The Calabi-Yau problem in torus bundles and generalized Monge-Ampère equations.

Luigi Vezzoni
Università di Torino

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vanishes.

In this setting a *Kähler structure* is a pair (Ω, J) where Ω is compatible with J and J is integrable.

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Given a Kähler structure (Ω, J) we define

$$\mathcal{C}_\Omega := \{\omega \in [\Omega]\,,\ \text{s.t. } \omega \text{ is compatible with } J\}$$

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dd^c-lemma. *Let $d^c := J^{-1}dJ$, then*

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Let Ric be the Ricci tensor of the metric g induced by (Ω, J). Then $\mathrm{Ric}(J\cdot, J\cdot) = \mathrm{Ric}(\cdot,\cdot)$ and $\rho(\cdot,\cdot) = \mathrm{Ric}(J\cdot,\cdot)$ is the *Ricci form* of (Ω, J).

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$$d\rho = 0\, , \quad [\rho] = 2\pi c_1(M, J).$$

Calabi-Yau's Theorem. *Let (M^{2n}, J, Ω) be a compact Kähler manifold and let $\tilde\rho \in \Lambda^{1,1}_{\mathbb{R}}$ be a closed form such that $[\tilde\rho] = 2\pi c_1(M, J)$. Then there exists a unique $\tilde\omega \in \mathcal{C}_\Omega$ such that $\tilde\rho$ is the Ricci form of $(\tilde\omega, J)$.*

THE ALMOST-KÄHLER CASE (DONALDSON/WEINKOVE)

Calabi-Yau's Theorem [Symplectic version]. *Let (M^{2n}, J, Ω) be a compact Kähler manifold and let σ be a volume form satisfying $\int_M \Omega^n = \int_M \sigma$. Then there exists a unique $\tilde{\omega} \in \mathcal{C}_\Omega$ such that*

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We can write $\sigma = e^F\,\Omega^n$, where F satisfies

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Then

$$\tilde{\omega}^n = \sigma \longleftrightarrow \begin{cases} (\Omega + d\alpha)^n = e^F \Omega^n \\ J d\alpha = d\alpha \end{cases} \quad \longleftrightarrow (\Omega + dd^c u)^n = e^F \Omega^n$$

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$(\Omega + dd^c u)^n = \mathrm{e}^F\Omega^n$ is a complex Monge-Ampère equation.

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The classical case

$$\begin{cases} \omega^n = \sigma \\ [\omega] = [\Omega]\,. \end{cases} \longrightarrow \begin{cases} (\Omega + d\alpha)^n = \mathrm{e}^F\,\Omega^n \\ Jd\alpha = d\alpha\,. \end{cases} \longrightarrow \begin{cases} (\Omega + dd^c u)^n = \mathrm{e}^F\,\Omega^n \\ d\alpha = dd^c u\,. \end{cases}$$

The case with torsion

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$$\text{CY Equation} \longleftrightarrow \begin{cases} (\Omega + d\alpha)^n = \mathrm{e}^F \Omega^n \\ Jd\alpha = d\alpha \\ d^*\alpha = 0 \end{cases} \qquad (*)$$

$(*)$ is not overdetermined for $n = 2$ and it is overdetermined for $n > 2$.

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Question: Can the Calabi-Yau Theorem be generalized to AK 4-manifolds? (At least in the special case $b^+ = 1$)

UNIQUENESS

Proposition. *In dimension 4 solutions to the CY equation are unique.*

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Proof. Let ω_1 and ω_2 be two solutions to the CY equation. Then

$$\begin{cases} \omega_1^2 = \omega_2^2 \,, \\ \omega_2 = \omega_1 + d\alpha \end{cases} \implies d\alpha^2 + 2\omega_1 \wedge d\alpha = 0\,.$$

Consider $\bar\omega = \omega_1 + \omega_2$. $\bar\omega$ is a symplectic form.

$$\bar\omega \wedge d\alpha = 0 \implies *_{\bar\omega} d\alpha = -d\alpha \implies \|d\alpha\|_{\bar\omega} = 0\,.$$

q.e.d.

S.K. Donaldson, in Inspired by S.S. Chern, World Sci. (2006)
B. Weinkove, *J.D.G.* (2006).

