Parabolic geometries and H-type Lie algebras

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Content

A special family of distributions

$\mathcal{D}$ vector distribution on a smooth *M* / $[\mathcal{D}, \mathcal{D}] = TM$.
For every never vanishing 1-form λ on *M* such that $\lambda(\mathcal{D}) = 0$ define a 2-form ω_λ on $\mathcal{D}$ by:

$$\omega_\lambda(X, Y) = \lambda([X, Y]) \quad \text{for } X,\ Y \in \mathcal{D}$$

The distribution is called **fat** if ω_λ is non-degenerate for every λ.

A fat distribution admits a **compatible subconformal structure** if there is a subconformal metric *g* on *D* that induces a metric compatible with ω_λ for every λ.

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Generally, a distribution $\mathcal{D}$ is called **regular** if there exists a family of distributions $\{\mathcal{D}^p\}_{p<0}$ such that:

- $\mathcal{D} = \mathcal{D}^{-1} \subset \cdots \subset \mathcal{D}^p \subset \mathcal{D}^{p-1} \subset \cdots$
- $\mathcal{D}^{p-1} = \mathcal{D}^p + \left[\mathcal{D}^{-1}, \mathcal{D}^p\right]$ for all $p < 0$.

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For every $x \in M$ the vector space

$$\mathfrak{n}(x) = \bigoplus_{i=-1}^{-\mu} \frac{\mathcal{D}^i(x)}{\mathcal{D}^{i+1}(x)} = \bigoplus_{i=-1}^{-\mu} \mathfrak{g}^i(x)$$

is endowed naturally with the structure of a graded nilpotent Lie algebra ($[\mathfrak{g}^i(x), \mathfrak{g}^j(x)] \subset \mathfrak{g}^{i+j}(x)$).

The Lie algebra $\mathfrak{n}(x)$ is called the **symbol** of $\mathcal{D}$ at x.

$\mathfrak{n}(x)$ is generated by $\mathfrak{g}^{-1}(x)$, a graded Lie algebra satisfying this property is called **fundamental**.

Fix a Lie algebra $\mathfrak{n} = \bigoplus_{i=-1}^{-\mu} \mathfrak{g}^i$, a distribution D is said **of constant type** $\mathfrak{n}$ if for any x the symbol $\mathfrak{n}(x)$ is isomorphic to $\mathfrak{n}$.

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Let $\mathfrak{n} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$ be a 2-graded nilpotent Lie algebra.

Definition

- $\mathfrak{g}$ is **non-singular** if *ad* $X : \mathfrak{g}^{-1} \to \mathfrak{g}^{-2}$ is onto for every $X \in \mathfrak{g}^{-1}$.
- $\mathfrak{g}$ is of **Heisenberg type** (or **H-type**) if there is a graded positive inner product such that $\mathfrak{g}^{-1}$ is a non-trivial real unitary module over the Clifford algebra $C(\mathfrak{g}^{-2})$ and the bracket is given by

$$< [x, y], z >_{\mathfrak{g}^{-2}} = < J_z x, y >_{\mathfrak{g}^{-1}}$$

Remark

- A distribution is Fat if and only if its symbol in each point is nonsingular.
- A distribution admits a compatible subconformal structure if and only if its symbol in each point is of Heisenberg type.

Parabolic subalgebras with H-type nilradical

The real division algebras $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ define naturally two classes of H-type Lie algebras:

$$\begin{aligned} \mathfrak{h}_n(\mathbb{F}) &= \mathbb{F}^{2n} \oplus \mathbb{F} \\ [(a, b), (c, d)] &= a^t d - c^t b, \end{aligned} \tag{1}$$

for a, b, c, $d \in \mathbb{F}^n$,
$\forall n \geq 1$ if $\mathbb{F} = \mathbb{C}, \mathbb{H}$, $\quad n = 1$ if $\mathbb{F} = \mathbb{O}$.

$$\begin{aligned} \mathfrak{h}'_{p,q}(\mathbb{F}) &= \mathbb{F}^{p+q} \oplus \Im(\mathbb{F}) \\ [(a, b), (c, d)] &= a^t \overline{c} - c^t \overline{a} + \overline{b}^t d - \overline{d}^t b, \end{aligned} \tag{2}$$

for a, $c \in \mathbb{F}^p$, b, $d \in \mathbb{F}^q$
$\forall p, q \geq 1$ if $\mathbb{F} = \mathbb{C}, \mathbb{H}$, $\quad (p, q) = (1, 0)$ if $\mathbb{F} = \mathbb{O}$.
Actually $h'_{p,q}(\mathbb{C}) = h'_{p+q,0}(\mathbb{C})$.

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Actually $h'_{p,q}(\mathbb{C}) = h'_{p+q,0}(\mathbb{C})$.

Theorem

Every real simple non-compact Lie algebra not isomorphic to $\mathfrak{so}(n,1)$ has a unique conjugacy class of parabolic subalgebras whose nilradical is isomorphic to

$$\mathfrak{h}_n(\mathbb{F}) = \mathbb{F}^{2n} \oplus \mathbb{F}, \qquad \mathfrak{h}'_{p,q}(\mathbb{F}) = \mathbb{F}^{p,q} \oplus \Im(\mathbb{F})$$

with $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$. Correspondingly, $\mathfrak{so}(n,1)$ has unique conjugacy class of parabolics with abelian nilradical, and is the unique simple algebra with this property.

