbb{R}^2$ and $\mathfrak{e}(2) \times \mathbb{R}$*

- $\text{aff}(\mathbb{R})$ is the 2 dimensional non abelian Lie algebra.
- $\mathfrak{e}(2)$ is the Lie algebra of the group of rigid motions in the plane.
Examples in Lie groups

Let $G$ be a **Lie group of dimension 4**, and $g$ a left invariant metric.
Any Killing-Yano tensor is also parallel (Andrada, Barberis and Moroianu).
In what follows we’ll search 4-dimensional metric Lie algebras that admit a parallel skew symmetric isomorphism.
Such a lie algebra must be **solvable** by a result of Chu.

**Theorem (Andrada-Dotti)**

*The only 4-dimensional solvable Lie algebras with non-trivial center that admits a metric and a parallel invertible tensor are* $\mathbb{R}^4$, $\text{aff}(\mathbb{R}) \times \mathbb{R}^2$ *and* $\mathfrak{e}(2) \times \mathbb{R}$

- $\text{aff}(\mathbb{R})$ is the 2 dimensional non abelian Lie algebra.
- $\mathfrak{e}(2)$ is the Lie algebra of the group of rigid motions in the plane.
We now consider the case with possible trivial centre.

**Theorem**

The unique 4-dimensional solvable metric Lie algebras admitting a parallel tensor are: \(\text{aff}(\mathbb{R}) \times \text{aff}(\mathbb{R}), \mathbb{R} \times e(2), \mathbb{R}^2 \times \text{aff}(\mathbb{R}), r'_4, \alpha, 0, \delta'_4, \lambda, \delta_4, \frac{1}{2}, \delta_4, 1, \delta_4, 2\).

\[-r'_4, \alpha, 0 = \mathbb{R} e_0 \ltimes_{ad_{e_0}} \mathbb{R}^3, \quad ad_{e_0} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & \beta & 0 \end{bmatrix} \text{ with } \alpha > 0, \beta \neq 0.\]

\[-\delta'_4, \lambda = \mathbb{R} e_0 \ltimes_{ad_{e_0}} h_3, \quad ad_{e_0} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & -1 \\ 0 & 1 & \beta \end{bmatrix}.\]

\(- \delta_4, 1/2, \delta_4, 1, \delta_4, 2\) are Lie algebras of the form \(\mathbb{R} e_0 \ltimes_{ad_{e_0}} h_3\) where \(ad_{e_0}\) acts as a diagonal matrix. For example, \(\delta_4, \frac{1}{2}\) is the Lie algebra of a Lie group which acts simply and transitively on the complex space \(\mathbb{C}H^2\).
We now consider the case with possible trivial centre.

**Theorem**

The unique 4-dimensional solvable metric Lie algebras admitting a parallel tensor are: \( \text{aff}(\mathbb{R}) \times \text{aff}(\mathbb{R}), \mathbb{R} \times \mathfrak{e}(2), \mathbb{R}^2 \times \text{aff}(\mathbb{R}), \mathfrak{r}'_4,\alpha,0,\delta'_4,\lambda, \delta_4,\frac{1}{2}, \delta_4,1, \delta_4,2. \)

- \( \mathfrak{r}'_4,\alpha,0 = \mathbb{R} \mathfrak{e}_0 \ltimes \text{ad}_{\mathfrak{e}_0} \mathbb{R}^3, \text{ad}_{\mathfrak{e}_0} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & \beta & 0 \end{bmatrix} \) with \( \alpha > 0, \beta \neq 0. \)

- \( \delta'_4,\lambda = \mathbb{R} \mathfrak{e}_0 \ltimes \text{ad}_{\mathfrak{e}_0} \mathfrak{h}_3, \text{ad}_{\mathfrak{e}_0} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & -1 \\ 0 & 1 & \beta \end{bmatrix}. \)