Existence of a solution

Donaldson's Conjecture. *Let* (M, Ω, J, σ) *be a compact symplectic 4-manifold with an acs* J *tamed by* Ω *and a normailized volume form* σ*. If* $\tilde{\omega} \in [\Omega]$ *is a symplectic form on* M *which is compatible with* J *and solving the CY equation*

$$\tilde{\omega}^2 = \sigma$$

then there are C^∞ *a priori bounds on* $\tilde{\omega}$ *depending only on* Ω, J *and* σ*.*

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Applications:

- Calabi-Yau's theorem holds on compact 4-dimensional AK manifolds with $b^+ = 1$.
- If $b^+ = 1$ and there exists Ω taming J, then there exists $\tilde{\omega}$ which is compatible with J.

S.K. Donaldson, in Inspired by S.S. Chern, World Sci. 2006

AK POTENTIAL (WEINKOVE)

Let (M, Ω, J) be a 4-dim. AK manifold and let $\tilde{\omega}$ be a J-compatible symplectic form such that $[\Omega] = [\tilde{\omega}]$. Then there exits $u \in C^\infty(M)$ (AK potential) and $a \in \Omega^1(M)$ s.t.

$$(\tilde{\omega} - \Omega) \wedge \tilde{\omega} = dd^c u \wedge \tilde{\omega}\,, \quad \tilde{\omega} = \Omega + dd^c u + da\,, \quad d^*_{\tilde{\omega}} a = 0\,,$$

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Theorem. [Weinkove]. *In order to show the solvability of the CY equation on 4-dimensional AK manifolds its enough to prove a C^0 a priori bound on the AK potential.*
That can be done if the L^1-norm of N_J is small enough.

B. Weinkove, *J.D.G.* (2006).

THE CASE OF POSITIVE CURVATURE (TOSATTI-WEINKOVE-YAU)

Given an almost-Hermitian manifold (M, g, J), there exists a unique connection ∇^C (Chen connection) satisfying

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Theorem. [Tosatti,Weinkove,Yau] *Let (M, Ω, J) be a compact AK manifold. Assume $\mathcal{R} > 0$, then Donaldson's conjecture holds.*

V. Tosatti, B. Weinkove, S.T. Yau, *Proc. London Math. Soc.*, 2008

THE CY EQUATION ON THE KODAIRA-THRUSTON MANIFOLD (TOSATTI-WEINKOVE)

The Kodaira-Thurston manifold is defined as $M = \Gamma \backslash Nil^3 \times S^1$, where

$$Nil^3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \, : \, x, y, z \in \mathbb{R} \right\}, \quad \Gamma = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \, : \, x, y, z \in \mathbb{Z} \right\}$$

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M has a global left-invariant coframe $\{e^1, e^2, e^3, e^4\}$

$$de^i = 0 , \quad i = 1, 2, 3 , \quad de^4 = e^1 \wedge e^2 , \quad (0, 0, 0, 12) .$$

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$b_1(M) = 3$ and M has no Kähler structures

[K] K.Kodaira, *Amer. J. Math.*, 1964

M is a T^2-bundle over a $\mathbb{T}^2$

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Argument of the proof:

- Writing $\sigma = e^F\,\Omega_0^2$, then every solution $\tilde\omega = \Omega_0 + d\alpha$ of the CY equation satisfies $\boxed{\mathrm{tr}_{g_0}\,\tilde g \leq \mathrm{Min}_M \Delta\, F}$
- The continuity method gives the result.

[TV] V. Tosatti, B. Weinkove, *J. Inst. Math. Jussieu*, 2011.

CY EQUATION ON THE KODAIRA-THURSTON MANIFOLD II

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$$\boxed{(1+v_{xx})(1+v_{yy}) - v_{xy}^2 = \mathrm{e}^F}$$

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Theorem. [Li]. *The Monge-Ampère equation on the standard torus $\mathbb{T}^n$ has always a solution.*

[Li] Y.Y. Li, *Comm. Pure Appl. Math.*, 1990.

