Parabolic geometries and H-type Lie algebras

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Joint work with A. Kaplan
1. A special family of distributions
2. Parabolic subalgebras with H-type nilradical
3. Tanaka’s prolongation
4. Parabolic geometries
A special family of distributions

$\mathcal{D}$ vector distribution on a smooth $M / [\mathcal{D}, \mathcal{D}] = TM$.

For every never vanishing 1-form $\lambda$ on $M$ such that $\lambda(\mathcal{D}) = 0$ define a 2-form $\omega_\lambda$ on $\mathcal{D}$ by:

$$\omega_\lambda(X, Y) = \lambda([X, Y]) \quad \text{for } X, Y \in \mathcal{D}$$

The distribution is called **fat** if $\omega_\lambda$ is non-degenerate for every $\lambda$.

A fat distribution admits a **compatible subconformal structure** if there is a subconformal metric $g$ on $D$ that induces a metric compatible with $\omega_\lambda$ for every $\lambda$. 
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Generally, a distribution $\mathcal{D}$ is called **regular** if there exists a family of distributions $\{\mathcal{D}^p\}_{p < 0}$ such that:

- $\mathcal{D} = \mathcal{D}^{-1} \subset \cdots \subset \mathcal{D}^p \subset \mathcal{D}^{p-1} \subset \cdots$
- $\mathcal{D}^{p-1} = \mathcal{D}^p + [\mathcal{D}^{-1}, \mathcal{D}^p]$ for all $p < 0$.

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We assume that $\mathcal{D}$ is **bracket-generating**, i.e. there exists $\mu \in \mathbb{N}$ such that $\mathcal{D}^{-\mu} = TM$. 
For every $x \in M$ the vector space

$$n(x) = \bigoplus_{i=-1}^{-\mu} \frac{D^i(x)}{D^{i+1}(x)} = \bigoplus_{i=-1}^{-\mu} g^i(x)$$

is endowed naturally with the structure of a graded nilpotent Lie algebra ($[g^i(x), g^j(x)] \subset g^{i+j}(x)$).

The Lie algebra $n(x)$ is called the **symbol** of $D$ at $x$.

$n(x)$ is generated by $g^{-1}(x)$, a graded Lie algebra satisfying this property is called **fundamental**.

Fix a Lie algebra $n = \bigoplus_{i=-1}^{-\mu} g^i$, a distribution $D$ is said of **constant type** $n$ if for any $x$ the symbol $n(x)$ is isomorphic to $n$. 
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Let \( n = g^{-1} \oplus g^{-2} \) be a 2-graded nilpotent Lie algebra.

**Definition**

- \( g \) is **non-singular** if \( \text{ad} \ X : g^{-1} \rightarrow g^{-2} \) is onto for every \( X \in g^{-1} \).
- \( g \) is of **Heisenberg type** (or **H-type**) if there is a graded positive inner product such that \( g^{-1} \) is a non-trivial real unitary module over the Clifford algebra \( C(g^{-2}) \) and the bracket is given by

\[
\langle [x, y], z \rangle_{g^{-2}} = \langle J_z x, y \rangle_{g^{-1}}
\]
Remark

- A distribution is Fat if and only if its symbol in each point is nonsingular.
- A distribution admits a compatible subconformal structure if and only if its symbol in each point is of Heisenberg type.
The real division algebras $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ define naturally two classes of H-type Lie algebras:

$$h_n(\mathbb{F}) = \mathbb{F}^{2n} \oplus \mathbb{F}$$ (1)

$$[(a, b), (c, d)] = a^t d - c^t b,$$

for $a, b, c, d \in \mathbb{F}^n$,

$\forall n \geq 1$ if $\mathbb{F} = \mathbb{C}, \mathbb{H}$, $n = 1$ if $\mathbb{F} = \mathbb{O}$.

$$h'_{p,q}(\mathbb{F}) = \mathbb{F}^{p+q} \oplus \mathcal{S}(\mathbb{F})$$ (2)

$$[(a, b), (c, d)] = a^t c - c^t a + b^t d - d^t b,$$

for $a, c \in \mathbb{F}^p, b, d \in \mathbb{F}^q$

$\forall p, q \geq 1$ if $\mathbb{F} = \mathbb{C}, \mathbb{H}$, $(p, q) = (1, 0)$ if $\mathbb{F} = \mathbb{O}$.

Actually $h'_{p,q}(\mathbb{C}) = h'_{p+q,0}(\mathbb{C})$. 
Parabolic subalgebras with H-type nilradical

The real division algebras $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ define naturally two classes of H-type Lie algebras:

$$h_n(\mathbb{F}) = \mathbb{F}^{2n} \oplus \mathbb{F}$$

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$$h'_{p,q}(\mathbb{F}) = \mathbb{F}^{p+q} \oplus \mathbb{S}(\mathbb{F})$$

$$(a, b), (c, d) = a^t \overline{c} - c^t \overline{a} + b^t d - \overline{d}^t b,$$

for $a, c \in \mathbb{F}^p, b, d \in \mathbb{F}^q$

$\forall p, q \geq 1$ if $\mathbb{F} = \mathbb{C}, \mathbb{H}$,   $(p, q) = (1, 0)$ if $\mathbb{F} = \mathbb{O}$.