- \( \delta_4,\frac{1}{2}, \delta_4,1, \delta_4,2 \) are Lie algebras of the form \( \mathbb{R} \mathfrak{e}_0 \ltimes \text{ad}_{\mathfrak{e}_0} \mathfrak{h}_3 \) where \( \text{ad}_{\mathfrak{e}_0} \) acts as a diagonal matrix. For example, \( \delta_4,\frac{1}{2} \) is the Lie algebra of a Lie group which acts simply and transitively on the complex space \( \mathbb{C}H^2. \)
<table>
<thead>
<tr>
<th>Lie algebras</th>
<th>brackets</th>
<th>metrics</th>
</tr>
</thead>
</table>
| aff(\(\mathbb{R}\)) \times aff(\(\mathbb{R}\)) | \[
\begin{align*}
[e_0, e_1] &= 0 \\
[e_0, e_2] &= \alpha e_2 \\
[e_0, e_3] &= \frac{1}{\alpha \left\| e_1 \right\|} e_3 \\
[e_2, e_3] &= 0 \\
[e_1, e_2] &= e_2 \\
[e_1, e_3] &= -e_3
\end{align*}
\] | \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & r_1 & 0 & 0 \\
0 & 0 & r_2 & 0 \\
0 & 0 & 0 & r_3
\end{bmatrix}
\] |
| \(\mathbb{R} \times \mathfrak{e}(2)\) | \[
\begin{align*}
[e_0, e_2] &= \alpha e_3 \\
[e_0, e_3] &= -\alpha e_2 \\
[e_2, e_3] &= [e_0, e_1] = 0 \\
[e_1, e_2] &= e_3 \\
[e_3, e_1] &= e_2
\end{align*}
\] | \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] |
| \(\mathbb{R}^2 \times \text{aff}(\mathbb{R})\) | \[
\begin{align*}
[e_0, e_1] &= 0 \\
[e_0, e_2] &= 0 \\
[e_0, e_3] &= \alpha e_3 \\
[e_2, e_3] &= 0 \\
[e_1, e_2] &= 0 \\
[e_1, e_3] &= 0
\end{align*}
\] | \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] |
<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>bracket</th>
<th>metrics</th>
</tr>
</thead>
</table>
| $\mathbb{R} \times \mathfrak{r}_4'$, $\alpha$, 0 | $[e_0, e_1] = \beta e_1$  
$[e_0, e_2] = \alpha e_3$  
$[e_0, e_3] = -\alpha e_2$  
$[e_2, e_3] = 0$  
$[e_1, e_2] = 0$  
$[e_1, e_3] = 0$ | $g_{e_0, ae_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$ |
| $\delta_4'$, $\lambda$ | $[e_0, e_1] = \lambda e_1$  
$[e_0, e_2] = \lambda e_2 + e_3$  
$[e_0, e_3] = -e_2 + \lambda e_3$  
$[e_2, e_3] = 0$  
$[e_1, e_2] = ke_3$ $k \neq 0$  
$[e_1, e_3] = 0$  
$\lambda = \frac{k}{2\alpha}$ | $g_{e_0, e_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\delta_4$, $2\lambda_1$ | $[e_0, e_1] = -2\lambda_2 e_1$  
$[e_0, e_2] = 2\lambda_2 e_2$  
$[e_0, e_3] = -\lambda_2 e_3$  
$[e_2, e_3] = 0$  
$[e_1, e_2] = ke_3$ $k \neq 0$  
$[e_1, e_3] = 0$  
$\lambda_2 = \pm \frac{k}{2}$ | $g_{e_0, e_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\delta_4$, 1 | $[e_0, e_1] = -2\lambda_2 e_1$  
$[e_0, e_2] = 2\lambda_2 e_2$  
$[e_0, e_3] = -\lambda_2 e_3$  
$[e_2, e_3] = 0$  
$[e_1, e_2] = ke_3$ $k \neq 0$  
$[e_1, e_3] = 0$  
$\lambda_2 = \pm \frac{k}{2}$ | $g_{e_0, e_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\delta_4$, $\frac{1}{2}$ | $[e_0, e_1] = \lambda_1 e_1$  
$[e_0, e_2] = \lambda_1 e_2$  
$[e_0, e_3] = 2\lambda_1 e_3$  
$[e_2, e_3] = 0$  
$[e_1, e_2] = ke_3$ $k \neq 0$  
$[e_1, e_3] = 0$  
$\lambda_1 = \pm \frac{k}{2}$ | $g_{e_0, e_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
Some comments about the proof

- Solvable Lie algebras of dimension 4 are classified. They can be written in the following way:

\[ g = \mathbb{R}e_0 \ltimes \varphi \mathcal{U} \]

with \( \mathcal{U} \) an **unimodular** ideal of codimension 1. So, \( \mathcal{U} \) is isomorphic to one of the following Lie algebras.

\[ \mathcal{U} \cong \mathbb{R}^3, h_3, e(2) \oplus e(1,1) \]

- Given a metric \( g \) in \( g \), we can choose \( e_0 \) so that \( e_0 \perp \mathcal{U} \).

- The restriction of \( g \) to \( \mathcal{U} \) is a metric in a 3-dimensional unimodular Lie algebra. By a result of Milnor, every 3-dimensional unimodular metric Lie algebra has an orthonormal basis such that

\[
[e_1, e_2] = \lambda_3 e_3, \ [e_2, e_3] = \lambda_1 e_1, \ [e_3, e_1] = \lambda_2 e_2
\]

The values \( \lambda_i \) determine the metric Lie algebra.
Some comments about the proof

- Solvable Lie algebras of dimension 4 are classified. They can be written in the following way:

\[ g = \mathbb{R}e_0 \ltimes \varphi \mathcal{U} \]

with \( \mathcal{U} \) an unimodular ideal of codimension 1. So, \( \mathcal{U} \) is isomorphic to one of the following Lie algebras.

\[ \mathcal{U} \cong \mathbb{R}^3, \mathfrak{h}_3, \mathfrak{e}(2) \oplus \mathfrak{e}(1,1) \]

- Given a metric \( g \) in \( g \), we can choose \( e_0 \) so that \( e_0 \perp \mathcal{U} \).

