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Isometries on pseudo-Riemannian nilmanifolds

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A pseudo-Riemannian nilmanifold is a pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$ admitting a transitive action by isometries of a nilpotent Lie group.

In the Riemannian situation WOLF'62:

• Let $N \subseteq Iso(M)$ be a connected nilpotent Lie group acting transitively on M, then

- N is unique: it is the nilradical of $Iso(M)$,
- the action is simple.

So M is identified with $(N, \langle\,,\,\rangle)$ with a metric invariant by translations on the left. Moreover

$$Iso(M) = N \rtimes H,$$

where H denotes the isotropy subgroup at the identity and it coincides with

$$H = Aut(N) \cap Iso(N)$$

How is the situation for homogeneous pseudo-Riemannian nilmanifolds?

The previous theorem does not hold

Example: We have a 2-step nilpotent Lie group N of dimension four acting simply and transitively on a Lorentzian manifold M such that

- $Iso(M) \cap Aut(N) \subsetneq H$
- $N \subset Iso(M)$ is not normal
- The action of the nilradical $\tilde{N} \subset Iso(M)$ is not transitive.

$$Iso(N) = N \cdot H$$

The isotropy subgroup H is not connected.

The subgroup of isometric automorphisms is not connected.

The example

Take $M = \mathbb{R}^4$ together with the following metric

$$g = dt(dz + \frac{1}{2}ydx - \frac{1}{2}xdy) + dx^2 + dy^2,$$

where (t, x, y, z) are usual coordinates for $\mathbb{R}^4$. Consider the maps

$$L^N_{(t_1,v_1,z_1)}(t_2, v_2, z_2) = (t_1 + t_2, v_1 + v_2, z_1 + z_2 + \frac{1}{2}v_1^t J v_2)$$

$$L^G_{(t_1,v_1,z_1)}(t_2, v_2, z_2) = (t_1 + t_2, v_1 + R(t_1)v_2, z_1 + z_2 + \frac{1}{2}v_1^t J R(t_1)v_2)$$

where J and $R(t)$ are the linear maps on $\mathbb{R}^2$ given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \ t \in \mathbb{R}.$$

For all $(t_1, v_1, z_1) \in \mathbb{R}^4$ the sets

$$\{L^N_{(t_1,v_1,z_1)}\} := N \quad \text{and} \quad \{L^G_{(t_1,v_1,z_1)}\} := G$$

build Lie groups acting simply and transitively on $(\mathbb{R}^4, g)$ so that M can be represented

$$M \simeq G \quad \text{or} \quad M \simeq N$$

G is a solvable Lie group known as the *oscillator group* and $N = \mathbb{R} \times H_3$.

By inducing the metrics to G and N respectively we get that g is bi-invariant on G but only left-invariant on N and

$$(N, g) \simeq (G, g) \qquad \text{isometric}$$

and we have $Iso(M) = H \cdot N$ and $Iso(M) = H \cdot G$ where

$$H \simeq (\{1, -1\} \times O(2)) \ltimes \mathbb{R}^2 \quad G \text{ is symmetric}$$

and

$$L^G_{(t_1,v_1,z_1)} = L^N_{(t_1,v_1,z_1)} \circ \chi_{(t_1,0,0)} = \chi_{(t_1,0,0)} \circ L^N_{(t_1,R(-t_1)v_1,z_1)}$$

where $\chi_{(t,v,z)}$ denotes the map representing the conjugation in G by the element (t, v, z).

The nilradical has dimension five and its action is not transitive.

Notice that the connected component of the group of isometric automorphisms of N is given by

$$H_0^{aut}(N) = \{\chi_{(s,0,0)} \,:\, s \in \mathbb{R}\}$$

where $\chi_{(s,0,0)}(t, v, z) = (t, R(s)v, z)$. It has dimension ONE and so

$$H^{aut}(N) \subseteq H$$

Moreover

- $H^{aut}(N)$ has two connect components.
- N acts transitively on M but it is not a normal subgroup of $Iso_0(M)$.

At the Lie algebra level.

