Parallel 2-forms and Killing Yano 2-forms in low dimensional Lie groups and some examples in flag manifolds

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Let (M, g, J) be an **almost Hermitian manifold**, where g is the metric and $J : TM \longrightarrow TM$ is the almost complex structure. Recall that

$$J^2 = -I$$

Let ∇ be the Levi-Civita connection.

J is said **parallel** if $\nabla J = 0$, that is

$$(\nabla_X J) Y := \nabla_X JY - J\nabla_X JY = 0, \ \ \forall X, Y \ \in \mathcal{X}(M).$$

and (M, g, J) is said to be a **Kähler manifold**.

If J satisfies only the weaker condition

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Comments:

- Kähler manifolds are nearly Kähler manifolds.
- The converse is true in dimension 2 and 4.
- There are examples of homogeneous spaces of dimension 6 that are strict nearly Kähler manifolds.
  1. The maximal flag manifold $\mathbb{F} = SU(3)/S^1 \times S^1$
  2. $S^3 \times S^3$
  3. $\mathbb{C}P(3)$
  4. S^6

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Generalizing Kähler and nearly Kähler manifolds

Let (M, g) be a Riemannian manifold with a skew-symmetric isomorphism $E : TM \to TM$ (i.e, $g(EX, Y) = -g(X, EY)$. **We are not requiring the condition $E^2 = -I$.**

Consider the associated 2-forms

$$\omega(X, Y) := g(EX, Y)$$
$$\mu(X, Y) := g\left(E^{-1}X, Y\right).$$

and the Nijenhuis tensor

$$N_E(X, Y) := [EX, EY] - E([X, EY] + [EX, Y]) + E^2[X, Y].$$

E is **parallel** if one of the following **equivalent conditions** holds.

$$1)\ \nabla E = 0, \quad 2)\ d\omega = d\mu = N_E = 0, \quad 3)\ \nabla\omega = 0.$$

This generalizes the notion of Kähler manifolds.

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Requiring only the condition

$$(\nabla_X E) X = 0, \; \forall X \in \mathcal{X}(M),$$

we generalize the notion of nearly Kähler manifold. This condition is also equivalent to

$$d\omega(X, Y, Z) = 3\nabla_X \omega(Y, Z), \; \forall X, Y, Z \in \mathcal{X}(M).$$

A 2-form satisfying this condition is called a **Killing-Yano form** and E is called a **Killing-Yano tensor**.

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- If E is parallel, it is a Killing-Yano tensor. The converse is also true in dimension 2 and 4 if ω has constant norm.
- The Iwasawa Manifold $\Gamma \setminus H_3(\mathbb{C})$, where $H_3(\mathbb{C})$ is the complex Heisenberg group of dimension 3 and $\Gamma = H_3(\mathbb{Z} + i\mathbb{Z})$ admit Killing Yano tensor and it hasn't nearly Kähler structure.

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Examples in Lie groups

Let G be a **Lie group of dimension 4**, and g a left invariant metric.

Any Killing-Yano tensor is also parallel (Andrada, Barberis and Moroianu).

In what follows we'll search 4-dimensional metric Lie algebras that admit a parallel skew symmetric isomorphism.

Such a lie algebra must be **solvable** by a result of Chu.

Theorem (Andrada-Dotti)

The only 4-dimensional solvable Lie algebras with non-trivial center that admits a metric and a parallel invertible tensor are $\mathbb{R}^4, \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}^2$ *and* $\mathfrak{e}(2) \times \mathbb{R}$

-$\mathfrak{aff}(\mathbb{R})$ is the 2 dimensional non abelian Lie algebra.

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We now consider the case with possible trivial centre.

Theorem

The unique 4-dimensional solvable metric Lie algebras admitting a parallel tensor are : $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, $\mathbb{R} \times \mathfrak{e}(2)$, $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{r}'_{4,\alpha,0}$, $\delta'_{4,\lambda}$, $\delta_{4,\frac{1}{2}}$, $\delta_{4,1}$, $\delta_{4,2}$.

-$\mathfrak{r}'_{4,\alpha,0} = \mathbb{R}e_0 \ltimes_{ad_{e_0}} \mathbb{R}^3$, $ad_{e_0} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & \beta & 0 \end{bmatrix}$ with $\alpha > 0$, $\beta \neq 0$.