- The restriction of \( g \) to \( \mathcal{U} \) is a metric in a 3-dimensional unimodular Lie algebra. By a result of Milnor, every 3-dimensional unimodular metric Lie algebra has an orthonormal basis such that

\[ [e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2 \]

The values \( \lambda_i \) determine the metric Lie algebra.
Some comments about the proof

- Solvable Lie algebras of dimension 4 are classified. They can be written in the following way:

\[ g = \mathbb{R} e_0 \ltimes \varphi \mathcal{U} \]

with \( \mathcal{U} \) an unimodular ideal of codimension 1. So, \( \mathcal{U} \) is isomorphic to one of the following Lie algebras.

\[ \mathcal{U} \cong \mathbb{R}^3, \mathfrak{h}_3, \mathfrak{e}(2) \oplus \mathfrak{e}(1,1) \]

- Given a metric \( g \) in \( g \), we can choose \( e_0 \) so that \( e_0 \perp \mathcal{U} \).

- The restriction of \( g \) to \( \mathcal{U} \) is a metric in a 3-dimensional unimodular Lie algebra. By a result of Milnor, every 3-dimensional unimodular metric Lie algebra has an orthonormal basis such that

\[ [e_1, e_2] = \lambda_3 e_3, \ [e_2, e_3] = \lambda_1 e_1, \ [e_3, e_1] = \lambda_2 e_2 \]

The values \( \lambda_i \) determine the metric Lie algebra.
Some comments about the proof

- Solvable Lie algebras of dimension 4 are classified. They can be written in the following way:

\[ g = \mathbb{R}e_0 \ltimes \varphi U \]

with \( U \) an **unimodular** ideal of codimension 1. So, \( U \) is isomorphic to one of the following Lie algebras.

\[ U \cong \mathbb{R}^3, \mathfrak{h}_3, \mathfrak{e}(2) \oplus \mathfrak{e}(1,1) \]

- Given a metric \( g \) in \( g \), we can choose \( e_0 \) so that \( e_0 \perp U \).

- The restriction of \( g \) to \( U \) is a metric in a 3-dimensional unimodular Lie algebra. By a result of Milnor, every 3-dimensional unimodular metric Lie algebra has an orthonormal basis such that

\[ [e_1, e_2] = \lambda_3 e_3, [e_2, e_3] = \lambda_1 e_1, [e_3, e_1] = \lambda_2 e_2 \]

The values \( \lambda_i \) determine the metric Lie algebra.
In all the metric Lie algebras of the theorem, except \(\text{aff}(\mathbb{R}) \times \text{aff}(\mathbb{R})\), we found examples of tensors \(E\) such that \(E^2 = -I\) (complex structure).

4-dimension Lie algebras having Kähler structures with a pseudo riemannian metric were studied by Ovando in [Ov].

Any \((g, g, E)\) allows us to built a Conformal Killing Yano tensor in Lie algebras of dimension 5.

\[
g' = g \oplus \mathbb{R}e_0, \quad g(e_0, g) = 0, \quad [[e_0, g]] = 0 \quad \text{and} \quad [[x, y]] = [x, y] + \mu(x, y)e_0, \quad \forall \ x, y \in g
\]

This is a result from a work in progress by Andrada and Dotti [AD].
In all the metric Lie algebras of the theorem, except $\text{aff}(\mathbb{R}) \times \text{aff}(\mathbb{R})$, we found examples of tensors $E$ such that $E^2 = -I$ (complex structure).

4-dimension Lie algebras having Kähler structures with a pseudo riemannian metric were studied by Ovando in [Ov].

Any $(g, g, E)$ allows us to built a Conformal Killing Yano tensor in Lie algebras of dimension 5.

$$g' = g \oplus \mathbb{R}e_0, \ g(e_0, g) = 0, \ [[e_0, g]] = 0 \text{ and } [[x, y]] = [x, y] + \mu(x, y)e_0, \ \forall \ x, y \in g$$

This is a result from a work in progress by Andrada and Dotti [AD].
In all the metric Lie algebras of the theorem, except $\text{aff}(\mathbb{R}) \times \text{aff}(\mathbb{R})$, we found examples of tensors $E$ such that $E^2 = -I$ (complex structure).

4-dimension Lie algebras having Kähler structures with a pseudo riemannian metric were studied by Ovando in [Ov].