CHANGING THE FIBRATION IN THE PREVIOUS CASE

Consider $(M = \Gamma\backslash Nil^3 \times S^1, J_0, \Omega_0)$ the T^2-fibration

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Here we can use the ansatz

$$\underline{\alpha = d^c v - ve^1} = (-v_t - v)e^1 - v_x e^4\,, \quad v \in C^\infty(\mathbb{T}^2_{xt})\,.$$

which implies

$$d\alpha = -v_{tx}e^{12} + (v_{tt} + v_t)e^{13} - v_{xx}e^{24} + (-v_{tx})e^{34} \in \Lambda^{1,1}_{\mathbb{R}}$$

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CY EQUATION ON T^2-BUNDLES OVER $\mathbb{T}^2$

Theorem [Fino, Li, Salamon, V / Buzano, Fino, V] *Let M be a T^2-bundle over a $\mathbb{T}^2$ equipped with an* invariant *AK structure (Ω, J). Then for every T^2-*invariant *normalized volume form $\sigma = e^F\,\Omega^2$ with $F \in C^\infty(\mathbb{T}^2)$, the corresponding CY equation has a unique solution.*

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Remarks:

1. Every orientable T^2-bundle over a $\mathbb{T}^2$ is an *infra-solvmanifold*, i.e. a finite quotient of a solvmanifold. ([Ue])

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Remarks:

1. Every orientable T^2-bundle over a $\mathbb{T}^2$ is an *infra-solvmanifold*, i.e. a finite quotient of a solvmanifold. ([Ue])

2. If $M = G$ is a 4-dimensional infra-solvmanifold equipped with an *invariant* AK structure (Ω, J). Then condition $\mathcal{R} > 0$ holds if and only if J is integrable.

CY EQUATION ON T^2-BUNDLES OVER $\mathbb{T}^2$

Theorem [Fino, Li, Salamon, V / Buzano, Fino, V] *Let M be a T^2-bundle over a $\mathbb{T}^2$ equipped with an* invariant *AK structure (Ω, J). Then for every* T^2-invariant *normalized volume form $\sigma = e^F\,\Omega^2$ with $F \in C^\infty(\mathbb{T}^2)$, the corresponding CY equation has a unique solution.*

Remarks:

1. Every orientable T^2-bundle over a $\mathbb{T}^2$ is an *infra-solvmanifold*, i.e. a finite quotient of a solvmanifold. ([Ue])

2. If $M = G$ is a 4-dimensional infra-solvmanifold equipped with an *invariant* AK structure (Ω, J). Then condition $\mathcal{R} > 0$ holds if and only if J is integrable. In particular the Tosatti-Weinkove-Yau theorem cannot be applied to the case of a T^2-bundle over a $\mathbb{T}^2$.

[Ue] M. Ue, *J. Math. Soc. Japan*, 2009.

CY EQUATION ON T^2-BUNDLES OVER $\mathbb{T}^2$

Theorem [Fino, Li, Salamon, –/ Buzano, Fino, –]. *Let M be a T^2-bundle over a $\mathbb{T}^2$ equipped with an* invariant *AK structure* (Ω, J)*. Then for every* T^2-invariant *normalized volume form $\sigma = e^F\,\Omega^2$ with $F \in C^\infty(\mathbb{T}^2)$, the corresponding CY equation has a unique solution.*

Layout of the proof:

- Using the classification of orientable T^2-bundles over $\mathbb{T}^2$;
- Classifying in each case *invariant Lagrangian* AK structures and *invariant Symplectic* AK structures;
- Rewriting the problem in terms of a Monge-Ampère equation;
- Showing that such an equation has solution.

The classification of T^2-bundles over $\mathbb{T}^2$

	G	Structure equations
i, *ii*	$\mathbb{R}^4$	$(0,0,0,0)$
iii	$Nil^3 \times \mathbb{R}$	$(0,0,0,12)$
iv, *v*	$Sol^3 \times \mathbb{R}$	$(0,0,13,41)$
vi, *vii*, *viii*	$Nil^3 \times \mathbb{R}$	$(0,0,0,12)$
ix	Nil^4	$(0,13,0,12)$

- The Lie group G is called *the geometry type*. M has Kähler structures only in the cases i, ii [G];
- in the cases iv, v M has no complex structures [FG].

[G] H. Geiges, *Duke Math. J.*, 1992.
[FG] M. Fernandez, A. Gray, *Geom. Dedicata*, 1990.