Let $\mathfrak{n}$ denote the Lie algebra of N, equipped with the Lorentzian metric g. Here $\dim \mathfrak{z} = 2$ and it is non-degenerate so that

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \qquad \text{where} \qquad \mathfrak{v} = \mathfrak{z}^{\perp}$$

This defines a distribution on TN by

$$TN = \mathfrak{v}N \oplus \mathfrak{z}N$$

In the Riemannian situation, at every $n \in N$

- $\mathfrak{v}N$ is the subspace of T_nN generated by the eigenvectors of the Ricci operator corresponding to negative eigenvalues;
- $\mathfrak{z}N$ is the subspace of T_nN generated by the eigenvectors of the Ricci operator corresponding to non-negative eigenvalues.

In the pseudo-Riemannian case: This is NOT TRUE.

The translations on the left preserve the distributions. So it is interesting to determine which isometries fixing e preserve the splitting $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Denote by

$$H^{split}(N) = \{ \text{ isometries of } N \text{ fixing } e\}.$$

In the example above the eigenvalue 0 has eigenspace intersecting both $\mathfrak{v}$ and $\mathfrak{z}$.

Lemma

Del Barco - O. '14 *Let* $(N, \langle \, , \, \rangle)$ *be a 2-step nilpotent Lie group such that* $\langle \, , \, \rangle$ *is a pseudo-Riemannian left-invariant metric for which the center is non-degenerate. Assume*

$$\mathfrak{v}^{\mathbb{C}} = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_j}, \quad \mathfrak{z}^{\mathbb{C}} = V_{\lambda_{j+1}} \oplus \ldots \oplus V_{\lambda_s}$$

for the different eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_s$ *of the Ricci operator with corresponding eigenspace* V_{λ_i}*. Then every isometry of* N *preserves the splitting* $TN = \mathfrak{v}N \oplus \mathfrak{z}N$*.*

In the general case for NON-DEGENERATE CENTER it holds

$$H^{spl}(N) = H^{aut}(N)$$

In the Riemannian situation

$$H = H^{aut}(N) = H^{spl}(N) \quad \text{Kaplan'81}$$

Define H-type Lie groups analogously to the Riemannian case

$$j(u)^2 = -\langle u, u\rangle Id \quad \forall u \in \mathfrak{z}$$

Theorem

Let denote a pseudo-H-type Lie group. Then

- $H^{aut}(N) = H^{spl}(N) = H$
- *the scalar curvature of* $(N, \langle\ ,\ \rangle)$ *is negative.*

For 2-step nilpotent Lie algebras with DEGENERATE CENTER the situation is complicated:
There exists a 2-step nilpotent Lie group with a left-invariant pseudo-Riemannian metric, degenerate center such that

- It admits an isometric automorphism which does not preserve any decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ (for any $\mathfrak{v}$!).
- It is no relationship between $H^{spl}(N)$ and $H^{aut}(N)$.

Example:*ad*-invariant metrics on 2-step nilpotent Lie algebras.
If $\mathfrak{z} = C(\mathfrak{n})$ then

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \quad \text{with} \quad \dim \mathfrak{z} = \dim \mathfrak{v}$$

both isotropic. Then for the ad-invariant metric of neutral signature (n, n):

$$H(N) = O(n, n).$$

OPEN QUESTION

HOW TO DESCRIBE THE ISOMETRY GROUP OF $(N, \langle\, , \,\rangle)$ IN ANY CASE.

At least, the case of non-degenerate center.

Dimension four Work with Justin Ryan

N is isomorphic to $H_3 \times \mathbb{R}$ whose Lie algebra is $\mathfrak{n} = \mathfrak{h}_3 \oplus \mathbb{R} = span\{e_1, e_2, e_3, e_4\}$ and

$$[e_1, e_2] = e_3$$

To study the left-invariant metrics on N, we reduce to $\mathfrak{n}$. And we have the cases

- non-degenerate center or
- degenerate center

Non degenerate center

$$g_0 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad g_1 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \varepsilon_4 \end{pmatrix}.$$

where $\varepsilon_i = \pm 1$ independently and $\mu \neq 0$.

So the Ricci operator for g_1 has the form

$$\mathrm{Rc} = \frac{\varphi}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \varphi = \varepsilon_1\varepsilon_2\mu.$$

See the eigenspaces... Not scalar flat...