Any $(g, g, E)$ allows us to built a Conformal Killing Yano tensor in Lie algebras of dimension 5.

$$g' = g \oplus \mathbb{R}e_0, \quad g(e_0, g) = 0, [[e_0, g]] = 0 \quad \text{and} \quad [[x, y]] = [x, y] + \mu(x, y)e_0, \quad \forall \ x, y \in g$$

This is a result from a work in progress by Andrada and Dotti [AD].
Further comments

We can also consider non invertible skew symmetric endomorphisms $E$. We obtained:

- The only 4-dimensional Lie algebra that admit a parallel non invertible endomorphism $\mathbb{R}^2 \times \text{aff}(\mathbb{R})$, $\mathbb{R} \times \mathfrak{e}(2)$, $r'_4,\alpha,0$.
- The only 3-dimensional metric Lie algebras admitting a parallel skew symmetric endormorphism are: $\mathfrak{e}(2)$ and $\mathbb{R} \times \text{aff}(\mathbb{R})$.
- $\mathbb{R} \times \mathfrak{h}_3$ doesn’t admit Killing Yano tensors of any type.
Further comments

We can also consider non invertible skew symmetric endomorphisms $E$. We obtained:

- The only 4-dimensional Lie algebra that admit a parallel non invertible endomorphism $\mathbb{R}^2 \times \text{aff}(\mathbb{R})$, $\mathbb{R} \times \mathfrak{e}(2)$, $\mathfrak{r}_4',\alpha,0$.
- The only 3-dimensional metric Lie algebras admitting a parallel skew symmetric endomorphism are: $\mathfrak{e}(2)$ and $\mathbb{R} \times \text{aff}(\mathbb{R})$.
- $\mathbb{R} \times \mathfrak{h}_3$ doesn’t admit Killing Yano tensors of any type.
Further comments

We can also consider non invertible skew symmetric endomorphisms $E$. We obtained:

- The only 4-dimensional Lie algebra that admit a parallel non invertible endomorphism $\mathbb{R}^2 \times \text{aff}(\mathbb{R})$, $\mathbb{R} \times \mathfrak{e}(2)$, $\mathfrak{r}'_{4,\alpha,0}$.
- The only 3-dimensional metric Lie algebras admitting a parallel skew symmetric endormorphism are: $\mathfrak{e}(2)$ and $\mathbb{R} \times \text{aff}(\mathbb{R})$.
- $\mathbb{R} \times \mathfrak{h}_3$ doesn't admit Killing Yano tensors of any type.
An example in Flag Manifolds

Some known results:

- San Martin and Negreiros proved that any invariant nearly Kähler structure in a Flag manifold is Kähler except when $F = SL(3, \mathbb{C})/P$, where $P$ is a parabolic subgroup of $G$.

- Cassia and San Martin shows that a Nearly Kähler structure in $F = G/P$ is strict if and only if $G/P$ has height 2.

We are looking for parallel structures and Killing Yano structures on Flag Manifolds.
Some known results:

- San Martin and Negreiros proved that any invariant nearly Kähler structure in a Flag manifold is Kähler except when $F = SL(3, \mathbb{C})/P$, where $P$ is a parabolic subgroup of $G$.
- Cassia and San Martin shows that a Nearly Kähler structure in $F = G/P$ is strict if and only if $G/P$ has height 2.

We are looking for parallel structures and Killing Yano structures on Flag Manifolds.
Some known results:

- San Martin and Negreiros proved that any invariant nearly Kähler structure in a Flag manifold is Kähler except when $F = SL(3, \mathbb{C})/P$, where $P$ is a parabolic subgroup of $G$.
- Cassia and San Martin shows that a Nearly Kähler structure in $F = G/P$ is strict if and only if $G/P$ has height 2.

We are looking for parallel structures and Killing Yano structures on Flag Manifolds.
Some known results:

- San Martin and Negreiros proved that any invariant nearly Kähler structure in a Flag manifold is Kähler except when $F = SL(3, \mathbb{C})/P$, where $P$ is a parabolic subgroup of $G$.

- Cassia and San Martin shows that a Nearly Kähler structure in $F = G/P$ is strict if and only if $G/P$ has height 2.

We are looking for parallel structures and Killing Yano structures on Flag Manifolds.
Let $g = sl(3, \mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra, so

$$\mathfrak{h} = \text{span} \{ e_{11} - e_{22}, e_{22} - e_{33} \}$$

Let $R = \{ \alpha_{i,j} = e_i - e_j \}$ be the root system, where $e_i : \mathfrak{h} \rightarrow \mathbb{R}$ is defined as $e_i(H) = H_{ii}$.