Geometry type $G = Nil^3 \times \mathbb{R}$

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In this case all the total spaces are *nilmanifolds*, all the invariant AK structures are *Lagrangian* and we can work as in the Kodaira-Thurston manifold.

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In this case the total spaces could be *infra-nilmanifolds*, invariant AK structures could be either *Lagrangian* or non-Lagrangian and the argument used in the Kodaira-Thurston case has to be modified.

Geometry type $G = Sol^3 \times \mathbb{R}$

	G	Structure equations
i, ii	$\mathbb{R}^4$	$(0,0,0,0)$
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ix	Nil^4	$(0,13,0,12)$

In this case the total space could be an *infra-solvmanifold*, all invariant AK structures are *non-Lagrangian* and the CY equation reduces to a Monge-Ampère equation.

Geometry type $G = Nil^4$

	G	Structure equations
i, *ii*	$\mathbb{R}^4$	$(0,0,0,0)$
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ix	Nil^4	$(0,13,0,12)$

In this case all total spaces are *nilmanifolds*, all invariant AK structures are *Lagrangian* and the CY reduces to the same Monge-Ampère equation for *Lagrangian* AK structures in the families *vi*), *vii*), *viii*) associated to $Nil^3 \times \mathbb{R}$.

The Monge-Ampère equation

The following equation covers all the cases

$$\boxed{A_{11}[u]A_{22}[u] - \big(A_{12}[u]\big)^2 = E_1 + E_2\, \mathrm{e}^F}$$

where

$$\begin{aligned} A_{11}[u] &= u_{xx} + B_{11}u_y + C_{11} + Du, \\ A_{12}[u] &= u_{xy} + B_{12}u_y + C_{12}, \\ A_{22}[u] &= u_{yy} + B_{22}u_y + C_{22}, \end{aligned}$$

and B_{ij}, C_{ij}, D, E_i are constants.

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In the Lagrangian case $D = 0$

Solutions to the Monge-Ampère equation

Goal: Show that $\boxed{A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + E_2\, \mathrm{e}^F}$ has a solution on $\mathbb{T}^2$.

We apply the continuity method to

$$A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-t)E_2 + tE_2\, \mathrm{e}^F \quad (*_t).$$

by defining $S := \{t \in [0,1] \ : \ (*_t) \text{ has a solution } u \in C^{2,\alpha}_0(\mathbb{T}^2)\}$ and showing that S is open and closed in $[0,1]$.

In this way we show the existence of a $C^{2,\alpha}$ solution u and a theorem of Nirenberg implies that u is C^∞.

L. Nirenberg, *Comm. Pure Appl. Math.* 1953.

Solutions to the Monge-Ampère equation

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- S is open by the implicit function theorem.
- in order to show that S is closed it's enough to give an priori bound on the first derivatives of the solutions to $(*_t)$ in view of an interior estimates proved by Heinz.

E. Heinz, in *Proc. Sympos. Pure Math.*, 1961.

CY EQUATION ON S^1-FIBRATIONS OVER A $\mathbb{T}^3$

The Kodaira-Thurston manifold has a natural structure of principal S^1-bundle over a $\mathbb{T}^3$

$$\begin{array}{ccc} S^1 & \hookrightarrow & \Gamma\backslash Nil^3 \times S^1 = M \\ & & \downarrow \\ & & \mathbb{T}^2 \times S^1 = \mathbb{T}^3_{xyt}\,. \end{array}$$

CY equation on S^1-fibrations over a $\mathbb{T}^3$

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We can study the CY problem for S^1-invariant volume forms (instead that T^2-invariant).