For g_0 we have null commutator and the Ricci operator follows

$$\mathrm{Rc} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\varepsilon_1\varepsilon_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which are scalar flat.

Degenerate center

$$g_2 = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_4 \end{pmatrix} \quad g_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad g_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & 0 \\ 1 & 0 & 0 & \end{pmatrix}.$$

- g_2 corresponds to the trivial extension of the flat metric on H_3, so is flat.
- g_3 is flat since the center is completely null (Cordero and Parker).
- For g_4 the Ricci operator is

$$\mathrm{Rc} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}\varepsilon_1\varepsilon_2 & 0 & 0 & 0 \end{pmatrix}.$$

we have

Metric	Signature			H
g_0	$(1,3)$	$\mathbb{R}$	$\mathbb{R}$	$(\{-1,1\} \times \mathrm{O}(2)) \ltimes \mathbb{R}^2$
	$(2,2)$	$\mathbb{R}$	$\mathbb{R}$	$(\{-1,1\} \times \mathrm{O}(1,1)) \ltimes \mathbb{R}^2$
g_1	all	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
g_2	$(1,3)$	0	$\mathbb{R}^2$	$\mathrm{O}(1,3)$
	$(2,2)$	0	$\mathbb{R}^2$	$\mathrm{O}(2,2)$
g_3	$(2,2)$	N^2	N^3	$\mathrm{O}(2,2)$
g_4	$(1,3)$	0	$\mathbb{R}$	$\mathbb{R}^2$
	$(2,2)$	0	$\mathbb{R}$	$\mathbb{R}^2$

FIGURA: Isotropy subgroups of each possible metric on N. The groups N^2 and N^3 are 2- and 3-dimensional solvable groups.

Cordero - Parker' 09, del Barco - O. '14, N. Bokan, T. Šukilović, and S. Vukmirović'14, Šukilović'16.

- The Lemma holds only for g_1
- It holds "if and only if"
- Moreover, it holds "if and only if" and "if and only if scalar flat".

Questions:

(1) Does the kernel of the Ricci operator play a role?

(2) What about the “if and only if” in the Lemma?

(3) What about the relationship:

scalar flat if and only if big isometry group ($H > H^{aut}$)?

Answers in dimension five?

Dimension five

Study 2-step nilpotent Lie groups with Lie algebras:

$$\mathfrak{h}_5 \qquad \text{and} \qquad \mathfrak{h}_3 \oplus \mathbb{R}^2.$$

- Case $\mathfrak{h}_5$: Generated by e_1, e_2, e_3, e_4, e_5 with the non-trivial Lie brackets:

$$[e_1, e_2] = [e_3, e_4] = e_5$$

On H_5 take the left-invariant pseudo-Riemannian metrics induced by the following metrics on $\mathfrak{h}_5$:

$$g_{\lambda,\mu} = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix},$$

where $\varepsilon_i = \pm 1$ independently and $\lambda, \mu \neq 0$.

The Ricci operator follows

$$\mathrm{R}c_{\lambda,\mu} = \frac{1}{2}\mu \begin{pmatrix} -\varepsilon_1\varepsilon_2 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon_1\varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\varepsilon_3}{\lambda} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\varepsilon_3}{\lambda} & 0 \\ 0 & 0 & 0 & 0 & \left(\varepsilon_1\varepsilon_2 + \frac{\varepsilon_3}{\lambda}\right) \end{pmatrix}$$

The scalar curvature

$$S_{\lambda,\mu} \quad = \quad -\mu\left(\varepsilon_1\varepsilon_2 + \tfrac{\varepsilon_3}{\lambda}\right),$$

In particular, the metric $g_{\lambda,\mu}$ is scalar-flat when

$$\lambda = -\varepsilon_1\varepsilon_2\varepsilon_3.$$

Moreover, the Lemma applies to the metric $g_{\lambda,\mu}$ and $H = H^{aut}$.

For $\lambda = -\varepsilon_1\varepsilon_2\varepsilon_3$ we have

- a metric for which the assumptions of Lemma hold, but
- the metric is scalar-flat.

Question (3) is false!

Study other metrics ($\Longrightarrow$ Lots of computations)

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THANK YOU!