Set

$$g_{\alpha_{i,j}} = \{ X \in g : [H, X] = \alpha(H) X, \; \forall \; H \in \mathfrak{h} \}$$

The following set is a Weyl basis, namely, it satisfies

$$B(X_\alpha, X_{-\alpha}) = 1,$$

where $B$ is the Killing form.

$$\{ X_\alpha \} = \left\{ \frac{e_{1,2}}{\sqrt{6}}, \frac{e_{2,1}}{\sqrt{6}}, \frac{e_{1,3}}{\sqrt{6}}, \frac{e_{3,1}}{\sqrt{6}}, \frac{e_{2,3}}{\sqrt{6}}, \frac{e_{3,2}}{\sqrt{6}} \right\}$$

We let $u = \text{span} \{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}}, i\mathfrak{h}_\mathbb{R} \}$ be a compact real form of $g$, where

$$A_{\alpha_{i,j}} = \frac{e_{i,j}}{\sqrt{6}} - \frac{e_{j,i}}{\sqrt{6}}, \quad iS_{\alpha_{i,j}} = i \left( \frac{e_{i,j}}{\sqrt{6}} + \frac{e_{j,i}}{\sqrt{6}} \right).$$
Let $g = sl(3, \mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra, so

$$\mathfrak{h} = \text{span} \{ e_{11} - e_{22}, e_{22} - e_{33} \}$$

Let $R = \{ \alpha_{i,j} = e_i - e_j \}$ be the root system, where $e_i : \mathfrak{h} \longrightarrow \mathbb{R}$ is defined as $e_i(H) = H_{ii}$.

Set

$$\mathfrak{g}_{\alpha_{i,j}} = \{ X \in \mathfrak{g} : [H, X] = \alpha(H) X, \ \forall \ H \in \mathfrak{h} \}$$

The following set is a Weyl basis, namely, it satisfies $B(X_\alpha, X_{-\alpha}) = 1$, where $B$ is the Killing form.

$$\{ X_\alpha \} = \left\{ \frac{e_{1,2}}{\sqrt{6}}, \frac{e_{2,1}}{\sqrt{6}}, \frac{e_{1,3}}{\sqrt{6}}, \frac{e_{3,1}}{\sqrt{6}}, \frac{e_{2,3}}{\sqrt{6}}, \frac{e_{3,2}}{\sqrt{6}} \right\}$$

We let $u = \text{span} \{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}}, i\mathfrak{h}_\mathbb{R} \}$ be a compact real form of $g$, where

$$A_{\alpha_{i,j}} = \frac{e_{i,j}}{\sqrt{6}} - \frac{e_{j,i}}{\sqrt{6}}, \quad iS_{\alpha_{i,j}} = i \left( \frac{e_{i,j}}{\sqrt{6}} + \frac{e_{j,i}}{\sqrt{6}} \right)$$
Let $g = sl(3, \mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra, so

$$\mathfrak{h} = \text{span} \{ e_{11} - e_{22}, e_{22} - e_{33} \}$$

Let $R = \{ \alpha_{i,j} = e_i - e_j \}$ be the root system, where $e_i : \mathfrak{h} \rightarrow \mathbb{R}$ is defined as $e_i(H) = H_{ii}$.

Set

$$g_{\alpha_{i,j}} = \{ X \in g : [H, X] = \alpha(H) X, \ \forall \ H \in \mathfrak{h} \}$$

The following set is a Weyl basis, namely, it satisfies

$$B(X_{\alpha}, X_{-\alpha}) = 1,$$

where $B$ is the Killing form.

$$\{ X_{\alpha} \} = \left\{ \frac{e_{1,2}}{\sqrt{6}}, \frac{e_{2,1}}{\sqrt{6}}, \frac{e_{1,3}}{\sqrt{6}}, \frac{e_{3,1}}{\sqrt{6}}, \frac{e_{2,3}}{\sqrt{6}}, \frac{e_{3,2}}{\sqrt{6}} \right\}$$

We let $u = \text{span} \{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}}, i\mathfrak{h}_R \}$ be a compact real form of $g$, where

$$A_{\alpha_{i,j}} = \frac{e_{i,j}}{\sqrt{6}} - \frac{e_{j,i}}{\sqrt{6}}, \quad iS_{\alpha_{i,j}} = i \left( \frac{e_{i,j}}{\sqrt{6}} + \frac{e_{j,i}}{\sqrt{6}} \right),$$

Cecilia Herrera Parallel 2-forms and Killing Yano 2-forms in low dimensional Lie
Let $\mathfrak{g} = sl(3, \mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra, so

$$\mathfrak{h} = \text{span} \{ e_{11} - e_{22}, e_{22} - e_{33} \}$$

Let $R = \{ \alpha_{i,j} = e_i - e_j \}$ be the root system, where $e_i : \mathfrak{h} \rightarrow \mathbb{R}$ is defined as $e_i(H) = H_{ii}$.