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$\text{Mauro Subils (CONICET / UNR)}$
**Theorem**

Every real simple non-compact Lie algebra not isomorphic to $\mathfrak{so}(n, 1)$ has a unique conjugacy class of parabolic subalgebras whose nilradical is isomorphic to

$$\mathfrak{h}_n(F) = F^{2n} \oplus F,$$
$$\mathfrak{h}_{p,q}'(F) = F^{p,q} \oplus \mathcal{S}(F)$$

with $F = \mathbb{C}, \mathbb{H}, \mathbb{O}$. Correspondingly, $\mathfrak{so}(n, 1)$ has unique conjugacy class of parabolics with abelian nilradical, and is the unique simple algebra with this property.
**Theorem**

Every real simple non-compact Lie algebra not isomorphic to $\mathfrak{so}(n, 1)$ has a unique conjugacy class of parabolic subalgebras whose nilradical is of H-type.
Theorem
Every real simple non-compact Lie algebra not isomorphic to $\mathfrak{so}(n,1)$ has a unique conjugacy class of parabolic subalgebras whose nilradical is non-singular.
| $h'_n(\mathbb{C})$ | $\mathfrak{sl}(n+2, \mathbb{R})$, $\mathfrak{su}(p, n+2-p)$, $\mathfrak{sp}(2n+2, \mathbb{R})$, $\mathfrak{so}(q, n+4-q)$ $\mathfrak{so}^*(2n+4)$ for even $n$, $EI$, $Ell$, $EIII$ for $n=10$ $EV$, $EVI$, $EVII$ for $n=16$, $EVIII$, $EXI$ for $n=28$ $FII$ for $n=7$, $G$ for $n=2$ |
| $h_n(\mathbb{C})$ | $\mathfrak{sl}(n+2, \mathbb{C})$, $\mathfrak{so}(n+4)$, $\mathfrak{sp}(2n+2, \mathbb{C})$, $E_6$ for $n=10$, $E_7$ for $n=16$, $E_8$ for $n=28$ $F_4$ for $n=7$, $G_2$ for $n=2$ $\mathfrak{sp}(p+1, q+1)$ $\mathfrak{sl}(n+2, \mathbb{H})$ $FII$ $EIV$ |
| $h'_{\rho,q}(\mathbb{H})$ | |
| $h_n(\mathbb{H})$ | |
| $h'_1(\mathbb{O})$ | |
| $h'_1(\mathbb{O})$ | |
Let \( m = \bigoplus_{i < 0} g^i \) be a fundamental Lie algebra. The **Tanaka’s prolongation** of \( m \) is a graded Lie algebra

\[
g = \bigoplus_{i \in \mathbb{Z}} g^i(m) = \bigoplus_{i \in \mathbb{Z}} g^i,
\]

satisfying:

1. \( g^i(m) = g^i \) for all \( i \leq 0 \);
2. if \( X \in g^i(m) \) with \( i > 0 \) satisfies \( [X, g_{-1}] = 0 \), then \( X = 0 \);
3. \( g \) is the maximal graded Lie algebra, satisfying 1 y 2.
\( \mathfrak{m} \) is **of finite type** if \( \mathfrak{g} \) is of finite dimension.

**Theorem [Tanaka, 70]**

Let \( \mathcal{D} \) a distribution of constant type \( \mathfrak{m} \). Assume that \( \mathfrak{m} \) is of finite type then the Lie algebra of all infinitesimal automorphisms of a \( \mathcal{D} \) is finite dimensional and of dimension \( \leq \dim \mathfrak{g} \).
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Proposition

For every non-singular Lie algebra whose automorphism group acts irreducibly on $g/z$, the following conditions are equivalent:

1. to be the nilradical of a parabolic subalgebra of a simple Lie algebra;
2. to have non-trivial Tanaka prolongation;
3. to be isomorphic to one of the Lie algebras $\mathfrak{h}_n(\mathbb{C})$, $\mathfrak{h}'_n(\mathbb{C})$, $\mathfrak{h}_n(\mathbb{H})$, $\mathfrak{h}'_{p,q}(\mathbb{H})$, $\mathfrak{h}_1(\mathbb{O})$ and $\mathfrak{h}'_{1,0}(\mathbb{O})$. 
Definition

Let $H \subset G$ Lie subgroup, $\mathfrak{h} = \text{Lie}(H)$, $\mathfrak{g} = \text{Lie}(G)$. A Cartan geometry of type $(G, H)$ on $M$ is

1. an $H$-principal fiber bundle $p : \mathcal{P} \to M$,
2. a $g$-valued 1-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called Cartan connection, that verifies:
   1. $(R_h)^* \omega = h^{-1} \cdot \omega$ for all $h \in H$,
   2. $\omega(X^\dagger(\lambda)) = x$ for all $x \in \mathfrak{h}$, $\lambda \in \mathcal{P}$,
   3. $\omega(\lambda) : T_\lambda \mathcal{P} \to \mathfrak{g}$ is an isomorphism for every $\lambda \in \mathcal{P}$.

A parabolic geometry is a Cartan geometry of type $(G, P)$ where $G$ is a semisimple Lie group and $P$ a parabolic subgroup.
So we associate to every real simple non-compact Lie algebra a parabolic geometry with an underlying fat distribution that admits a compatible subconformal structure.

For example,
\( G = \mathfrak{sl}(n + 2, \mathbb{R}) \) Lagrangean contact structures,
\( G = \mathfrak{su}(p + 1, q + 1) \), non-degenerate partially integrable hypersurface type almost CR-structures of signature \((p, q)\),
\( G = \mathfrak{sp}(2n + 2, \mathbb{R}) \), contact projective structures,
\( G = \mathfrak{sp}(p + 1, q + 1) \), quaternionic contact structures of signature \((p, q)\).
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GRACIAS!