Submanifolds of Einstein solvmanifolds

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Noncompact Homogeneous Einstein Manifolds: Big Questions

- **Question**: Which manifolds admit a metric of constant negative Ricci curvature?
- **Easier question**: Which manifolds admit a homogeneous metric of constant negative Ricci curvature?

**Alekseevskii conjecture [1975]**: Every noncompact homogeneous Einstein manifold is a *solvmanifold*, a Riemannian manifold with a solvable group of isometries.
Solvmanifolds

All our homogeneous examples will be solvable Lie groups with a left invariant metric

Definition
A simply connected solvmanifold $S$ with a left invariant metric $g$ is completely determined by its **metric Lie algebra** $(\mathfrak{s}, \langle , \rangle)$.

Definition
Write a metric solvable algebra $(\mathfrak{s}, \langle , \rangle)$ as $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$. We say $\mathfrak{s}$ is of **Iwasawa type** if

- $\mathfrak{a}$ is abelian
- $\text{ad}_A$ is symmetric relative to $\langle , \rangle$, for all $A$ in $\mathfrak{a}$
- for some $A$ in $\mathfrak{a}$, $\text{ad}_A|_{\mathfrak{n}}$ is positive definite.
For a solvmanifold $\mathfrak{s}$, the algebraic rank of $\mathfrak{s}$ is the dimension of $\mathfrak{a}$

- The nilpotency of the nilradical $\mathfrak{n}$ of $\mathfrak{s}$ is the step-size:
- $\mathfrak{n}$ is $k$-step if the $(k - 1)$ derived algebra is non-zero but the $k^{th}$ derived algebra vanishes ($[[\mathfrak{n}, [\mathfrak{n}, \ldots [\mathfrak{n}, \mathfrak{n}]..]]] = 0$).

In general, it seems difficult to find explicit examples of Einstein solvmanifolds with higher nilpotency.
Rank One Reduction

- Given any Einstein solvmanifold \((\mathfrak{s}, \langle \cdot, \cdot \rangle)\), we can find a rank one sub-solvmanifold,

\[
\mathfrak{s}' = \langle A_0 \rangle + \mathfrak{n}.
\]

- Endowed with the induced metric, the sub-solvmanifold \(\mathfrak{s}'\) is not only also Einstein, it inherits its Einstein constant from \(\mathfrak{s}\).

- This suggests the structure of \(\mathfrak{n}\) determines the Ricci geometry of \(\mathfrak{s}\).

We will see that in some cases, a comparable reduction can be done on \(\mathfrak{n}\) (and \(\alpha\)), so that again, the constant Ricci curvature is inherited.
Let $\mathfrak{g}$ be a semisimple Lie algebra.

Use a Cartan involution $\sigma$ to decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ ($\sigma|_{\mathfrak{k}} = \text{Id}, \sigma|_{\mathfrak{p}} = -\text{Id}$).

Define $B_{\sigma}(X, Y) := -B(X, \sigma(Y))$, an $\text{ad}_{\mathfrak{k}}$-invariant inner product on $\mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal torus in $\mathfrak{p}$.

$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ is root space decomposition, $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$,

$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\text{ad} A)X = \alpha(A)X \text{ for all } A \in \mathfrak{a}\}$.

In $\Delta^+$, take simple roots $\Lambda \subset \Delta^+$. 
Define $\mathfrak{n} := \sum_{\alpha \in \Delta^+} g_\alpha$, the nilradical of $g$.

Define $s := a + n$, with inner product

$$\langle , \rangle = 2B_\sigma |_{a \times a} + B_\sigma |_{n \times n}.$$

The corresponding simply connected solvmanifold $(S, g)$ is a symmetric space.
Tamaru’s construction

- Let $H^i$ in $\mathfrak{a}$ be the dual to $\alpha_i$ in $\mathfrak{a}^*$, so that $\alpha_i(H^j) = \delta_{ij}$.
- Choose a subset of fundamental roots

$$\Gamma' = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\} \subset \Lambda.$$ 

- Let $Z = H^{i_1} + H^{i_2} + \cdots + H^{i_k}$, the characteristic element.