Set

$$\mathfrak{g}_{\alpha_{i,j}} = \{ X \in \mathfrak{g} : [H, X] = \alpha(H) X, \ \forall \ H \in \mathfrak{h} \}$$

The following set is a Weyl basis, namely, it satisfies

$$B(X_\alpha, X_{-\alpha}) = 1,$$

where $B$ is the Killing form.

$$\{ X_\alpha \} = \left\{ \frac{e_{1,2}}{\sqrt{6}}, \frac{e_{2,1}}{\sqrt{6}}, \frac{e_{1,3}}{\sqrt{6}}, \frac{e_{3,1}}{\sqrt{6}}, \frac{e_{2,3}}{\sqrt{6}}, \frac{e_{3,2}}{\sqrt{6}} \right\}$$

We let $u = \text{span} \{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}}, i\mathfrak{h}[\mathbb{R}] \}$ be a compact real form of $\mathfrak{g}$, where

$$A_{\alpha_{i,j}} = \frac{e_{i,j}}{\sqrt{6}} - \frac{e_{j,i}}{\sqrt{6}}, \quad iS_{\alpha_{i,j}} = i \left( \frac{e_{i,j}}{\sqrt{6}} + \frac{e_{j,i}}{\sqrt{6}} \right).$$
Let $g = sl(3, \mathbb{C})$, $\mathfrak{h}$ its **Cartan subalgebra**, so

$$\mathfrak{h} = \text{span} \left\{ e_{11} - e_{22}, e_{22} - e_{33} \right\}$$

Let $R = \{ \alpha_{i,j} = e_{i} - e_{j} \}$ be the **root system**, where $e_{i} : \mathfrak{h} \to \mathbb{R}$ is defined as $e_{i}(H) = H_{ii}$.

Set

$$g_{\alpha_{i,j}} = \{ X \in g : [H, X] = \alpha(H) X, \forall H \in \mathfrak{h} \}$$

The following set is a **Weyl basis**, namely, it satisfies $B(X_{\alpha}, X_{-\alpha}) = 1$, where $B$ is the Killing form.

$$\{ X_{\alpha} \} = \left\{ \frac{e_{1,2}}{\sqrt{6}}, \frac{e_{2,1}}{\sqrt{6}}, \frac{e_{1,3}}{\sqrt{6}}, \frac{e_{3,1}}{\sqrt{6}}, \frac{e_{2,3}}{\sqrt{6}}, \frac{e_{3,2}}{\sqrt{6}} \right\}$$

We let $u = \text{span} \left\{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}}, i\mathfrak{h}_{\mathbb{R}} \right\}$ be a **compact real form** of $g$, where

$$A_{\alpha_{i,j}} = \frac{e_{i,j}}{\sqrt{6}} - \frac{e_{j,i}}{\sqrt{6}}, \quad iS_{\alpha_{i,j}} = i \left( \frac{e_{i,j}}{\sqrt{6}} + \frac{e_{j,i}}{\sqrt{6}} \right),$$
Consider the **maximal flag** $F = U/T$, where $U$ is the Lie group $\mathfrak{u}$ and $T$ that of $i\mathfrak{h}_R$.

We let $q$ be the tangent space in the identity of $F$, so

$$q = \text{span} \{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}} \}$$

and

$$\dim q = 6$$

All the $U$-invariants metric $g$ are of the form:

$$g (X, Y) = -B (QX, Y)$$

where $Q$ is a positive endomorphism such that for all $\alpha_{i,j} \in \mathbb{R}$:

$$Q (A_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} A_{\alpha_{i,j}}$$

$$Q (iS_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} iS_{\alpha_{i,j}}$$

If $E$ is an $U$-invariant skew-symmetric isomorphism then:

$$E (A_{\alpha_{i,j}}) = \varepsilon_{\alpha_{i,j}} iS_{\alpha_{i,j}}$$

$$E (iS_{\alpha_{i,j}}) = -\varepsilon_{\alpha_{i,j}} A_{\alpha_{i,j}}$$
Consider the **maximal flag** \( F = U/T \), where \( U \) is the Lie group \( \mathfrak{u} \) and \( T \) that of \( i\mathfrak{h}_{\mathbb{R}} \).

We let \( q \) be the tangent space in the identity of \( F \), so

\[
q = \text{span} \left\{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}} \right\}
\]

and

\[
\text{dim } q = 6
\]

All the \( U \)-invariants metric \( g \) are of the form:

\[
g(X, Y) = -B(QX, Y)
\]

where \( Q \) is a positive endomorphism such that for all \( \alpha_{i,j} \in \mathbb{R} \):

\[
Q(A_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} A_{\alpha_{i,j}}
\]

\[
Q(iS_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} iS_{\alpha_{i,j}}
\]

If \( E \) is an \( U \)-invariant skew-symmetric isomorphism then:

\[
E(A_{\alpha_{i,j}}) = \varepsilon_{\alpha_{i,j}} iS_{\alpha_{i,j}}
\]

\[
E(iS_{\alpha_{i,j}}) = -\varepsilon_{\alpha_{i,j}} A_{\alpha_{i,j}}
\]
Consider the **maximal flag** \( \mathbb{F} = U/T \), where \( U \) is the Lie group \( u \) and \( T \) that of \( i\mathfrak{h}_\mathbb{R} \).