Theorem [Buzano-Fino- V]. *The CY equation on (M, J_0, Ω_0) can be solved for every S^1-invariant normlized volume form σ.*

PROOF OF THE THEOREM

Step 1. The system reduces to a single equation

Let $u \in C_0^\infty(\mathbb{T}^3)$. If

$$\alpha = d^c u - ue^1$$

then

$$J d\alpha = d\alpha \quad (i.e.\ d\alpha \text{ is } (1,1))$$

and the CY equation reduces to

$$(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1) - u_{xy}^2 - u_{xt}^2 = e^F.$$

PROOF OF THE THEOREM

Step 2. C^0-a priori estimates

Let $u \in C^2_0(\mathbb{T}^3)$ be such that $(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=\mathrm{e}^F$

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- $|u_x| < 1$
- $\left\|\nabla |u|^{p/2}\right\|^2_{L^2} \leq \frac{p^2}{16} \, \|u\|^p_{L^p} + \frac{5p^3}{16} \, \|1+e^F\|_{C^0} \, \|u\|^{p-1}_{L^p}$

  $\left[\text{In Yau's proof: } \left\|\nabla |\varphi|^{p/2}\right\|^2_{L^2} \leq \frac{np^2}{4p-1} \, \left(\|1-e^F\|_{C^0}\right) \|\varphi\|^{p-1}_{L^p} \right]$

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- $\|u\|_{L^2} \leq \|1+e^F\|_{C^0}$,

Finally:

PROOF OF THE THEOREM

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- $\|u\|_{L^2} \le \|1+e^F\|_{C^0}$,

Finally:

- $\|u\|_{C^0} \le C$, where $C = C(\|F\|_{C^0})$.

PROOF OF THE THEOREM

Step 3. First order estimates

Let $u \in C^4_0(\mathbb{T}^3)$ solving $(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=\mathrm{e}^F$, then

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PROOF OF THE THEOREM

Step 3. First order estimates

Let $u \in C_0^4(\mathbb{T}^3)$ solving $(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=\mathrm{e}^F$, then

- $\|\Delta u\|_{C^0} \leq C_1\big(1+\|u\|_{C^1}\big)$, where $C_1 = C_1(\|F\|_{C^2})$
- $\|u\|_{C^1} \leq C_2$, where $C_2 = C_2(\|F\|_{C^2})$.

PROOF OF THE THEOREM

Step 4. $C^{2,\rho}$ estimates

Theorem [Tosatti-Wang-Weinkove-Yang]. *Let $\tilde{\Omega}$ be be the solution of the Calabi-Yau equation. Assume there are two constants $\tilde{C}_0 > 0$ and $0 < \rho_0 < 1$ such that $F \in C^{\rho_0}(M^{2n})$ and*

$$\mathrm{tr}\, \tilde{g} \leq \tilde{C}_0,$$

Then there exist two constants $\tilde{C} > 0$ and $0 < \rho < 1$, depending only on M^{2n}, Ω, J, C_0 and $\|F\|_{C^{\rho_0}}$, such that $\|\tilde{g}\|_{C^{\rho}} \leq \tilde{C}$.

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Proposition. *Let $u \in C^4_0(\mathbb{T}^3)$ solving $(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1) - u_{xy}^2 - u_{xt}^2 = \mathrm{e}^F$. Then there exist constants $C_3 > 0$ and $\rho > 0$, both depending only on $\|F\|_{C^2}$, such that*

$$\|u\|_{C^{2,\rho}} \leq C_3.$$

PROOF OF THE THEOREM

Step 5. Continuity Method

Let S be the set of $\tau \in [0,1]$ such that

$$(u_{yy} + u_{tt} + u_t + 1)(u_{xx} + 1) - u_{xy}^2 - u_{xt}^2 = 1 - \tau + \tau\, \mathrm{e}^F$$

has a solution in $C_0^\infty(\mathbb{T}^3)$.

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Then $1 \in S$ and the claim follows. $\square$

A NEW PROOF OF OUR THEOREM (TOSATTI-WEINKOVE)

Recently Tosatti and Weinkove have provided a simplified proof of the C^0-a priori estimate for solution to

$$(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=\mathrm{e}^F$$

on $\mathbb{T}^3$ based on the Aleksandrov-Bakelman-Pucci estimate.

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Proposition [Székelyhidi]. *Let $v\colon \bar{B}_r(0)\to\mathbb{R}$ be a smooth map satisfying*

$$v(0)+\varepsilon\leq\inf_{\partial B_r(0)} v$$

for some $\varepsilon>0$. Then

$$\varepsilon^n\leq C_0\int_P \det(D^2v)$$

where
$P=\{x\in B_r(0)\ :\ |Dv(x)|<\varepsilon/2,\, v(y)>v(x)+Dv(x)(y-x)\forall y\in B_r(0)\}$
and $C_0=C_0(n)$.