**Definition**

- Let $\mathfrak{a}' = \text{span}\{H^{i_1}, H^{i_2}, \ldots, H^{i_k}\} \subset \mathfrak{a}$
- Let $n' = \sum_{\alpha(Z) > 0} n_{\alpha}$
- Take $s' = \mathfrak{a}' + n'$, with $\langle , \rangle' = \langle , \rangle$, restricted.
Ricci curvature

Tamaru’s original solvmanifold \((\mathfrak{s}, \langle , \rangle)\) corresponds to a noncompact symmetric space with the symmetric metric: \textit{Einstein}.

The subalgebra is constructed so that the constant Ricci curvature is unchanged:

- For any \(A, A' \in \mathfrak{a}'\) and any \(X, Y \in \mathfrak{n}'\),
- \(\text{Ric}^{s'}(A, A') = \text{Ric}^{s}(A, A')\)
- \(\text{Ric}^{s'}(A, X) = \text{Ric}^{s}(A, X) = 0\)
- \(\text{Ric}^{s'}(X, Y) = \text{Ric}^{s}(X, Y)\).
Outline of Tamaru’s Proof:

**Theorem (Wolter)**

Let \((s = \mathfrak{a} + \mathfrak{n}, \langle , \rangle)\) be a solvable metric Lie algebra of Iwasawa type. Then the Ricci curvature satisfies

1. \(\text{Ric}^s(A, A') = \text{tr}(\text{ad}_A) \circ (\text{ad}_{A'})\) for all \(A, A' \in \mathfrak{a}\),
2. \(\text{Ric}^s(X, A) = 0\) for all \(A \in \mathfrak{a}\) and \(X \in \mathfrak{n}\),
3. \(\text{Ric}^s(X, Y) = \text{Ric}^n(X, Y) - \langle \text{ad}_{H_0} X, Y \rangle\) for all \(X, Y \in \mathfrak{n}\).

Here \(H_0\) is the mean curvature vector for \(s\). Thanks to Wolter,

**Corollary**

\(\text{Ric}^s(A_1, A_2) = \text{Ric}^s(A_1, A_2)\) for every \(A_1, A_2 \in \mathfrak{a}'\).
Theorem (Alekseevskii)

Let \( \{E_i\} \) be an orthonormal basis for the nilpotent metric Lie algebra \((n, \langle , \rangle)\). The Ricci endomorphism \( \text{Ric}^n \) is given by

\[
\text{Ric}^n = \frac{1}{4} \sum (\text{ad}_{E_i}) \circ (\text{ad}_{E_i})^* - \frac{1}{2} \sum (\text{ad}_{E_i})^* \circ (\text{ad}_{E_i}).
\]

Lemma

Let \( X \in n' \) and let \( H_0^\perp = H_0 - H'_0 \). Then

\[
\text{Ric}^n(X) - \text{Ric}^{n'}(X) = [H_0^\perp, X].
\]

As above, \( H_0 \) is the MCV for \( s \), and \( H'_0 \) denotes the MCV for \( s' \).
Using the Lemma, \( \text{Ric}^n(X) - \text{Ric}^{n'}(X) = [H_0^\perp, X] \).

Combining this with Wolter’s result,

\[
\text{Ric}^s(X, Y) - \text{Ric}^{s'}(X, Y) = \text{Ric}^n(X, Y) - \text{Ric}^{n'}(X, Y)
- (\langle [H_0, X], Y \rangle - \langle [H_0', X], Y \rangle)
= \langle [H_0^\perp, X], Y \rangle - \langle [H_0 - H_0', X], Y \rangle
= \langle [H_0^\perp, X], Y \rangle - \langle [H_0^\perp, X], Y \rangle
= 0.
\]
Other consequences

- $S'$ is a minimal submanifold of $S$.
- The Einstein condition is not needed here. The trace of the second fundamental form vanishes.
- $S'$ in $S$ is not totally geodesic, as long as $\Gamma'$ and $\Lambda \setminus \Gamma'$ are not orthogonal.
Extending Tamaru’s construction

Let $s = a \oplus n$ be an Einstein solvable Lie algebra.

\[ s = a + \sum_{\alpha \in \Delta} n_\alpha \]

where $n_\alpha = \{ X \in n \mid [A, X] = \alpha(A)X \text{ for all } A \in a \}$.

\[ \Delta = \{ \alpha \in a^* \mid n_\alpha \neq 0 \} \]

Let $\Lambda = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \}$ be a set of fundamental roots.

