Construction of locally conformal geometric structures on compact solvmanifolds

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Outline

In this talk we study left invariant locally conformal symplectic (LCS) and locally conformal Kähler (LCK) structures on solvable Lie groups. Beginning with an LCS (or LCK) Lie algebra and a suitable representation, we give a construction of an LCS (or LCK) structure in the semidirect product of the Lie algebra with the representation space.

A bit of Hermitian geometry:
Let (M, J, g) be a Hermitian manifold, where J is a complex structure and g is a Hermitian metric. (M, J, g) is a locally conformally Kähler (LCK) manifold if there exists an open cover $\{U_i\}$ and smooth functions f_i on U_i such that each local metric

$$g_i = exp(-f_i)g$$

is Kähler on U_i. This condition is equivalent to requiring that

$$d\omega = \theta \wedge \omega$$

for some closed 1-form θ, called the Lee form.
The Lee form θ is determined by

$$\theta = -\frac{1}{n-1}(\delta\omega) \circ J,$$

where $2n$ is the dimension of M.

A locally conformally symplectic (LCS) form on a manifold M is a non-degenerate 2-form ω such that there exists an open cover $\{U_i\}$ and smooth functions f_i on U_i such that

$$\omega_i = exp(-f_i)\omega$$

is a symplectic form on U_i.

This condition is equivalent to requiring that

$$d\omega = \theta \wedge \omega$$

for some closed 1-form θ, called the Lee form.

When the manifold is a Lie group and the LCS structure is left-invariant, then θ and ω are left invariant, and therefore we obtain $\rightsquigarrow$ an LCS structure on the Lie algebra.

Given a simply connected Lie group G, a lattice on G is a discrete subgroup Γ such that $\Gamma \backslash G$ is a compact manifold. If G is solvable (nilpotent), we have a solvmanifold (nilmanifold).

A left-invariant LCK (LCS) structure on a $G \rightsquigarrow$ an LCK (LCS) structure on $\Gamma \backslash G$. Some results:

- Sawai (2007): if a non-toral nilmanifold admits an invariant LCK structure, then it is a quotient of $\mathbb{R} \times H_{2n+1}$, where H_{2n+1} is the $(2n+1)$-dimensional Heisenberg Lie group.
- Bazzoni (2016) proved that if $\Gamma \backslash G$ is a compact nilmanifold with a Vaisman structure, then G is isomorphic to $H \times \mathbb{R}$.
- Kasuya (2013) proved the non-existence of Vaisman metrics on some solvmanifolds (O.T. manifolds).
- Andrada, O. (2014): if $\mathfrak{g}$ is a unimodular Lie algebra with an LCK structure where the complex structure is abelian, that is $[X, Y] = [JX, JY]$ then $\mathfrak{g} \simeq \mathbb{R} \times \mathfrak{h}_{2n+1}$.
- Bazzoni, Marrero (2015) proved an structure theorem for Lie algebras with LCS structures of the first kind.

OT-manifolds

Oeljeklaus and Toma (2005) constructed complex compact manifolds, called OT-manifolds of type (s, t) using algebraic number theory, and they proved that any OT-manifold of type $(s, 1)$ with $s > 0$ admit LCK metrics,
and the case $(2, 1)$ is a counterexample to a Vaisman's conjecture.
Kasuya (2013) proved that OT-manifolds of type $(s, 1)$
admit no Vaisman metric, and also proved that these manifolds
are in fact solvmanifolds.

We can see that the Lie algebra associated to these solvmanifolds are of the form $\mathfrak{g} = \mathit{aff}(\mathbb{R})^n \ltimes \mathbb{R}^2$, where $\mathit{aff}(\mathbb{R})$ is the only two dimensional non-abelian Lie algebra.
This fact motivated us to consider Lie algebras of the form $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^{2n}$.

Given a Lie algebra $\mathfrak{h}$ with an LCK structure $(J_1, \langle \cdot\,,\cdot \rangle_1)$, that is, $d\omega_1 = \theta_1 \wedge \omega_1$, we consider $\mathbb{R}^{2n}$ with the standard Kähler structure $(J_0, \langle \cdot\,,\cdot \rangle_0)$.

Let $\pi : \mathfrak{h} \to gl(2n, \mathbb{R})$ be a Lie algebra morphism.

We consider the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \ltimes_\pi \mathbb{R}^{2n}$$

where the Lie bracket are $[(X, U), (Y, V)] = ([X, Y]_{|\mathfrak{h}}, \pi(X)V - \pi(Y)U)$
We consider on $\mathfrak{g}$:

- the almost complex structure J given by $J_{|\mathfrak{h}} = J_1$ and $J_{|\mathbb{R}^{2n}} = J_0$.
- the metric $\langle \cdot\,,\cdot \rangle$ given by $\langle \cdot\,,\cdot \rangle_{|\mathfrak{h}} = \langle \cdot\,,\cdot \rangle_1$, $\langle \cdot\,,\cdot \rangle_{|\mathbb{R}^{2n}} = \langle \cdot\,,\cdot \rangle_0$, $\mathfrak{h}$ and $\mathbb{R}^{2n}$ are orthogonal.

If $\pi(X) \circ J = J \circ \pi(X)$ for all $X \in \mathfrak{h}$, then J is a complex structure on $\mathfrak{g}$.

When is $(J, \langle \cdot , \cdot \rangle)$ an LCK structure on $\mathfrak{g}$?