-$\delta'_{4,\lambda} = \mathbb{R}e_0 \ltimes_{ad_{e_0}} \mathfrak{h}_3$, $ad_{e_0} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & -1 \\ 0 & 1 & \beta \end{bmatrix}$.

- $\delta_{4,1/2}, \delta_{4,1}, \delta_{4,2}$ are Lie algebras of the form $\mathbb{R}e_0 \ltimes_{ad_{e_0}} \mathfrak{h}_3$ where ad_{e_0} acts as a diagonal matrix. For example, $\delta_{4,\frac{1}{2}}$ is the Lie algebra of a Lie group which acts simply and transitively on the complex space $\mathbb{C}H^2$.

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Lie algebras	brackets	metrics
$\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$	$[e_0, e_1] = 0$ $[e_0, e_2] = \alpha e_2$ $[e_0, e_3] = \frac{1}{\alpha \Vert e_1 \Vert} e_3$ $[e_2, e_3] = 0$ $[e_1, e_2] = e_2$ $[e_1, e_3] = -e_3$	$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_3 \end{bmatrix}$
$\mathbb{R} \times \mathfrak{e}(2)$	$[e_0, e_2] = \alpha e_3$ $[e_0, e_3] = -\alpha e_2$ $[e_2, e_3] = [e_0, e_1] = 0$ $[e_1, e_2] = e_3$ $[e_3, e_1] = e_2$	$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$	$[e_0, e_1] = 0$ $[e_0, e_2] = 0$ $[e_0, e_3] = \alpha e_3$ $[e_2, e_3] = 0$ $[e_1, e_2] = 0$ $[e_1, e_3] = 0$	$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Lie algebra	bracket	metrics
$\mathbb{R} \times \mathfrak{r}'_{4,\alpha,0}$	$[e_0, e_1] = \beta e_1$ $[e_0, e_2] = \alpha e_3$ $[e_0, e_3] = -\alpha e_2$ $[e_2, e_3] = 0$ $[e_1, e_2] = 0$ $[e_1, e_3] = 0$	$g_{e_0, ae_1, e_2, fe_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$
$\delta'_{4,\lambda}$	$[e_0, e_1] = \lambda e_1$ $[e_0, e_2] = \lambda e_2 + e_3$ $[e_0, e_3] = -e_2 + \lambda e_3$ $[e_2, e_3] = 0$ $[e_1, e_2] = ke_3 \;\; k \neq 0$ $[e_1, e_3] = 0$ $\lambda = \frac{k}{2b}$	$g_{e_0, e_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\delta_{4,2}$	$[e_0, e_1] = -2\lambda_2 e_1$ $[e_0, e_2] = \lambda_2 e_2$ $[e_0, e_3] = -\lambda_2 e_3$ $[e_2, e_3] = 0$ $[e_1, e_2] = ke_3 \;\; k \neq 0$ $[e_1, e_3] = 0$ $\lambda_2 = \pm\frac{k}{2}$	$g_{e_0, e_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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$\delta_{4,\frac{1}{2}}$	$[e_0, e_1] = \lambda_1 e_1$ $[e_0, e_2] = \lambda_1 e_2$ $[e_0, e_3] = 2\lambda_1 e_3$ $[e_2, e_3] = 0$ $[e_1, e_2] = ke_3 \;\; k \neq 0$ $[e_1, e_3] = 0$ $\lambda_1 = \pm\frac{k}{2}$	$g_{e_0, e_1, e_2, e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Some comments about the proof

- Solvable Lie algebras of dimension 4 are classified. They can be written in the following way:

$$\mathfrak{g} = \mathbb{R}e_0 \ltimes_\varphi \mathcal{U}$$

with $\mathcal{U}$ an **unimodular** ideal of codimension 1. So, $\mathcal{U}$ is isomorphic to one of the following Lie algebras.

$$\mathcal{U} \cong \mathbb{R}^3,\ \mathfrak{h}_3,\ \mathfrak{e}(2) \text{ ó } \mathfrak{e}(1,1)$$

- Given a metric g in $\mathfrak{g}$, we can choose e_0 so that $e_0 \perp \mathcal{U}$.
- The restriction of g to $\mathcal{U}$ is a metric in a 3-dimensional unimodular Lie algebra. By a result of Milnor, every 3 dimensional unimodular metric Lie algebra has an orthonormal basis such that

$$[e_1, e_2] = \lambda_3 e_3,\ [e_2, e_3] = \lambda_1 e_1,\ [e_3, e_1] = \lambda_2 e_2$$

The values λ_i determine the metric Lie algebra.