Székelyhidi, preprint 2015. Tosatti and Weinkove, preprint 2016.

A NEW PROOF OF OUR THEOREM (TOSATTI-WEINKOVE)

Let $u \in C^\infty(\mathbb{T}^3)$ be such that

$$(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=\mathrm{e}^F \quad u \leq 0\,, \quad \min u < -1\,.$$

A new proof of our theorem (Tosatti-Weinkove)

Let $u \in C^\infty(\mathbb{T}^3)$ be such that

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Let $x_0 \in M$ be such that $\min_M u = u(x_0)$ and regard u as a map $u\colon B_r(0)\to\mathbb{R}$ with $0\equiv x_0$.

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$$\varepsilon^n \leq C_0 \int_P \det(D^2 v) \quad \text{and} \quad \det(D^2 v(x)) \leq C\,, \quad \forall x \in P$$

for a uniform C.

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$$\varepsilon^n \leq C_0\int_P \det(D^2v) \quad \text{and} \quad \det(D^2v(x))\leq C\,, \quad \forall x\in P$$

for a uniform C. Therefore $\varepsilon^n \leq C|P|$ and

$$\|u\|_{C^0}\leq \frac{C^{1/p}}{\varepsilon^{n/p}}\|u\|_{L^p}+1\,.$$

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On the other hand, $\Delta u + u_t > -2$ which implies that $\|u\|_{L^p}$ is uniformally buonded.

A NEW PROOF OF OUR THEOREM (TOSATTI-WEINKOVE)

Theorem [Tosatti-Weinkove]. *Let (Ω, J) be an invariant AK structure on the Kodaira-Thurston manifold M inducing the standard metric. Then the CY equation on (M, J, Ω) can be solved for every S^1-invariant normlized volume form σ.*

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Problem. *Generalize the previous theorem to every invariant AK on M.*

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Problem. *Generalize the previous theorem to every invariant AK on M.*

Proposition. It is possible to generalize the theorem if we assume $\langle e_1, e_2, e_3 \rangle$ orthogonal to e_4.

The general case on the Kodaira-Thurston manifold

(work in progress with E. Buzano, A. Fino and Y.Y. Li)

Now we consider the CY problem on the Kodaira-Thurston manifold (M, Ω_0, J_0) when σ is *not invariant*.

The general case on the Kodaira-Thurston manifold

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Now we consider the CY problem on the Kodaira-Thurston manifold (M, Ω_0, J_0) when σ is *not invariant*.

Functions on M can be regarded as functions $u \colon \mathbb{R}^4 \to \mathbb{R}$ satisfying

$$u(x + j, y + k, z + jy + m, t + n) = u(x, y, z, t),$$

for all (x, y, z, t) in $\mathbb{R}^4$ and (j, k, m, n) in $\mathbb{Z}^4$.

THE EQUATION ON THE HEISENBERG GROUP

(WORK IN PROGRESS WITH E. BUZANO, A. FINO AND Y.Y. LI)

Theorem. *Assume* $\sigma = \mathrm{e}^F\,\Omega_0^2$ *be such that* $F \in C_0^\infty(Nil^3/\Gamma)$. *Assume that*

$$\begin{aligned}&[u_y + xu_x + 1]^2(u_{xx} + u_{zz}) + [u_x^2 + u_z^2 + \mathrm{e}^F]\,[u_{yy} + x^2u_{zz} + 2xu_{yz}]\\ &- 2u_x[u_y + xu_z + 1]\,[u_{xy} + xu_{xz}] - 2u_z[u_y + xu_z + 1]\,[u_{yz} + xu_{zz}]\\ &- \mathrm{e}^F[F_y + xF_z]\,[u_y + xu_z + 1] = 0,\end{aligned}$$

has a solution u. *Then there exist* $v, w \in C_0^\infty(Nil^3/\Gamma)$ *such that*

$$\alpha = v\,e^1 + \partial_z w\,e^2 + u\,e^3 - \partial_x w\,e^4$$

solve

$$\begin{cases}(\Omega + d\alpha)^2 = \mathrm{e}^F\,\Omega^2\\ Jd\alpha = d\alpha\,.\end{cases}$$