If we decompose

$$\pi(X) = S(X) + \rho(X),$$

with $\rho(X)$ skew-symmetric and $S(X)$ symmetric, then $(\mathfrak{g}, J, \langle \cdot , \cdot \rangle)$ is LCK if and only if

$$S(X) = -\frac{1}{2}\theta_1(X) Id$$

for all $X \in \mathfrak{h}$ and ρ is a representation.

Proposition

If $\mathfrak{h}$ is an LCK Lie algebra with Lie form θ_1, $\rho : \mathfrak{h} \to \mathfrak{u}(n)$ a representation. Then $\mathfrak{g} = \mathfrak{h} \ltimes_\pi \mathbb{R}^{2n}$ has a LCK structure where $\pi : \mathfrak{h} \to gl(2n, \mathbb{R})$ is given by $\pi(X) = -\frac{1}{2}\theta(X) + \rho(X)$.

Is $\mathfrak{g}$ unimodular?

$$\mathrm{ad}_X^{\mathfrak{g}} = \left(\begin{array}{c|c} \mathrm{ad}_X^{\mathfrak{h}} & 0 \\ \hline 0 & \pi(X) \end{array}\right)$$

$\mathrm{tr}(\mathrm{ad}_X^{\mathfrak{g}}) = \mathrm{tr}(\mathrm{ad}_X^{\mathfrak{h}}) - n\theta_1(X)$ for all $X \in \mathfrak{h}$.

$\mathfrak{g}$ is unimodular if and only if $\mathrm{tr}(\mathrm{ad}_X^{\mathfrak{h}}) = n\theta_1(X)$ for all $X \in \mathfrak{h}$.

Remark

If $\mathfrak{h}$ is solvable, then $\rho(\mathfrak{h}) \subset \mathfrak{u}(n)$ is abelian.

If $\mathfrak{h}$ is a Lie algebra with an LCS structure (ω_1, θ_1) and $\mathbb{R}^{2n}$ has the canonical symplectic form ω_0.

We can build an LCS structure on

$$\mathfrak{g} = \mathfrak{h} \ltimes_\pi \mathbb{R}^{2n},$$

in the same way as in the LCK case.

In this case π has the form

$$\pi(X) = -\frac{1}{2}\theta_1(X) + \rho(X),$$

where $\rho(X) \in sp(n, \mathbb{R})$ for all $X \in \mathfrak{h}$.

Set $\mathfrak{g}_\omega = \{x \in \mathfrak{g} : L_x\omega = 0\}$, where $L_x\omega$ is the Lie derivative of ω, or equivalently

$$\mathfrak{g}_\omega = \{x \in \mathfrak{g} : \omega([x, y], z) + \omega(y, [x, z]) = 0 \text{ for all } y, z \in \mathfrak{g}\}.$$

- If $\theta|_{\mathfrak{g}_\omega} : \mathfrak{g}_\omega \to \mathbb{R}$ is surjective, then the LCS structure (θ, ω) is of the *first kind*.
- If $\theta|_{\mathfrak{g}_\omega}$ is identically zero, then the LCS structure is of the *second kind*.

[Bazzoni, Marrero] (2015) studied LCS structures of the first kind.

In our construction, all the LCS structures obtained are of the second kind.

Almost abelian Lie groups

A Lie algebra $\mathfrak{g}$ is called almost abelian if it has an abelian ideal of codimension one:

$$\mathfrak{g} = \mathbb{R} \ltimes_M \mathbb{R}^n.$$

The associated simple conected Lie group $G = \mathbb{R} \ltimes_\phi \mathbb{R}^{2n+1}$, where $\phi(t) = e^{tM}$.

Theorem

If $\Gamma \backslash G$ is a solvmanifold, with G almost abelian, admitting an invariant LCK structure, then $\dim G = 4$.

Theorem

For any $n \geq 1$ there exists a solvmanifold $\Gamma \backslash G$, with G a $(2n+2)$-dimensional almost abelian Lie group, admitting an invariant LCS structure.

Let $\mathfrak{g}$ be a $2n+2$-dimensional almost abelian Lie algebra given by $\mathfrak{g} = \mathbb{R}f_1 \ltimes_M \mathbb{R}^{2n+1}$ with

$$M = \left(\begin{array}{c|ccccccc} 1 & & & & & & & \\ \hline & 0 & & & & & & \\ & & -\frac{1}{n} & & & & & \\ & & & \frac{1}{n} & & & & \\ & & & & -\frac{2}{n} & & & \\ & & & & & \frac{2}{n} & & \\ & & & & & & \ddots & \\ & & & & & & & -1 \end{array}\right). \tag{1}$$

The simply connected Lie group associated to $\mathfrak{g}$ is $G = \mathbb{R} \ltimes_\phi \mathbb{R}^{2n+1}$ where ϕ is given by

$$\phi(t) = e^{tM} = \left(\begin{array}{c|cccccccc} e^t & & & & & & & \\ \hline & 1 & & & & & & \\ & & e^{-\frac{t}{n}} & & & & & \\ & & & e^{\frac{t}{n}} & & & & \\ & & & & e^{-\frac{2t}{n}} & & & \\ & & & & & e^{\frac{2t}{n}} & & \\ & & & & & & \ddots & \\ & & & & & & & e^{-t} \end{array}\right).$$

For each $m > 2$, let

$$t_m = n \operatorname{arccosh}\left(\frac{m}{2}\right), \quad t_m > 0.$$

$$\Gamma_m =: t_m\mathbb{Z} \ltimes_\phi P^{-1}\mathbb{Z}^{2n+1},$$

define a lattice in G, where P satisfies $P\phi(t_m)P^{-1} = B$ and
For all $m > 2$, $\Gamma_m \backslash G$ is a solvmanifold with an LCS structure.

Gracias!!!

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