We let \( q \) be the tangent space in the identity of \( \mathbb{F} \), so

\[
q = \text{span} \left\{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}} \right\}
\]

and

\[
\dim q = 6
\]

All the \( U \)-invariants metric \( g \) are of the form:

\[
g(X, Y) = -B(QX, Y)
\]

where \( Q \) is a positive endomorphism such that for all \( \alpha_{i,j} \in \mathbb{R} \):

\[
Q(A_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} A_{\alpha_{i,j}}
\]

\[
Q(iS_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} iS_{\alpha_{i,j}}
\]

If \( E \) is an \( U \)-invariant skew-symmetric isomorphism then:

\[
E(A_{\alpha_{i,j}}) = \varepsilon_{\alpha_{i,j}} iS_{\alpha_{i,j}}
\]

\[
E(iS_{\alpha_{i,j}}) = -\varepsilon_{\alpha_{i,j}} A_{\alpha_{i,j}}
\]
Consider the maximal flag $\mathbb{F} = U/T$, where $U$ is the Lie group $\mathfrak{u}$ and $T$ that of $i\mathfrak{h}\mathbb{R}$.

We let $q$ be the tangent space in the identity of $\mathbb{F}$, so

$$q = \text{span} \{ A_{\alpha_{i,j}}, iS_{\alpha_{i,j}} \}$$

and

$$\dim q = 6$$

All the $U$-invariants metric $g$ are of the form:

$$g (X, Y) = -B (QX, Y)$$

where $Q$ is a positive endomorphism such that for all $\alpha_{i,j} \in \mathbb{R}$:

$$Q (A_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} A_{\alpha_{i,j}}$$

$$Q (iS_{\alpha_{i,j}}) = \lambda_{\alpha_{i,j}} iS_{\alpha_{i,j}}$$

If $E$ is an $U$-invariant skew-symmetric isomorphism then:

$$E (A_{\alpha_{i,j}}) = \varepsilon_{\alpha_{i,j}} iS_{\alpha_{i,j}}$$

$$E (iS_{\alpha_{i,j}}) = -\varepsilon_{\alpha_{i,j}} A_{\alpha_{i,j}}$$
The $U$-invariant isomorphism $E$ is parallel if and only if $E$ satisfies one of the following conditions for all $\alpha, \beta, \gamma \in R$ such that $\alpha + \beta + \gamma = 0$:

- $E(A_\alpha) = \varepsilon_\alpha iS_\alpha$, $E(A_\beta) = -\varepsilon_\alpha iS_\beta$, $E(A_\gamma) = -\varepsilon_\alpha iS_\gamma$ and $\lambda_\gamma = \lambda_\beta - \lambda_\alpha$
- $E(A_\alpha) = \varepsilon_\alpha iS_\alpha$, $E(A_\beta) = \varepsilon_\alpha iS_\beta$, $E(A_\gamma) = -\varepsilon_\alpha iS_\gamma$ and $\lambda_\gamma = \lambda_\beta + \lambda_\alpha$
- $E(A_\alpha) = -\varepsilon_\beta iS_\alpha$, $E(A_\beta) = \varepsilon_\beta iS_\beta$, $E(A_\gamma) = -\varepsilon_\beta iS_\gamma$ and $\lambda_{\alpha+\beta} = \lambda_\beta - \lambda_\alpha$
- $E(A_\alpha) = \varepsilon_\beta iS_\alpha$, $E(A_\beta) = \varepsilon_\beta iS_\beta$, $E(A_\gamma) = -\varepsilon_\beta iS_\gamma$ and $\lambda_{\alpha+\beta} = \lambda_\beta + \lambda_\alpha$
\[ \alpha_{1,2} + \alpha_{2,3} + \alpha_{3,1} = 0 \]

\[
E = \begin{bmatrix}
0 & -\varepsilon_{\alpha_{1,2}} & 0 & 0 & 0 & 0 \\
\varepsilon_{\alpha_{1,2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{\alpha_{1,2}} & 0 & 0 \\
0 & 0 & -\varepsilon_{\alpha_{1,2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\varepsilon_{\alpha_{1,2}} \\
0 & 0 & 0 & 0 & -\varepsilon_{\alpha_{1,2}} & 0 \\
\end{bmatrix}
\]

\[ Q = \text{diag} \left\{ \lambda_{\alpha_{1,2}}, \lambda_{\alpha_{1,2}}, \lambda_{\alpha_{2,3}}, \lambda_{\alpha_{2,3}}, \lambda_{\alpha_{1,2}} - \lambda_{\alpha_{2,3}} \lambda_{\alpha_{1,2}} - \lambda_{\alpha_{2,3}} \right\} \]

\[
E = \begin{bmatrix}
0 & -\varepsilon_{\alpha_{1,2}} & 0 & 0 & 0 & 0 \\
\varepsilon_{\alpha_{1,2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{\alpha_{1,2}} & 0 & 0 \\
0 & 0 & -\varepsilon_{\alpha_{1,2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\varepsilon_{\alpha_{1,2}} \\
0 & 0 & 0 & 0 & -\varepsilon_{\alpha_{1,2}} & 0 \\
\end{bmatrix}
\]

