Effective separability of lattices in nilpotent Lie groups

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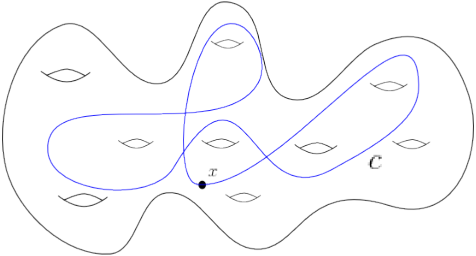
Motivation

- Let M be a connected, smooth manifold, and let c be a non-trivial closed curve based at x.



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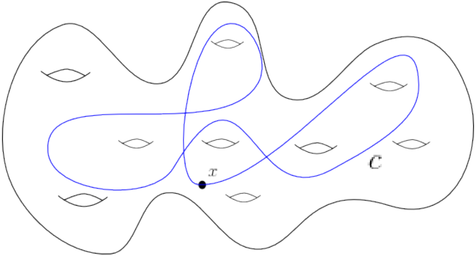

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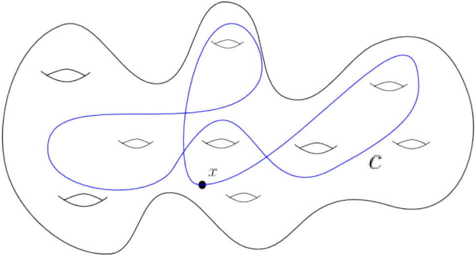

- Can we find a finite normal cover $\rho : \widetilde{M} \to M$ where c does not lift?
- Can we bound the minimal index of a finite normal cover where c does not lift?

Connection to group theory

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- When $\pi_1(M,x)$ is residually finite, then there exists an algorithm that can tell in finite time whether a based loop c is null-homotopic or not (Mal'tsev 1958).

Manifolds with residually finite fundamental groups

Some examples include

- Surfaces and hyperbolic 3-manifolds

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Our main interest are compact nilmanifolds which can be realized as $\Gamma \backslash G$ where G is a connected, simply connected nilpotent Lie group and $\Gamma \subset G$ is a cocompact lattice.

Lifting closed curves

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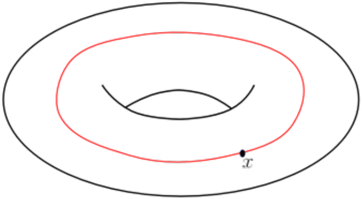

Figure: Based closed loop where it is easy to find a normal finite cover where it doesn't lift

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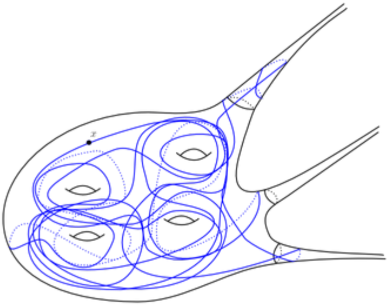

Figure: Based closed loop where it is difficult to find a normal finite cover where it doesn't lift

Complexity of lifting closed curves

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Definition

Let M be a connected, smooth manifold with $x \in M$, and let $\Gamma = \pi_1(M, x)$. Let c a closed curve based at x. Following Bou-Rabee 2010, we define

$$D_\Gamma([c]) = \min_{[\Gamma:\Delta|<\infty} \{|\Gamma : \Delta| : \Delta \trianglelefteq \Gamma \text{ and } [c] \notin \Delta\}.$$

Complexity of lifting closed curves

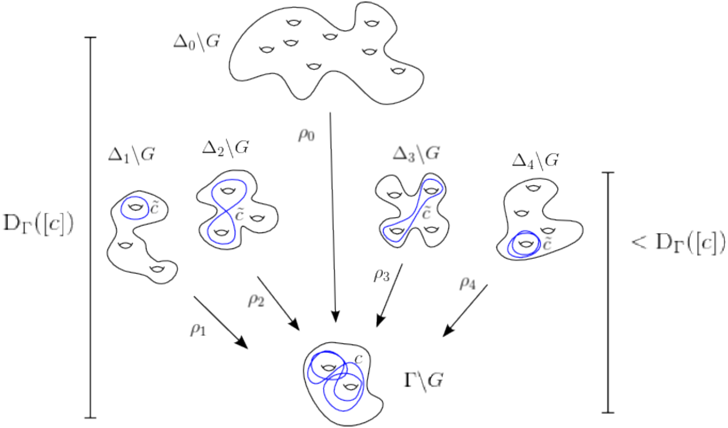

Figure: A topological interpretation of $D_\Gamma([c])$

Complexity function

Definition

Let Γ be a finitely presentable residually finite group with finite generating subset S. We define the function $\mathrm{F}_{\Gamma,S} : \mathbb{N} \to \mathbb{N}$ as

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Suppose S_1 and S_2 are two different finite generating subsets for Γ. Then there exists $C_1, C_2 \in \mathbb{N}$ such that

$$C_1 \, \mathrm{F}_{\Gamma,S_1}(n) \leq \mathrm{F}_{\Gamma,S_2}(n) \leq C_2 \, \mathrm{F}_{\Gamma,S_2}(n)$$

Survey of Results

- Nilmanifolds - Bou-Rabee 10
- Surfaces with punctures - Bou-Rabee 10, Kassabov–Matucci 11, Thom 15
- Linear groups - Bou-Rabee–McReynolds 15, Franz 16

Main Result

Theorem (P. 2016)

Let G be a connected, simply connected nilpotent Lie group, and let $\Gamma \subset G$ be a cocompact lattice with a finite generating subset S. There exists an effectively computable $\psi(G) \in \mathbb{N}$ such that $F_{\Gamma,S}(n) \approx (\log(n))^{\psi(G)}$.

What is $\psi(G)$?

Let G be a connected, simply connected nilpotent Lie group and let $\mathfrak{g}$ be its Lie algebra. Suppose that $\mathfrak{g}$ admits a basis $\{X_i\}_{i=1}^{d(\mathfrak{g})}$ with rational structure constants, and suppose that $\{X_i\}_{i=1}^{d(Z(\mathfrak{g}))}$ is a basis for $Z(\mathfrak{g})$.

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- For each $1 \leq i \leq d(Z(\mathfrak{g}))$, there exists a Lie ideal $\mathfrak{h}_i \subseteq \mathfrak{g}$ such that $Z(\mathfrak{g}/\mathfrak{h}_i) = \pi(\mathbb{R}X_i)$.

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- $\psi(G) = \max\{d(\mathfrak{g}/\mathfrak{h}_i) : 1 \leq i \leq d(Z(\mathfrak{g}))\}$.

Corollaries

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Let G be a connected, simply connected nilpotent Lie group, and let $\Gamma \subset G$ be a cocompact lattice with a finite generating subset S. Then $F_{\Gamma,S}(n) \approx (\log(n))^{dim(G)}$ if and only if $dim(Z(G)) = 1$.

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Corollary

Let G be a connected, simply connected nilpotent Lie group, and let $\Gamma \subset G$ be a cocompact lattice with finite generating subset S. Then $\Gamma \backslash G$ is a torus if and only if $\mathrm{F}_{\Gamma,S}(n) \precnapprox (\log(n))^3$.

Thank you!