Let $H^i$ in $a$ be the dual to $\alpha_i$ in $a^*$.

Choose a subset of fundamental roots

\[ \Gamma' = \{ \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k} \} \subset \Lambda. \]
Let $Z = H^{i_1} + H^{i_2} + \cdots + H^{i_k} \in \mathfrak{a}$.

**Definition**

The subset $\Gamma'$ defines the following subalgebras of $\mathfrak{s}$:

- $\mathfrak{a}' = \text{span}\{H^{i_1}, H^{i_2}, \ldots, H^{i_k}\} \subset \mathfrak{a}$
- $\mathfrak{n}' = \sum_{\alpha(Z) > 0} \mathfrak{n}_\alpha$
- $\mathfrak{s}' = \mathfrak{a}' + \mathfrak{n}'$

Restrict the inner product $\langle \ , \ \rangle$ on $\mathfrak{s}$ to $\mathfrak{s}'$. We say that $(\mathfrak{s}', \langle \ , \ \rangle)$ is an *attached* metric Lie subalgebra to $(\mathfrak{s}, \langle \ , \ \rangle)$. 
Lemma

Let $X \in \mathfrak{n}'$ and let $H_0 \perp = H_0 - H'_0$. Then

$$\text{Ric}^{n}(X) - \text{Ric}^{n'}(X) = [H_0 \perp, X]$$

if and only if

(i) $\sum_i (\text{ad}_{E_i'})^* E_i' \text{ is in } \alpha'$ (the mean curvature vector for $s'$), and

(ii)

$$\sum_j (\text{ad}_{E_j \perp})^* (\text{ad}_{E_j \perp})(X) - \sum_j (\text{ad}_{E_j \perp})(\text{ad}_{E_j \perp})^*(X) = \sum_j \text{ad}(\text{ad}_{E_j \perp})^* E_j \perp(X).$$
(i) Mean curvature vector

The MCV for $s$ is $H_0 = - \sum_k (\text{ad}_{E_k})^* E_k$, where the sum is over an orthonormal basis $\{E_k\}$ for all of $\mathfrak{n}$.

We choose $\{E_k\} = \{E'_i\} \cup \{E_{\perp j}\}$ where each basis vector is in a root space and $\{E'_i\}$ is a basis for $\mathfrak{n}'$, while $\{E_{\perp j}\}$ is a basis for $\mathfrak{n}^0$.

$$H_0 = - \sum_k (\text{ad}_{E_k})^* E_k = - \sum_i (\text{ad}_{E'_i})^* E'_i + - \sum_j (\text{ad}_{E_{\perp j}})^* E_{\perp j}.$$  

To preserve constant Ricci curvature, we need $- \sum_i (\text{ad}_{E'_i})^* E'_i \in \mathfrak{a}'$. This is not always the case.
(ii) When is the “ad-star” condition true?

If \((\text{ad}_{E_j^\perp})^* = \text{ad}_X\) for some \(X\) (in some larger algebra), then write \(\text{ad}_{E_j^\perp} = \text{ad}_Y\). Our condition (before summing)

\[
(\text{ad}_{E_j^\perp})^*(\text{ad}_{E_j^\perp}) - (\text{ad}_{E_j^\perp})(\text{ad}_{E_j^\perp})^* = \text{ad}(\text{ad}_{E_j^\perp})^*E_j^\perp
\]

is exactly the Jacobi Identity: \(\text{ad}_X \text{ad}_Y - \text{ad}_Y \text{ad}_X = \text{ad}[X, Y]\).

\[
\sum_j (\text{ad}_{E_j^\perp})^*(\text{ad}_{E_j^\perp})(\cdot) - \sum_j (\text{ad}_{E_j^\perp})(\text{ad}_{E_j^\perp})^*(\cdot) = \sum_j \text{ad}(\text{ad}_{E_j^\perp})^*E_j^\perp(\cdot).
\]

For our condition, we need only sum over \(n^0\) and apply to \(n'\). Although this is weaker, it does not hold in all cases.