\[ Q = \text{diag} \left\{ \lambda_{\alpha_{1,2}}, \lambda_{\alpha_{1,2}}, \lambda_{\alpha_{2,3}}, \lambda_{\alpha_{2,3}}, \lambda_{\alpha_{1,2}} + \lambda_{\alpha_{2,3}}, \lambda_{\alpha_{1,2}} + \lambda_{\alpha_{2,3}} \right\} \]
\[
E = \begin{bmatrix}
0 & \varepsilon_{\alpha_2,3} & 0 & 0 & 0 & 0 \\
-\varepsilon_{\alpha_2,3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{\alpha_2,3} & 0 & 0 \\
0 & 0 & \varepsilon_{\alpha_2,3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon_{\alpha_2,3} \\
0 & 0 & 0 & 0 & -\varepsilon_{\alpha_2,3} & 0 \\
\end{bmatrix}
\]

\[
Q = \text{diag} \left\{ \lambda_{\alpha_1,2}, \lambda_{\alpha_2,3}, \lambda_{\alpha_2,3}, \lambda_{\alpha_2,3} - \lambda_{\alpha_1,2}, \lambda_{\alpha_2,3} - \lambda_{\alpha_1,2} \right\}
\]

\[
E = \begin{bmatrix}
0 & \varepsilon_{\alpha_2,3} & 0 & 0 & 0 & 0 \\
0 & \varepsilon_{\alpha_2,3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{\alpha_2,3} & 0 & 0 \\
0 & 0 & \varepsilon_{\alpha_2,3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon_{\alpha_2,3} \\
0 & 0 & 0 & 0 & -\varepsilon_{\alpha_2,3} & 0 \\
\end{bmatrix}
\]

\[
Q = \text{diag} \left\{ \lambda_{\alpha_1,2}, \lambda_{\alpha_1,2}, \lambda_{\alpha_2,3}, \lambda_{\alpha_2,3}, \lambda_{\alpha_2,3} + \lambda_{\alpha_1,2}, \lambda_{\alpha_2,3} + \lambda_{\alpha_1,2} \right\}
\]
The $U$-invariant isomorphism $E$ is Killing Yano if and only if $E$ satisfies one of the following conditions for all $\alpha, \beta, \gamma \in R$ such that $\alpha + \beta + \gamma = 0$

- $E(A_\alpha) = \varepsilon_\alpha iS_\alpha$,  $E(A_\beta) = \varepsilon_\alpha iS_\beta$,  $E(A_\gamma) = -\varepsilon_\alpha iS_\gamma$ and $\lambda_\gamma = \lambda_\alpha - \lambda_\beta$
- $E(A_\alpha) = -\varepsilon_\beta iS_\alpha$,  $E(A_\beta) = \varepsilon_\beta iS_\beta$,  $E(A_\gamma) = \varepsilon_\beta iS_\gamma$ and $\lambda_\gamma = \lambda_\beta - \lambda_\alpha$
- $E(A_\alpha) = -\varepsilon_\beta iS_\alpha$,  $E(A_\beta) = \varepsilon_\beta iS_\beta$,  $E(A_\gamma) = \varepsilon_\beta iS_\gamma$ and $\lambda_\gamma = \lambda_\beta$
- $E(A_\alpha) = \varepsilon_\beta iS_\alpha$,  $E(A_\beta) = \varepsilon_\beta iS_\beta$,  $E(A_\gamma) = \varepsilon_\beta iS_\gamma$ and $\lambda_\gamma = \lambda_\beta = \lambda_\alpha$
\[ Q = \text{diag} \{ \lambda_{1,2}, \lambda_{1,2}, \lambda_{2,3}, \lambda_{2,3}, \lambda_{1,2} - \lambda_{2,3}, \lambda_{2,3} - \lambda_{1,2} \} \]
\[ E = \begin{bmatrix}
0 & \varepsilon_{2,3} & 0 & 0 & 0 & 0 & 0 \\
-\varepsilon_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{2,3} & 0 & 0 & 0 \\
0 & 0 \varepsilon_{2,3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon_{2,3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_{2,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ Q = \text{diag} \left\{ \lambda_{1,2}, \lambda_{1,2}, \lambda_{2,3}, \lambda_{2,3}, \lambda_{2,3}, \lambda_{2,3} \right\} \]

\[ E = \begin{bmatrix}
0 & \varepsilon_{2,3} & 0 & 0 & 0 & 0 & 0 \\
-\varepsilon_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{2,3} & 0 & 0 & 0 \\
0 & 0 \varepsilon_{2,3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon_{2,3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_{2,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ Q = \text{diag} \left\{ \lambda_{1,2}, \lambda_{1,2}, \lambda_{1,2}, \lambda_{1,2}, \lambda_{1,2}, \lambda_{1,2}, \lambda_{1,2} \right\} \]

A. Andrada, ML Barberis and A. Moroianu, *Conformal Killing 2-forms on 4-dimensional Manifolds*

A. Andrada, I. Dotti, *Conformal Killing 2-forms*, work in production

Gracias!