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Comments

- In all the metric Lie algebras of the theorem, except $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, we found examples of tensors E such that $E^2 = -I$ (complex structure).
- 4-dimension Lie algebras having Kähler structures with a pseudo riemannian metric were studied by Ovando in [**Ov**].
- Any $(\mathfrak{g}, g, E)$ allows us to built a Conformal Killing Yano tensor in Lie algebras of dimension 5.

$$\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{R}e_0,\ g(e_0, \mathfrak{g}) = 0,\ [[e_0, \mathfrak{g}]] = 0 \text{ and}$$
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- The only 4-dimensional Lie algebra that admit a parallel non invertible endomorphism $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathbb{R} \times \mathfrak{e}\,(2)$, $\mathfrak{r}'_{4,\alpha,0}$.
- The only 3-dimensional metric Lie algebras admitting a parallel skew symmetric endormorphism are: $\mathfrak{e}(2)$ and $\mathbb{R} \times \mathfrak{aff}\,(\mathbb{R})$.
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Some known results:

- San Martin and Negreiros proved that any invariant nearly Kähler structure in a Flag manifold is Kähler except when $\mathbb{F} = SL(3, \mathbb{C})/P$, where P is a parabolic subgroup of G.
- Cassia and San Martin shows that a Nearly Kähler structure in $\mathbb{F} = G/P$ is strict if and only if G/P has height 2.

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We are looking for parallel structures and Killing Yano structures on Flag Manifolds.

Let $\mathfrak{g} = sl\,(3, \mathbb{C})$, $\mathfrak{h}$ its **Cartan subalgebra**, so

$$\mathfrak{h} = \text{span}\left\{e_{11} - e_{22}, e_{22} - e_{33}\right\}$$

Let $R = \{\alpha_{i,j} = e_i - e_j\}$ be the **root system**, where $e_i : \mathfrak{h} \longrightarrow \mathbb{R}$ is defined as $e_i(H) = H_{ii}$.

Set

$$\mathfrak{g}_{\alpha_{i,j}} = \{X \in \mathfrak{g} : [H, X] = \alpha\,(H)\,X, \ \forall\ H \in \mathfrak{h}\}$$

The following set is a **Weyl basis**, namely, it satisfies $B\,(X_\alpha, X_{-\alpha}) = 1$, where B is the Killing form.

$$\{X_\alpha\} = \left\{\frac{e_{1,2}}{\sqrt{6}}, \frac{e_{2,1}}{\sqrt{6}}, \frac{e_{1,3}}{\sqrt{6}}, \frac{e_{3,1}}{\sqrt{6}}, \frac{e_{2,3}}{\sqrt{6}}, \frac{e_{3,2}}{\sqrt{6}}\right\}$$

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We let $\mathfrak{q}$ be the tangent space in the identity of $\mathbb{F}$, so

$$\mathfrak{q} = \operatorname{span}\left\{A_{\alpha_{i,j}}, iS_{\alpha_{i,j}}\right\}$$

and

$$\dim \mathfrak{q} = 6$$

All the U-invariants metric g are of the form:

$$g(X,Y) = -B(QX,Y)$$

where Q is a positive endomorphism such that for all $\alpha_{i,j} \in R$:

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The U-invariant isomorphism E is parallel if and only if E satisfies one of the following conditions for all $\alpha, \beta, \gamma \in R$ such that $\alpha + \beta + \gamma = 0$:

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$\alpha_{1,2} + \alpha_{2,3} + \alpha_{3,1} = 0$

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- A. Andrada, ML Barberis, I.Dotti, *Invariant solutions to the conformal Killing-Yano equation on Lie groups*, Journal of Geo. and Physis 94(2015) 199-208.
- A. Andrada, ML Barberis and A. Moroianu, *Conformal Killing 2-forms on 4-dimensional Manifolds*
- A. Andrada, I. Dotti, *Conformal Killing 2-forms*, work in production
- G. Ovando, *Invariant pseudo Kähler metrics*, Differential Geometry and its appl Vol 36(2014) pags 44-55.

Gracias!