Theorem

Every real simple non-compact Lie algebra not isomorphic to $\mathfrak{so}(n, 1)$ has a unique conjugacy class of parabolic subalgebras whose nilradical is of H-type.

Theorem

Every real simple non-compact Lie algebra not isomorphic to $\mathfrak{so}(n, 1)$ has a unique conjugacy class of parabolic subalgebras whose nilradical is non-singular.

$\mathfrak{h}'_n(\mathbb{C})$	$\mathfrak{sl}(n+2,\mathbb{R})$, $\mathfrak{su}(p,n+2-p)$, $\mathfrak{sp}(2n+2,\mathbb{R})$,
	$\mathfrak{so}(q,n+4-q)$
	$\mathfrak{so}^*(2n+4)$ for even n, *EI*, *EII*, *EIII* for $n=10$
	EV, *EVI*, *EVII* for $n=16$, *EVIII*, *EIX* for $n=28$
	FI for $n=7$, *G* for $n=2$
$\mathfrak{h}_n(\mathbb{C})$	$\mathfrak{sl}(n+2,\mathbb{C})$, $\mathfrak{so}(n+4)$, $\mathfrak{sp}(2n+2,\mathbb{C})$,
	E_6 for $n=10$, E_7 for $n=16$, E_8 for $n=28$
	F_4 for $n=7$, G_2 for $n=2$
$\mathfrak{h}'_{p,q}(\mathbb{H})$	$\mathfrak{sp}(p+1,q+1)$
$\mathfrak{h}_n(\mathbb{H})$	$\mathfrak{sl}(n+2,\mathbb{H})$
$\mathfrak{h}'_{1,0}(\mathbb{O})$	*FII*
$\mathfrak{h}_1(\mathbb{O})$	*EIV*

Tanaka's prolongation

Let $\mathfrak{m} = \bigoplus_{i<0} \mathfrak{g}^i$ be a fundamental Lie algebra. The **Tanaka's prolongation** of $\mathfrak{m}$ is a graded Lie algebra

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i(\mathfrak{m}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i,$$

satisfying:

1. $\mathfrak{g}^i(\mathfrak{m}) = \mathfrak{g}^i$ for all $i \leq 0$;
2. if $X \in \mathfrak{g}^i(\mathfrak{m})$ with $i > 0$ satisfies $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$;
3. $\mathfrak{g}$ is the maximal graded lie algebra, satisfying 1 y 2.

$\mathfrak{m}$ is **of finite type** if $\mathfrak{g}$ is of finite dimension.

Theorem [Tanaka, 70]

Let $\mathcal{D}$ a distribution of constant type $\mathfrak{m}$. Assume that $\mathfrak{m}$ is of finite type then the Lie algebra of all infinitesimal automorphisms of a $\mathcal{D}$ is finite dimensional and of dimension $\leq \dim \mathfrak{g}$.

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Proposition

For every non-singular Lie algebra whose automorphism group acts irreducibly on $\mathfrak{g}/\mathfrak{z}$, the following conditions are equivalent:

1. to be the nilradical of a parabolic subalgebra of a simple Lie algebra;
2. to have non-trivial Tanaka prolongation;
3. to be isomorphic to one of the Lie algebras $\mathfrak{h}_n(\mathbb{C})$, $\mathfrak{h}'_n(\mathbb{C})$, $\mathfrak{h}_n(\mathbb{H})$, $\mathfrak{h}'_{p,q}(\mathbb{H})$, $\mathfrak{h}_1(\mathbb{O})$ and $\mathfrak{h}'_{1,0}(\mathbb{O})$.

Parabolic geometries

Definition

Let $H \subset G$ Lie subgroup, $\mathfrak{h} = Lie(H)$, $\mathfrak{g} = Lie(G)$. A *Cartan geometry of type* (G, H) on M is

1. an H-principal fiber bundle $p : \mathcal{P} \to M$,
2. a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called *Cartan connection*, that verifies:
    1. $(R_h)^*\omega = h^{-1} \cdot \omega$ for all $h \in H$,
    2. $\omega(X^\dagger(\lambda)) = x$ for all $x \in \mathfrak{h}$, $\lambda \in \mathcal{P}$,
    3. $\omega(\lambda) : T_\lambda\mathcal{P} \to \mathfrak{g}$ is an isomorphism for every $\lambda \in \mathcal{P}$.

A *parabolic geometry* is a Cartan geometry of type (G, P) where G is a semisimple Lie group and P a parabolic subgroup.

So we associate to every real simple non-compact Lie algebra a parabolic geometry with an underlying fat distribution that admits a compatible subconformal structure.

For example,
$G = \mathfrak{sl}(n+2, \mathbb{R})$ Lagrangean contact structures,
$G = \mathfrak{su}(p+1, q+1)$, non-degenerate partially integrable hypersurface type almost CR-structures of signature (p, q),
$G = \mathfrak{sp}(2n+2, \mathbb{R})$, contact projective structures,
$G = \mathfrak{sp}(p+1, q+1)$, quaternionic contact structures of signature (p, q).

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GRACIAS!